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SOLVING LINEAR FOURTH ORDER BOUNDARY VALUE PROBLEM BY USING A HYPERBOLIC SPLINES OF ORDER 4

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Abstract: In this paper, a numerical method is developed for solving a linear fourth order boundary value problem (4VBP) by using the hyperbolic uniform spline of order 4 (lower order). Experimental results demonstrate that our method is more effective for the problems where the exact solution is hyperbolic.

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Key Words: linear fourth order boundary value problem, hyperbolic uniform spline, interpolation

1. Introduction

Much attention have been given to solve the fourth-order boundary value problems, which have application in various branches of applied sciences. These problems are generally arise in the mathematical modeling of viscoelastic flows [5]. A spline has been widely applied for the numerical solutions of some ordinary and partial differential equations in the numerical analysis. Many authors have used numerical and approximate methods to solve third and fourth-order BVPs. Some of the details about the numerical methods can be found in references [1, 10]. In a series of paper by Caglar et al. [3, 4] BVPs of order two, third, fourth and fifth were solved using third, fourth and sixth-degree splines. Lamnii et al. [7, 8] discussed a boundary-value problems based on spline interpolation and quasi-interpolation with second order convergence. The numerical analysis literature contains little on the solution

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of boundary value problems by using the hyperbolic B-splines, generally we find the splines used in the above mentioned papers are all non hyperbolic B-splines with higher degrees, which effect the computational efficiency in pratical application. This motivates us to use hyperbolic B-splines of order 4 (lower order) to solve these problems. In this paper we study a method based on the hyperbolic B-splines of order 4 for constructing numerical solutions to fourth-order boundary value problems (4BVPs) of the form:

$$y^{(4)}(\theta) + f(\theta)y^{(2)}(\theta) + g(\theta)y(\theta) = p(\theta), \tag{1}$$

with boundary conditions:

$$y(a) = a_0,\ y'(a) = a_1,\ y(b) = b_0,\ y'(b) = b_1, \tag{2}$$

where $f(\theta)$, $g(\theta)$ and $p(\theta)$ are given continuous functions defined in the bounded interval $[a, b]$, $a_i (i = 0, 1)$, and $b_i (i = 0, 1)$ are real constants. The rest of paper is organized as follows. In Section 2, we give a explicit representation of B-splines of order 4. The interpolation hyperbolic B-splines is developed in Section 3. Solution is explicited in Section 4. Numerical examples are presented in Section 5 and 6.

2. Hyperbolic B-Splines of Order 4

In this section, we briefly give a explicit representation of B-splines of order 4 and we give the interesting properties of UAH B-splines of order 4, for more details see [2, 9]. To do this, we need the following notations. Suppose k one intergre such that $k \geq 1$. Let $m_k = 4.2^k + 3$ and $h_k = \frac{b-a}{m_k-3}$. Put

$$\begin{cases} \theta^k_{-3} = \theta^k_{-2} = \theta^k_{-1} = \theta^k_0 = a, \\ \theta^k_i = a + ih, i = 1...m_k - 4, \\ \theta^k_{m_k-3} = \theta^k_{m_k-2} = \theta^k_{m_k-1} = \theta_{m_k} = b, \end{cases} \tag{3}$$

the set of knots that subdivide the interval $I = [a, b]$ uniformly. The hyperbolic B-splines space of order 4 is defined as follows

$$\mathcal{V}_k = \{s \in \mathcal{C}^2(I) :\ s_{[\theta^k_i, \theta^k_{i+1}]} \in \Gamma_4\} \text{ where } \Gamma_4 = \{1, \theta, \cosh(\theta), \sinh(\theta)\}.$$

The dimension of $\mathcal{V}_k$ is m_k and the fourth-order hyperbolic B-splines are given by: for $i = 0, 1, ..., m_k - 7$,

$$\nu_{i,k}(\theta) = C_k \begin{cases} -\theta + \theta^k_i + \sinh(\theta - \theta^k_i), & \theta^k_i \leq \theta < \theta^k_{i+1}; \\ \theta - \theta^k_{i+2} + 2(\theta - \theta^k_{i+1})\cosh(h_k) + & \\ 2\sinh(\theta^k_{i+1} - \theta) + \sinh(\theta^k_{i+2} - \theta), & \theta^k_{i+1} \leq \theta < \theta^k_{i+2}; \\ -\theta + \theta^k_{i+2} + 2(\theta^k_{i+3} - \theta)\cosh(h_k) - & \\ \sinh(\theta^k_{i+2} - \theta) - 2\sinh(\theta^k_{i+3} - \theta), & \theta^k_{i+2} \leq \theta < \theta^k_{i+3}; \\ \theta - \theta^k_{i+4} + \sinh(\theta^k_{i+4} - \theta), & \theta^k_{i+3} \leq \theta < \theta^k_{i+4}; \\ 0, & \text{otherwise.} \end{cases}$$

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where $C_k = \frac{1}{4h\sinh(\frac{h_k}{2})^2}$. with the respective left and right hand side boundary hyperbolic B-splines are

$$\nu_{-3,k}(\theta) = \begin{cases} \frac{-h_k+\theta-a+\sinh(h_k-\theta+a)}{-h_k+\sinh(h_k)}, & \theta_0^k \le \theta < \theta_1^k \\ 0, & \text{otherwise.} \end{cases}$$

$$\nu_{-2,k}(\theta) = \begin{cases} \frac{\theta-a-\sinh(h_k)+\sinh(h_k-\theta+a)}{h_k-\sinh(h_k)} + \\ \frac{a-\theta-2\theta\sinh(h_k)+2a\sinh(h_k)+2\sinh(h_k)-2\sinh(h_k-\theta+a)+\sinh(\theta-a)}{2h_k\cosh(h_k)-2\sinh(h_k)}, & \theta_0^k \le \theta < \theta_1^k \\ \frac{\theta-2h_k-a+\sinh(2h_k-\theta+a)}{2h_k\cosh(h_k)-2\sinh(h_k)}, & \theta_1^k \le \theta < \theta_2^k \\ 0, & \text{otherwise.} \end{cases}$$

$$\nu_{-1,k}(\theta) = \begin{cases} \frac{\theta-a-\sinh(\theta-a)}{-2h_k+2h_k\cosh(h_k)} + \\ \frac{\theta-a+2(\theta-a)\cosh(h_k)-2\sinh(h_k)+2\sinh(h_k-\theta+a)-\sinh(\theta-a)}{2h_k\cosh(h_k)-2\sinh(h_k)}, & \theta_0^k \le \theta < \theta_1^k \\ 1 + \frac{h_k-\sinh(h_k)}{-2h_k+2h_k\cosh(h_k)} + \frac{2h_k-\theta+a-\sinh(2h_k-\theta+a)}{2h_k\cosh(h_k)-2\sinh(h_k)} + \\ \frac{2(h_k-\theta+a)\cosh(h_k)-\sinh(h_k)+\sinh(h_k-\theta+a)+\sinh(2h_k-\theta+a)}{4h_k\sinh(\frac{h_k}{2})^2}, & \theta_1^k \le \theta < \theta_2^k \\ \frac{-3h_k+\theta-a+\sinh(3h_k-\theta+a)}{4h_k\sinh(\frac{h_k}{2})^2}, & \theta_2^k \le \theta < \theta_3^k \\ 0, & \text{otherwise.} \end{cases}$$

$$\nu_{m_k-6,k}(\theta) = \begin{cases} \frac{b-\theta-3h_k+\sinh(3h_k-b+\theta)}{4h_k\sinh(\frac{h_k}{2})^2}, & \theta_{m_k-6}^k \le \theta < \theta_{m_k-5}^k \\ \frac{h_k-\sinh(h_k)}{2h_k-2h_k\cosh(h_k)} - \frac{b-2h_k-\theta+\sinh(2h_k-b+\theta)}{2h_k\cosh(h_k)-2\sinh(h_k)} + \\ \frac{2(b-2h_k-\theta)\cosh(h_k)+\sinh(h_k)+\sinh(-b+\theta+h_k)+\sinh(2h_k+\theta-b)}{4h_k\sinh(h_k)^2}, & \theta_{m_k-5}^k \le \theta < \theta_{m_k-4}^k \\ \frac{h_k(b-\theta)\cosh(2h_k)+2h_k\cosh(\frac{3h_k}{2}-b+\theta)\sinh(\frac{h_k}{2})+2h_k\sinh(h_k)}{2h_k(-1+\cosh(h_k))(h_k\cosh(h_k)-\sinh(h_k))} + \\ \frac{-b\sinh(h_k)+\theta\sinh(h_k)-h_k\sinh(2h_k)+\sinh(h_k)\sinh(b-\theta)-h_k\sinh(h_k+b-\theta)}{2h_k(-1+\cosh(h_k))(h_k\cosh(h_k)-\sinh(h_k))}, & \theta_{m_k-4}^k \le \theta < \theta_{m_k-3}^k \\ 0, & \text{otherwise.} \end{cases}$$

$$\nu_{m_k-5,k}(\theta) = \begin{cases} \frac{b-\theta-2h_k+\sinh(2h_k-b+\theta)}{2h_k\cosh(h_k)-2\sinh(h_k)}, & \theta_{m_k-5}^k \le \theta < \theta_{m_k-4}^k \\ \frac{-\sinh(h_k)(b-\theta-2h_k+2(h_k+\theta)\cosh(h_k)+\sinh(-\theta+b))}{2(h_k-\sinh(h_k))(h_k\cosh(h_k)-\sinh(h_k))} + \\ \frac{h_k(b+\theta)+b\sinh(2h_k)-h_k(2\sinh(-b+\theta+h_k)+\sinh(2h_k+\theta-b)}{2(h_k-\sinh(h_k))(h_k\cosh(h_k)-\sinh(h_k))}, & \theta_{m_k-4}^k \le \theta < \theta_{m_k-3}^k \\ 0, & \text{otherwise.} \end{cases}$$

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$$\nu_{m_k-4,k}(\theta) \quad = \quad \begin{cases} \frac{h_k+\theta-b+\sinh(-h_k-\theta+b)}{h_k-\sinh(h_k)}, & \theta^k_{m_k-4} \leq \theta < \theta^k_{m_k-3} \\ 0, & \text{otherwise.} \end{cases}$$

The hyperbolic B-splines of order 4 possess all the desirable properties of classical polynomial B-splines, see [9]. In this paper, we limit ourselves to list some of them

- $\nu_{i,k}(\theta)$ is supported on the interval $[\theta^k_i, \theta^k_{i+1}]$;
- Positivity : $\nu_{i,k}(\theta) \geq 0, \quad \forall \theta \in [\theta^k_i, \theta^k_{i+1}]$;
- Partition of unity: $\sum_{i=-3}^{m_k-4} \nu_{i,k}(\theta) = 1$.

Table 1: The values of $\nu_{i,k}(\theta)$,$\nu'_{i,k}(\theta)$ and $\nu''_{i,k}(\theta)$ at the knots.

	θ^k_i	θ^k_{i+1}	θ^k_{i+2}	θ^k_{i+3}	θ^k_{i+4}	$else$
$\nu_{i,k}(\theta)$	0	$\frac{-h_k+\sinh(h_k)}{4h_k\sinh(\frac{h_k}{2})^2}$	$\frac{h_k\cosh(h_k)-\sinh(h_k)}{2h_k\sinh(\frac{h_k}{2})^2}$	$\frac{-h_k+\sinh(h_k)}{4h_k\sinh(\frac{h_k}{2})^2}$	0	0
$\nu'_{i,k}(\theta)$	0	$\frac{1}{2h_k}$	0	$\frac{-1}{2h_k}$	0	0
$\nu''_{i,k}(\theta)$	0	$\frac{1}{2\tanh(\frac{h_k}{2})}$	$\frac{-1}{\tanh(\frac{h_k}{2})}$	$\frac{1}{2\tanh(\frac{h_k}{2})}$	0	0

3. Hyperbolic B-Splines Interpolation

In this section, we will construct an approximate of $y^{(4)}(\theta^k_j)$ by using Taylor series expansion.

According to Schoenberg-Whitney theorem (see [6]), for a given function $y(\theta)$ sufficiently smooth there exists a unique hyperbolic spline $s(\theta) = \sum_{i=-3}^{m_k-4} \mu_i \nu_{i,k}(\theta) \in \mathcal{V}_k$ satisfying the interpolation conditions:

$$s(\theta^k_j) \quad = \quad y(\theta^k_j), \quad j = 0, 1, \cdots, m_k - 3; \tag{4}$$

$$s'(a) \quad = \quad y'(a), \quad s'(b) = y'(b). \tag{5}$$

For $j = 0, 1, \cdots, m_k - 3$, let $m_j = s'(\theta^k_j)$ and for $j = 0, 1, \cdots, m_k - 3$, let

$$M_j = \frac{\mu_{j-1} - 2\mu_{j-2} + \mu_{j-3}}{2h_k \tanh(\frac{h_k}{2})} \tag{6}$$

By using the Taylor series expansion we have:

$$m_j = s'(\theta^k_j) = y'(\theta^k_j) - \frac{1}{180} h^4_k y^{(5)}(\theta^k_j) + O(h^6_k); \tag{7}$$

$$M_j = y''(\theta_j^k) + \frac{1}{12}h_k^2 y^{(4)}(\theta_j^k) + \frac{1}{360}h_k^4 y^{(6)}(\theta_j^k) + O(h_k^6); \tag{8}$$

Now we can applied M_j to construct $y^{(3)}(\theta_j^k)$ and $y^{(4)}(\theta_j^k)$ for $j = 1, 2, \cdots, m_k - 4$, as follows, where the errors obtained by Taylor series expansion.

$$\frac{M_{j+1} - M_{j-1}}{2h_k} = y^{(3)}(\theta_j^k) + \frac{1}{4}h_k^2 y^{(5)}(\theta_j^k) + O(h_k^4); \tag{9}$$

$$\frac{M_{j+1} - 2M_j + M_{j-1}}{h_k^2} = y^{(4)}(\theta_j^k) + \frac{1}{6}h_k^4 y^{(6)}(\theta_j^k) + O(h_k^6); \tag{10}$$

By Table 1 and this equations, we get: where $v_{h_k} = \frac{2(h_k\cosh(h_k) - \sinh(h_k))}{\sinh(h_k) - h_k}$.

Table 2: The approximation values of $y(\theta_j^k)$, $y'(\theta_j^k)$ and $y''(\theta_j^k)$.

	$y(\theta_j^k)$	$y'(\theta_j^k)$	$y''(\theta_j^k)$
Approximate value	$s(\theta_j^k)$	m_j	M_j
Representation in μ_j	$\frac{(-h_k+\sinh(h_k))(\mu_{j-1}+v_{h_k}\mu_{j-2}+\mu_{j-3})}{4h_k\sinh(\frac{h_k}{2})^2}$	$\frac{\mu_{j-1}-\mu_{j-3}}{2h_k}$	$\frac{\mu_{j-1}-2\mu_{j-2}+\mu_{j-3}}{2h_k\tanh(\frac{h_k}{2})}$
Error order	$O(h_k^4)$	$O(h_k^4)$	$O(h_k^2)$

Table 3: The approximation values of $y^{(3)}(\theta_j^k)$, and $y^{(4)}(\theta_j^k)$.

	$y^{(3)}(\theta_j^k)$	$y^{(4)}(\theta_j^k)$
Approximate value	$\frac{M_{j+1}-M_{j-1}}{2h_k}$	$\frac{M_{j+1}-2M_j+M_{j-1}}{h_k^2}$
Representation in μ_j	$\frac{-\mu_{j-4}+2\mu_{j-3}-2\mu_{j-1}+\mu_j}{4h_k^2\tanh(\frac{h_k}{2})}$	$\frac{\mu_{j-4}-4\mu_{j-3}+6\mu_{j-2}-4\mu_{j-1}+\mu_j}{2h_k^3\tanh(\frac{h_k}{2})}$
Error order	$O(h_k^2)$	$O(h_k^4)$

4. Hyperbolic B-Splines Solutions of 4 BVP with $f(\theta) = 0$

Let $s(\theta) = \sum_{i=-3}^{m_k-4} \mu_i \nu_{i,k}(\theta)$ be the approximate of (11) and $\widetilde{s}(\theta) = \sum_{i=-3}^{m_k-4} \widetilde{\mu}_i \nu_{i,k}(\theta)$ be the approximate spline of $s(\theta)$. Discretize (11) at θ_j^k for $j = 1, 2, \cdots, m_k - 4$, and using $f(\theta_j^k) = 0$ for $j = 1, 2, \cdots, m_k - 4$ we get :

$$y^{(4)}(\theta_j^k) + g(\theta_j^k)y(\theta_j^k) = p(\theta_j^k), \quad j = 1, 2, \cdots, m_k - 4. \tag{11}$$

Now, by using Table 2 and 3, the equation (11) becomes

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$$\frac{\mu_{j-4}-4\mu_{j-3}+6\mu_{j-2}-4\mu_{j-1}+\mu_j}{2h_k^3\tanh(\frac{h_k}{2})}+g_jC_k(-h_k+\sinh(h_k))(\mu_{j-1}+v_{h_k}\mu_{j-2}+\mu_{j-3}) = p_j+O(h_k^4), \quad (12)$$

where $g_j = g(\theta_j^k)$ and $p_j = p(\theta_j^k)$. Consequently

$$(\mu_{j-4}-4\mu_{j-3}+6\mu_{j-2}-4\mu_{j-1}+\mu_j)+g_j\frac{h_k^2}{\sinh(h_k)}(-h_k+\sinh(h_k))(\mu_{j-1}+v_{h_k}\mu_{j-2}+\mu_{j-3}) = 2p_jh_k^3\tanh(\frac{h_k}{2})+O(2h_k^7\tanh(\frac{h_k}{2})). \quad (13)$$

By dropping $O(2h_k^7\tanh(\frac{h_k}{2}))$ from (13), we yield a linear system with m_k-4 linear equations in m_k unknowns $\mu_j,\ j=-3,-2,\cdots,m_k-4$. So four more equations are needed.
On the other hand, by using the fourth boundary conditions (2), we get

$$\begin{cases}\mu_{-3}=a_0;\\ \mu_{-1}-\mu_{-3}=2a_1h_k.\end{cases} \text{Thus, } \begin{cases}\mu_{-3}=a_0;\\ \mu_{-1}=a_0+2a_1h_k.\end{cases} \quad (14)$$

By using a similar technique, we get:

$$\begin{cases}\mu_{m_k-4}=b_0;\\ \mu_{m_k-4}-\mu_{m_k-6}=2b_1h_k.\end{cases} \text{Thus, } \begin{cases}\mu_{m_k-4}=b_0\\ \mu_{m_k-6}=b_0-2b_1h_k\end{cases} \quad (15)$$

Take (13), we get m_k-4 linear equations with $\mu_i,\ i=-2,0,1,\cdots,m_k-8,m_k-7,m_k-5$, as unknowns since μ_{-3}, μ_{-1}, μ_{m_k-6} and μ_{m_k-4} have been yielded from (14)and (15).

Let

$$C=[\mu_{-2},\mu_0,\mu_1,\cdots,\mu_{m_k-8},\mu_{m_k-7},\mu_{m_k-5}]^T,$$

$$\widetilde{C}=[\widetilde{\mu}_{-2},\widetilde{\mu}_0,\widetilde{\mu}_1,\cdots,\widetilde{\mu}_{m_k-8},\widetilde{\mu}_{m_k-7},\widetilde{\mu}_{m_k-5}]^T,$$

$D=[d_0,d_1,\cdots,d_{m_k-5}]^T$, $E=[e_0,e_1,\cdots,e_{m_k-5}]^T$ and using equations (13), (14), (15), we get

$$(A+\frac{h_k^2}{\sinh(h_k)}(-h_k+\sinh(h_k))GB)C=D+E; \quad (16)$$

$$(A+\frac{h_k^2}{\sinh(h_k)}(-h_k+\sinh(h_k))GB)\widetilde{C}=D, \quad (17)$$

where A and B are the following $(m_k-4)\times(m_k-4)$ matrix:

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$$A = \begin{pmatrix} -4 & -4 & 1 & & & & & & \\ 1 & 6 & -4 & 1 & & & & & \\ 0 & -4 & 6 & -4 & 1 & & & & \\ & 1 & -4 & 6 & -4 & 1 & & & \\ & \cdots & \cdots & \cdots & \cdots & & & & \\ & & \cdots & \cdots & \cdots & \cdots & & & \\ & & & 1 & -4 & 6 & -4 & 0 \\ & & & & 1 & -4 & 6 & 1 \\ & & & & & 1 & -4 & -4 \end{pmatrix},$$

$$B = \begin{pmatrix} 1 & 1 & & & \\ & v_{h_k} & 1 & & \\ & 1 & v_{h_k} & 1 & \\ & & \cdots & & \\ & & & \cdots & \\ & 1 & v_{h_k} & 1 & \\ & & 1 & v_{h_k} & \\ & & & 1 & 1 \end{pmatrix},$$

and where G and D are the following matrix

$$G = \begin{pmatrix} g_1 & & & & \\ & g_2 & & & \\ & & g_3 & & \\ & & & \ddots & \\ & & & & g_{m_k-4} \end{pmatrix}$$

$$D = \begin{pmatrix} 2h_k^3\tanh(\frac{h_k}{2})p_1 - \frac{h_k^2}{\sinh(h_k)}(-h_k+\sinh(h_k))v_{h_k}g_1\mu_{-1} - \mu_{-3} - 6\mu_{-1} \\ 2h_k^3\tanh(\frac{h_k}{2})p_2 - \frac{h_k^2}{\sinh(h_k)}(-h_k+\sinh(h_k))g_2\mu_{-1} + 4\mu_{-1} \\ 2h_k^3\tanh(\frac{h_k}{2})p_3 - \mu_{-1} \\ 2h_k^3\tanh(\frac{h_k}{2})p_4 \\ \vdots \\ 2h_k^3\tanh(\frac{h_k}{2})p_{m_k-7} \\ 2h_k^3\tanh(\frac{h_k}{2})p_{m_k-6} - \mu_{m_k-6} \\ 2h_k^3\tanh(\frac{h_k}{2})p_{m_k-5} - \frac{h_k^2}{\sinh(h_k)}(-h_k+\sinh(h_k))g_{m_k-5}\mu_{m_k-6} + 4\mu_{m_k-6} \\ 2h_k^3\tanh(\frac{h_k}{2})p_{m_k-4} - \frac{h_k^2}{\sinh(h_k)}(-h_k+\sinh(h_k))v_{h_k}g_{m_k-4}\mu_{m_k-6} - 6\mu_{m_k-6} - \mu_{m_k-4} \end{pmatrix}$$

and $e_i = O(2h_k^7\tanh(\frac{h_k}{2})), i = 0, 1, \cdots, m_k - 5$.

After solving the linear system (17), $\widetilde{\mu}_i,\ i = -2, 0, 1, \cdots, m_k-8, m_k-7, m_k-5$, $\widetilde{\mu}_{-3} = \mu_{-3}$, $\widetilde{\mu}_{-1} = \mu_{-1}$, $\widetilde{\mu}_{m_k-6} = \mu_{m_k-6}$, and $\widetilde{\mu}_{m_k-4} = \mu_{m_k-4}$ will be used together to get the approximation spline solution $\widetilde{s}(\theta) = \sum_{i=-3}^{m_k-4} \widetilde{\mu}_i \nu_{i,k}(\theta)$.

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Lemma 1. *The matrix A is inversible.*

Proof. It suffises to prove that for all $D = [d_0, d_1, \cdots, d_{m_k-5}]^T \in \mathbb{R}^{m_k-4}$ such that $A\ D = 0$, we have $D = 0$. Indeed, If we put $z(\theta) = \sum_{j=-3}^{m_k-8} d_{j+3}\nu_{j,k}(\theta) + \sum_{j=m_k-7}^{m_k-4} 0\ \nu_{j,k}(\theta)$, then $z^{(4)}(\theta_i^k) = 0$, for all $i = 4, 5, \cdots, m_k - 7$.
On the other hand, using the fact that z is hyperbolic spline function of $\mathcal{C}^2$, we deduce that $z(\theta) = \alpha + \beta\theta + \gamma\cosh(\theta) + \delta\sinh(\theta)$ in $[\theta_4^k, \theta_5^k]$. From $z^{(4)}(\theta_4^k) = 0$ and $z^{(4)}(\theta_5^k) = 0$, we have,

$$\begin{cases} \gamma\cosh(\theta_4^k) + \delta\sinh(\theta_4^k) = 0; \\ \gamma\cosh(\theta_5^k) + \delta\sinh(\theta_5^k) = 0; \end{cases}$$

we deduce $\gamma = 0$ and $\delta = 0$. Consequently, $z^{(4)}(\theta) = 0$ and $z^{(2)}(\theta) = 0$ in all the interval $[\theta_4^k, \theta_5^k]$. In a same way, we have in all the other subintervals of $[\theta_4^k, \theta_{m_k-7}^k]$, $z^{(4)}(\theta) = 0$, $z^{(2)}(\theta) = 0$ and $z^{'}(\theta) = \beta$. Consequently, we have

$$\begin{cases} z^{(2)}(\theta_4^k) = 0 \\ z^{(2)}(\theta_5^k) = 0 \\ \quad\vdots \\ z^{(2)}(\theta_{m_k-7}^k) = 0 \end{cases} \quad \text{thus} \quad \begin{cases} \frac{d_4+d_6}{2} = d_5 \\ \frac{d_5+d_7}{2} = d_6 \\ \quad\vdots \\ \frac{d_{m_k-7}+d_{m_k-5}}{2} = d_{m_k-6} \end{cases}$$

$$\begin{cases} z^{'}(\theta_4^k) = \beta \\ z^{'}(\theta_5^k) = \beta \\ \quad\vdots \\ z^{'}(\theta_{m_k-7}^k) = \beta \end{cases} \quad \text{thus} \quad \begin{cases} \frac{d_6-d_4}{2h_k} = \beta \\ \frac{d_7-d_5}{2h_k} = \beta \\ \quad\vdots \\ \frac{d_{m_k-5}-d_{m_k-7}}{2h_k} = \beta \end{cases}.$$

Consequently

$$\begin{cases} d_5 = d_4 + h_k\beta \\ d_6 = d_5 + h_k\beta \\ \quad\vdots \\ d_{m_k-6} = d_{m_k-7} + h_k\beta \end{cases}$$

we have also

$$\begin{cases} -4d_0 - 4d_1 + d_2 = 0 \\ d_0 + 6d_1 - 4d_2 + d_3 = 0 \\ -4d_1 + 6d_2 - 4d_3 + d_4 = 0 \\ d_1 - 4d_2 + 6d_3 - 4d_4 + d_5 = 0 \\ \qquad\qquad \vdots \\ d_{m_k-9} - 4d_{m_k-8} + 6d_{m_k-7} - 4d_{m_k-6} = 0 \\ d_{m_k-8} - 4d_{m_k-7} + 6d_{m_k-6} + d_{m_k-5} = 0 \\ d_{m_k-7} - 4d_{m_k-6} - 4d_{m_k-5} = 0 \end{cases} \tag{18}$$

thus $\beta = 0$, finally we have $d_0 = d_1 = d_2 = \ldots = d_{m_k-5} = 0$ which in turn gives $D = 0$. □

5. Numerical Results for $f(\theta) = 0$

To test our method, we considered three examples of fourth-order boundary value problems (4BVPs) of the form ((2),(11)) with f(θ)=0.

$$\begin{cases} y^{(4)} - y = -(8+8x)\exp(x), & x \in [0,1]; \\ y(0) = y(1) = 0,\ y'(0) = 1,\ y'(1) = -\exp(1). \end{cases} \tag{19}$$

The exact solution is $y(x) = x(1-x)\exp(x)$.

$$\begin{cases} y^{(4)} - y = 4((1-2x)\sinh(x) - 3\cosh(x)), & x \in [0,1]; \\ y(0) = y(1) = 0,\ y'(0) = 1,\ y'(1) = -\cosh(1). \end{cases} \tag{20}$$

The exact solution is $y(x) = x(1-x)\cosh(x)$.

$$\begin{cases} y^{(4)} - y = -4\cosh(x), & x \in [0,1]; \\ y(0) = y(1) = 0,\ y'(0) = 1,\ y'(1) = -sinh(1). \end{cases} \tag{21}$$

The exact solution is $y(x) = (1-x)\sinh(x)$.

In Table 4 we summarize the results of the problems 19, 20 and 21 for different values of k.

Table 4: Maximum absolute error for Problems (19), (20) and (21).

k	Error of problem (19)	Error of problem (20)	Error of problem (21)
3	6.826e-002	4.271e-002	3.559e-002
4	3.290e-002	2.076e-002	1.719e-002
5	1.581e-002	1.001e-002	8.239e-003
6	7.415e-003	4.690e-003	3.825e-003
7	3.254e-003	2.039e-003	1.634e-003
8	1.183e-003	7.172e-004	5.434e-004
9	1.499e-004	5.661e-005	1.139e-006

6. Hyperbolic B-Splines Solutions of 4 BVP of the Form ((1),(2)) and Numerical Results

Discretize (1) at θ_j^k for $j = 1, 2, \cdots, m_k - 4$,we get :

$$y^{(4)}(\theta_j^k) + f(\theta_j^k)y^{(2)}(\theta_j^k) + g(\theta_j^k)y(\theta_j^k) = p(\theta_j^k), \quad j = 1, 2, \cdots, m_k - 4. \tag{22}$$

In a same way, we have

$$\left(A_1 + h_k^2 F A_2 + \frac{h_k^2(\sinh(h_k) - h_k)}{\sinh(h_k))} GB\right) C = D_1 + E; \tag{23}$$

$$\left(A_1 + h_k^2 F A_2 + \frac{h_k^2(\sinh(h_k) - h_k)}{\sinh(h_k)} GB\right) \widetilde{C} = D_1, \tag{24}$$

where A_1 and A_2 are the following $(m_k - 4) \times (m_k - 4)$ matrix:

$$A_1 = \begin{pmatrix} -4 & -4 & 1 & & & & & & \\ 1 & 6 & -4 & 1 & & & & & \\ 0 & -4 & 6 & -4 & 1 & & & & \\ & 1 & -4 & 6 & -4 & 1 & & & \\ & \cdots & \cdots & \cdots & \cdots & & & & \\ & & \cdots & \cdots & \cdots & \cdots & & & \\ & & & 1 & -4 & 6 & -4 & 0 \\ & & & & 1 & -4 & 6 & 1 \\ & & & & & 1 & -4 & -4 \end{pmatrix},$$

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$$A_2 = \begin{pmatrix} 1 & 1 & & & & & & & \\ & -2 & 1 & & & & & & \\ & 1 & -2 & 1 & & & & & \\ & & 1 & -2 & 1 & & & & \\ & \cdots & \cdots & \cdots & \cdots & & & & \\ & & \cdots & \cdots & \cdots & \cdots & & & \\ & & & & 1 & -2 & 1 & & \\ & & & & & 1 & -2 & & \\ & & & & & & 1 & 1 \end{pmatrix}$$

and where F, G, B and D_1 are the following matrix

$$F = \begin{pmatrix} f_1 & & & & \\ & f_2 & & & \\ & & f_3 & & \\ & & & \ddots & \\ & & & & f_{m_k-4} \end{pmatrix},$$

$$G = \begin{pmatrix} g_1 & & & & \\ & g_2 & & & \\ & & g_3 & & \\ & & & \ddots & \\ & & & & g_{m_k-4} \end{pmatrix}$$

$$B = \begin{pmatrix} 1 & 1 & & & \\ & v_{h_k} & 1 & & \\ & 1 & v_{h_k} & 1 & \\ & & \cdots & & \\ & & & \cdots & \\ & 1 & v_{h_k} & 1 & \\ & & 1 & v_{h_k} & \\ & & & 1 & 1 \end{pmatrix}$$

and

$D_1 =$

$$\begin{pmatrix} 2h_k^3\tanh(\frac{h_k}{2})p_1+2h_k^2f_1\mu_{-1}-\frac{h_k^2}{\sinh(h_k)}(-h_k+\sinh(h_k))v_{h_k}g_1\mu_{-1}-\mu_{-3}-6\mu_{-1}\\ 2h_k^3\tanh(\frac{h_k}{2})p_2-h_k^2f_2\mu_{-1}-\frac{h_k^2}{\sinh(h_k)}(-h_k+\sinh(h_k))g_2\mu_{-1}+4\mu_{-1}\\ 2h_k^3\tanh(\frac{h_k}{2})p_3-\mu_{-1}\\ 2h_k^3\tanh(\frac{h_k}{2})p_4\\ \vdots\\ 2h_k^3\tanh(\frac{h_k}{2})p_{m_k-7}\\ 2h_k^3\tanh(\frac{h_k}{2})p_{m_k-6}-\mu_{m_k-6}\\ 2h_k^3\tanh(\frac{h_k}{2})p_{m_k-5}-h_k^2f_{m_k-5}\mu_{m_k-6}-\frac{h_k^2}{\sinh(h_k)}(-h_k+\sinh(h_k))g_{m_k-5}\mu_{m_k-6}+4\mu_{m_k-6}\\ 2h_k^3\tanh(\frac{h_k}{2})p_{m_k-4}+2h_k^2f_{m_k-4}\mu_{m_k-6}\\ -\frac{h_k^2}{\sinh(h_k)}(-h_k+\sinh(h_k))v_{h_k}g_{m_k-4}\mu_{m_k-6}-\mu_{m_k-4}-6\mu_{m_k-6} \end{pmatrix}.$$

After solving the linear system (24), $\widetilde{\mu}_i,\ i=-2,0,1,\cdots,m_k-8,m_k-7,m_k-5$, $\widetilde{\mu}_{-3}=\mu_{-3}$, $\widetilde{\mu}_{-1}=\mu_{-1}$, $\widetilde{\mu}_{m_k-6}=\mu_{m_k-6}$, and $\widetilde{\mu}_{m_k-4}=\mu_{m_k-4}$ will be used together to get the approximation spline solution $\widetilde{s}(\theta)=\sum_{i=-3}^{m_k-4}\widetilde{\mu}_i\nu_{i,k}(\theta)$.

We considered three examples of fourth-order boundary value problems (4BVPs) of the form ((1),(2)).

$$\begin{cases} y^{(4)}+y^{(2)}-y=-(x^2+11x+8)\exp(x), & x\in[0,1];\\ y(0)=y(1)=0,\ y'(0)=1,\ y'(1)=-\exp(1). \end{cases} \tag{25}$$

The exact solution is $y(x)=x(1-x)\exp(x)$.

$$\begin{cases} y^{(4)}+y^{(2)}-y=6(1-2x)\sinh(x)-(x^2-x+14)\cosh(x)), & x\in[0,1];\\ y(0)=y(1)=0,\ y'(0)=1,\ y'(1)=-\cosh(1). \end{cases} \tag{26}$$

The exact solution is $y(x)=x(1-x)\cosh(x)$.

$$\begin{cases} y^{(4)}+y^{(2)}-y=-6\cosh(x)+(1-x)\sinh(x), & x\in[0,1];\\ y(0)=y(1)=0,\ y'(0)=1,\ y'(1)=-sinh(1). \end{cases} \tag{27}$$

The exact solution is $y(x)=(1-x)\sinh(x)$.

In Table 5 we summarize the results of the problems 25, 26 and 27 for different values of k.

7. Conclusion

Experimental results demonstrate that our method is more effective for the problems where the exact solution is hyperbolic (see Tables 4 and 5). The proposed method

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Table 5: Maximum absolute error for Problems (25), (26) and (27).

k	Error of problem (25)	Error of problem (26)	Error of problem (27)
3	6.971e-002	4.416e-002	3.585e-002
4	3.347e-002	2.121e-002	1.726e-002
5	1.602e-002	1.014e-002	8.243e-003
6	7.482e-003	4.712e-003	3.802e-003
7	3.255e-003	2.025e-003	1.600e-003
8	1.154e-003	6.891e-004	5.046e-004
9	1.061e-004	2.240e-005	4.244e-005

can be extended to solve higher (i.e., more than 5th) order boundary value problems by using the hyperbolic (tension) B-splines of order more than 5, 6, $\cdots$.

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