SOME APPLICATION OF IDEALS OF AG-RING

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Abstract: In this paper we define c-prime, 3-prime and weakly prime ideal of AG-ring which we will study relation of c-prime, 3-prime, weakly prime ideal and prime ideal.

Key Words: c-prime 3-prime weakly prime ideal

1. Introduction

M.A. Kazim and MD. Naseeruddin [2, Proposition 2.1] asserted that, in every LA-semigroups $G$ a medial law hold

$$(a \cdot b) \cdot (c \cdot d) = (a \cdot c) \cdot (b \cdot d), \quad \forall a, b, c, d \in G.$$  

Q. Mushtaq and M. Khan [4, p.322] asserted that, in every LA-semigroups $G$ with left identity

$$(a \cdot b) \cdot (c \cdot d) = (d \cdot c) \cdot (b \cdot a), \quad \forall a, b, c, d \in G.$$  

Further M. Khan, Faisal, and V. Amjid [3], asserted that, if a LA-semigroup $G$ with left identity the following law holds

$$a \cdot (b \cdot c) = b \cdot (a \cdot c), \quad \forall a, b, c \in G.$$  

M. Sarwar (Kamran) [5, p.112] defined LA-group as the following; a groupoid $G$ is called a left almost group, abbreviated as LA-group, if (i) there exists $e \in G$ such that $e a = a$ for all $a \in G$, (ii) for every $a \in G$ there exists $a' \in G$ such that, $d'a = e$, (iii) $(ab)c = (cb)a$ for every $a, b, c \in G$.  

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S.M. Yusuf in [6, p.211] introduces the concept of a left almost ring (LA-ring). That is, a non-empty set $R$ with two binary operations “$+$” and “$\cdot$” is called a left almost ring, if $\langle R, + \rangle$ is an LA-group, $\langle R, \cdot \rangle$ is an LA-semigroup and distributive laws of “$\cdot$” over “$+$” holds. T. Shah and I. Rehman [6, p.211] asserted that a commutative ring $\langle R, +, \cdot \rangle$, we can always obtain an LA-ring $\langle R, \oplus, \cdot \rangle$ by defining, for $a, b, c \in R$, $a \oplus b = b - a$ and $a \cdot b$ is same as in the ring. We can not assume the addition to be commutative in an LA-ring. An LA-ring $\langle R, +, \cdot \rangle$ is said to be LA-integral domain if $a \cdot b = 0$, $a, b \in R$, then $a = 0$ or $b = 0$. Let $\langle R, +, \cdot \rangle$ be an LA-ring and $S$ be a non-empty subset of $R$ and $S$ is itself and LA-ring under the binary operation induced by $R$, the $S$ is called an LA-subring of $R$, then $S$ is called an LA-subring of $\langle R, +, \cdot \rangle$. If $S$ is an LA-subring of an LA-ring $\langle R, +, \cdot \rangle$, then $S$ is called a left ideal of $R$ if $RS \subseteq S$. Right and two-sided ideals are defined in the usual manner. An ideal $I$ of $R$ is called prime if $AB \in I$ implies $A \in I$ or $B \in I$.

In this note we prefer to called left almost rings (LA-rings) as Abel-Grassmann’s rings (abbreviated as an “AG-rings”).

In [1] An ideal $I$ of $N$ is called c-prime if $a, b \in N$ and $ab \in I$ implies $a \in I$ or $b \in I$. $R$ is called c-prime nearring if $\{0\}$ is a c-prime ideal of $R$.

An ideal $I$ of $R$ is called 3-prime if $a, b \in N$ and $anb \in I$ for all $n \in N$ implies $a \in I$ or $b \in I$.

The notions of c-ideal, 3-prime ideal and prime ideal coincide in rings.

In [7] A proper ideal $I$ of an ring $R$ to be weakly prime if $0 \neq AB \subseteq I$ implies either $A \subseteq I$ or $B \subseteq I$ for any ideals $A, B$ of $R$.

The following implications are well known in rings:

(1) c-prime ideal $\Rightarrow$ 3-prime ideal $\Rightarrow$ prime ideal;

(2) prime ideal $\Rightarrow$ weakly prime ideal.

2. Main Results

In this paper, we define c-prime and 3-prime of AG-ring.

**Definition 2.1.** An ideal $I$ of an AG-ring $R$ is called c-prime if $a, b \in R$ and $ab \in I$ implies $a \in I$ or $b \in I$.

**Definition 2.2.** An ideal $I$ of an AG-ring $R$ is called 3-prime if $a, b \in R$ and $arb \in I$ for all $r \in R$ implies $a \in I$ or $b \in I$.

The following lemmas and theorem we will study relation of c-prime, 3-prime and prime ideals.

**Lemma 2.3.** Every c-prime ideal is a 3-prime ideal.
Proof. Suppose that $I$ is a c-prime ideal of AG-ring $R$, let $a, b \in R$ and $arb \in I$ for all $r \in R$. Since $I$ is a c-prime ideal we have $a \in I$ and $b \in I$. Then $I$ is a 3-prime ideal of $R$.

Lemma 2.4. Every 3-prime ideal is a prime ideal

Proof. Suppose that $I$ is a 3-prime ideal of AG-ring $R$, let $a, b \in I$ and $ab \in I$. Since $I$ is a 3-prime ideal we have $a \in I$ and $b \in I$. Then $I$ is a prime ideal of $R$.

Lemma 2.5. Every c-prime ideal is a prime ideal

Proof. Suppose that $I$ is a c-prime ideal of AG-ring $R$, let $a, b \in I$ and $ab \in I$. Since $I$ is a c-prime ideal we have $a \in I$ and $b \in I$. Then $I$ is a prime ideal of $R$.

In [6, p.221], studied if $I$ is a prime ideal in AG-ring $R$ if and only if $R/I$ is an AG-integral domain. The following theorems are application by lemmas 2.4 and 2.5

Theorem 2.6. Let $R$ be an AG-ring. Then $I$ is a 3-prime ideal in $R$ if and only if $R/I$ is an AG-integral domain.

Proof. ($\Rightarrow$) Let $I$ is a 3-prime ideal in $R$. By Lemma 2.4 then $I$ is a prime ideal. Thus $R/I$ is an AG-integral domain.

($\Leftarrow$) Assume that $R/I$ is an AG-integral domain with $arb \in I$ for all $r \in R$. Then $I + arb = I$ so $(I + a)(I + b) = I$. Since $R/I$ is an AG-integral domain we have $I + a = I$ or $I + b = I$. Then $a \in I$ or $b \in I$. Thus $P$ is a 3-prime ideal of $R$.

Theorem 2.7. Let $R$ be an AG-ring. Then $I$ is a c-prime ideal in $R$ if and only if $R/I$ is an AG-integral domain.

Proof. ($\Rightarrow$) Let $I$ is a c-prime ideal in $R$. By Lemma 2.5 then $I$ is a prime ideal. Thus $R/I$ is an AG-integral domain.

($\Leftarrow$) Assume that $R/I$ is an AG-integral domain with $ab \in I$ for all $a, b \in R$. Then $I + ab = I$ so $(I + a)(I + b) = I$. Since $R/I$ is an AG-integral domain we have $I + a = I$ or $I + b = I$. Then $a \in I$ or $b \in I$. Thus $P$ is a c-prime ideal of $R$.

The next following we define of weakly prime ideal.

Definition 2.8. A proper ideal $I$ of an AG-ring $R$ to be weakly prime if $0 \neq AB \subseteq I$ implies either $A \subseteq I$ or $B \subseteq I$ for any ideals $A, B$ of $R$.

Clearly every prime ideal is weakly prime and $\{0\}$ is always weakly prime ideal of $R$. The following theorem we will study properties

Theorem 2.9. If $I$ is weakly prime but not prime, then $I^2 = 0$. 

Proof. Since $I$ is weakly prime (but not prime), there exist ideals $A \not\subseteq I$ and $B \not\subseteq P$ but $0 = AB \subseteq I$. Since $I \subseteq A + I$ and $I \subseteq B + I$. But if $I^2 \neq 0$. By distributive laws “.” over “+” of AG-ring we have

\[ 0 \neq I^2 = II \subseteq (A + I)(B + I) \]
\[ = [(A + I)B] + [(A + I)P] \]
\[ = AB + IB + AI + II \]
\[ \subseteq I \]

which implies $(A + I) \subseteq I$ and $(B + I) \subseteq I$, since $I$ is a weakly prime; that is $A \subseteq I$ or $B \subseteq I$, a contradiction. Hence, $I^2 = 0$.

If $R^2 = 0$, then it is evident that every ideal of $R$ is weakly prime. In particular, if an ideal $I$ of an AG-ring $R$ is weakly prime but not a prime ideal, then every ideal of $I$ as an AG-ring is weakly prime by Theorem 2.9.

**Theorem 2.10.** Every ideal of an AG-ring $R$ is weakly prime if and only if for any ideals $A$ and $B$ of $R$, $AB = A$, $AB = B$, or $AB = 0$.

Proof. Suppose that every ideal of $R$ is weakly prime. Let $A, B$ be ideals of $R$. Then $AB$ is a left ideals of $R$, if $AB \neq R$, then by hypothesis, $AB$ is weakly prime. We are consider two situation, that is $AB = 0$. or $AB \neq 0$. If $0 \neq AB \subseteq AB$, then by Definition 2.8, we have $A \subseteq AB$ or $B \subseteq AB$. Since $A$ and $B$ are ideals of $R$, we have $AB \subseteq A$ and $AB \subseteq B$. Therefore $A = AB$ or $B = AB$. If $AB = R$, then we have $A = B = R$ whence $R^2 = R$.

Conversely, let $K$ be any proper ideal of $R$ and suppose that $0 \neq AB \subseteq K$ for ideals $A$ and $B$ of $R$. Then we have either $A = AB \subseteq K$ or $B = AB \subseteq K$.

**Corollary 2.11.** Let $R$ be an AG-ring and every ideal of $R$ is weakly prime. Then for any ideal $I$ of $R$, either $I^2 = I$ or $I^2 = 0$.

The following theorem we will study relation of c-prime, 3-prime, weakly prime ideals.

**Theorem 2.12.** Every c-prime ideal is a weakly prime ideal

Proof. Suppose that $I$ is a c-prime ideal of AG-ring $R$. By Lemma 2.3 we have $I$ is a prime ideal. Since every prime ideal is weakly prime ideal we have $I$ is a weakly prime ideal of $R$.

**Theorem 2.13.** Every 3-prime ideal is a weakly prime ideal

Proof. Suppose that $I$ is a 3-prime ideal of AG-ring $R$. By Lemma 2.4 we have $I$ is a prime ideal. Since every prime ideal is weakly prime ideal we have $I$ is a weakly prime ideal of $R$. 
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References


