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MODELING TWO DYNAMICAL LANE NAVIGATION PROBLEMS

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Abstract: There are many dynamical problems that arise from naval crafts maneuvering in a lane of confined width. This paper proposes to describe two of these problems, pursuit and collision, then attempts to provide mathematical and suggested solutions, or approaches to resolving them. The initial section is an analysis of pursuit curve modeling with the *MATLAB* software to determine a trajectory for a pursuing craft to possibly overtake another with a key interest in how long it takes and at what coordinates. The second section constructs theoretically an ordinary collision scenario. The focus of this section is to determine the collision probability of two crafts based on a couple of acceleration assumptions.

Key Words: lane navigation, pursuit curve, dynamical

1. The Curve of Pursuit Problem

The essential description of a curve of pursuit problem is given in a paraphrase of Arthur Bernhart from his paper, *Curves of Pursuit* : "Let the point Q move along a given path $Q(t)$ while another point P move always along a curve $P(t)$ such that its velocity vector, dP/dt is ever normal to the curve $Q(t)$. Then $P(t)$ is called a curve of pursuit.", see [2].

Historically, the problem of the curve of pursuit was stated by Pierre Bouguer in 1732. He considered only the simplest case where the pursued point moved along a straight line.

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International Electronic Journal of Pure and Applied Mathematics – IEJPAM, Volume 7, No. 2 (2014)

2. General Pursuit Curve

Assumptions: The origin $(0,0)$ is assumed to be the position of the pursuer's craft in the (x,y) coordinate frame. The pursued follows a straight line relative to an $(X-Y)$ axes in which the Y-axis is the line of the pursued craft. The X-axis passes through the y-intercept of the pursued craft in the (x,y) frame. Some translation, or rotation of coordinates may be necessary to establish the relative coordinates.

Let α be the angle between the line of the pursued craft and the horizontal axis in the (x,y) coordinate frame. α will be given in radians by default in the *MATLAB* coding. The range of α is from $-\pi$ to $+\pi$. Let h be the distance from the line of the pursued and the starting point of the pursuer.

Let yzero be the y-intercept of the line of the pursued. This curve model is:

Formula: $h = (yzero)\cos(\alpha)$ which is an elementary derivation based on the classical formulation given by Barton and Eliezer, see [1].

The speed ratio c is the ratio of the pursuer's speed to the pursued's speed.

The equation of the pursuer's path is:

$$\frac{Y}{h} = \frac{x\sec\alpha + \tan\alpha}{c^2-1} + \frac{1}{2}\left\{\frac{c}{c+1}\omega\left(\frac{X}{h}\right)^{1+1/c} - \frac{c}{c-1}\omega^{-1}\left(\frac{X}{h}\right)^{1-1/c}\right\}, \quad (1)$$

in which $\omega = \sec\alpha - \tan\alpha$. There will be capture when $c > 1$ and no capture when $c < 1$.

Now set

$$\lambda = \frac{X}{2c}\left\{\omega\left(\frac{X}{h}\right)^{1/c} + \omega^{-1}\left(\frac{X}{h}\right)^{-1/c}\right\},$$

λ is a measure of the separation between pursuer and pursued. Note tht for $c > 1$, $\lambda \to 0$ as $X \to 0$. In fact, $\lambda = 0$ when $X = 0$. The coordinates of the point of capture are $X = 0$ and

$$Y = h\left(\frac{c\sec\alpha + \tan\alpha}{c^2-1}\right). \quad (2)$$

The corresponding $x-$ coordinate of the pursuer must be:

$$X = Y\cos\alpha = (\text{yzero})\left(\frac{c\cos\alpha + \sin\alpha\cos\alpha}{c^2-1}\right). \quad (3)$$

Note that for $c < 1$, $\lambda \to \infty$ and the chase continues indefinitely. Provided below is *MATLAB* coding for this scenario.

International Electronic Journal of Pure and Applied Mathematics – IEJPAM, Volume 7, No. 2 (2014)

3. MATLAB M–Script File Coding For General Curve of Pursuit

GeneralPursuit.M

```
%General Pursuit Curve Formulation
slope = input('Enter slope of the pursued craft navig line>');
α = atan(slope)
yzero = input('Enter y-intercept of pursued craft navig line>');
h = yzero*cos(α)
X = input('Enter X values>');
v1 = input('Enter pursuer craft speed>');
v2 = input('Enter pursued craft speed>');
c = v1/v2
omega = sec(α)-tan(α) onc = 1+(1/c)
negc = 1-(1/c)
seg1 = (c*sec(α)+tan(α))/((c.^2)-1);
seg2 = 0.5*((c/(c+1))*omega*((X/h).^onc));
seg3 = 0.5*((c/(ct1))*omega*((X/h).^negc));
Y = h*(seg1+seg2-seg3);
plot(X,Y,'r-'), title('General Curve of Pursuit');
grid on, box on;
Ycaptre = h*(c*sec(α)+tan(α))/((c.^2)-1);
xpursuer= Ycaptre*cos(α).
```

The next section offers comments on collision probability modeling and a specific analytical model is rendered for two crafts navigating towards an encounter based upon acceleration changes and other assumptions.

4. A Simple Probabilistic Collision Model

Assumptions:

1. A craft is moving along a path at speed v1; another craft is moving towards the other with a speed v2.
2. When the pilot of the first craft notices the other craft, the first craft is a distance x1 from the collision point.
3. If the first craft can pass the collision point without changing speed, then it does so, otherwise after a reaction time interval tp, the pilot of the first craft slows it down at constant deceleration a.
4. If the 2^{nd} craft arrives first at the collision point, then it stops there; whereas if the first craft reaches the collision point first, then the 2^{nd} craft stops before hitting the first.

International Electronic Journal of Pure and Applied Mathematics – IEJPAM, Volume 7, No. 2 (2014)

5. No other craft passes either the 1st or 2nd until either one stops.

6. The distance of the 2nd craft from the collision point is given by x2.

Obviously, $x_N = \dfrac{v_1 x_1}{v_2}$ is the maximum distance the first craft can be from the collision point and still avoid the collision. The stopping distance for the first craft is given by $x_S = v_1 t_p + \dfrac{v_1^2}{2a}$; hence, when $x_C = x_S - x_N > 0$ then there is the "collision zone" from $2a$ which the pilot of the first craft cannot avoid a collision by either decelerating or continuing at speed v_1. The mathematicians, Weiss and Herman [4], and Brieman [3] have shown that given a light density of craft in an unobstructed lane of transit, then the distribution of space headways converges to an exponential as time passes. Now if $f_1(v_1)$ denotes the probability distribution of the 1st craft's speed, and assuming that x_1 and v_1 are independent, then the exponential spacing model will show that the probability that the 2nd craft collides with the 1st craft is given by:

$$\text{Prob}\left\{ v_1 \frac{x_2}{v_2} < x_1 < v_1 t_p + \left(\frac{v_1^2}{2a}\right) \,\middle|\, x_2, v_2, a, t_p \right\} = \int_b^\infty \left\{ e^{-\rho \frac{v_1 x_2}{v_2}} - e^{\rho\left(\frac{v_1^2}{2a} + v_1 t_p\right)} \right\} f_1(v_1)\, dv_1, \quad (4)$$

where b equals the larger of 0 and $2a(x2/v2 - tp)$, and ρ is the space density of crafts in the lane. This density is given by $\rho = \frac{\kappa}{\psi}$, where κ is the mean craft traffic flow, and ψ is the mean craft speed. If the craft speeds are taken to be normally distributed with specification of a mean speed, a speed variance, and a mean craft traffic flow, along with v_2, x_2, t_p, and a, then the last equation may be evaluated.

Now let v_c denote the craft speed at collision. Now four cases may be distinguished:

Case 1. $x_N < x_1 < v_1 t_p$, resulting in $v_c = v_1$ (the first craft does not have time to stop).

Case 2. $x_1 t_p < x_1 < x_S$, resulting in $v_C = \sqrt{v_1^2 - 2a(x_1 - v_1 t_p)}$ (the first craft decelerates but hits the second craft).

Case 3. $x_1 \geq x_S$, resulting in $v = 0$ (the collision is avoided by deceleration).

Case 4. $x_1 \leq x_N$, resulting in $v_C = 0$ (collision avoided by first craft arriving before the second craft).

Now the conditional distribution function of the collision speeds is given by:

$$\text{Prob}\,\{v_C \leq \hat{\nu} \,|\, x_N < x_1 < x_S\} = \frac{\text{Prob}\,\{v_C \leq \hat{\nu} \text{ and } x_N < x_1 < x_S\}}{\text{Prob}\,\{x_N < x_1 < X_S\}}. \quad (6)$$

Here:

$$F_1(\hat{v}) = \int_{\hat{v}}^{\infty} \left\{ e^{-\rho\frac{v_1 x_2}{v_2}} - e^{-\rho\left(\frac{v_1^2}{2a}+v_1 t_p\right)} \right\} f_1(v_1)\, dv_1;\ I\left[\hat{v}^2 > \gamma(v_1)\right] f_1(v_1) dv_1, \qquad (7)$$

$$F_2(\hat{v}) = \int_{\hat{v}}^{\infty} \left\{ e^{-\rho\left(\frac{v_1^2-\hat{v}^2}{v_2}+vt_p\right)} - e^{-\rho\left(\frac{v_1^2}{2a}+vt_p\right)} \right\} f_1(v_1)\, dv_1;\ I\left[\hat{v}^2 > \gamma(v_1)\right] f_1(v_1) dv_1, \qquad (8)$$

$$F_3(\hat{v}) = \int_{b}^{\hat{v}} \left\{ e^{-\rho\frac{v_1 x_2}{v_2}} - e^{-\rho\left(\frac{v_1^2}{2a}+v_1 t_p\right)} \right\} f_1(v_1)\, dv_1;\ I\left[\hat{v}^2 > \gamma(v_1)\right] f_1(v_1) dv_1. \qquad (9)$$

and

$$\gamma(v_1) = v_1^2 + 2av_1\left(t_p - \frac{x_2}{v_2}\right),$$

also

$$I[x] = \begin{cases} 1, & \text{if statement } x \text{ is true,} \\ 0, & \text{if statement } x \text{ is false.} \end{cases}$$

Again, nice representations of the three integrals above is dependent upon the specification of the speed density $f_1(v_1)$; yet numerical evaluations of the above integrals is certainly possible.

References

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[4] G. Weiss, R. Herman, Statistical properties of low density traffic, *Quarterly of Applied Mathematics*, **20** (1962), 121-130.