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ON SPECTRAL PROPERTIES OF λ-COMMUTING OPERATORS IN HILBERT SPACES

J.M. Khalagai[1], M. Kavila[2]

[1,2]School of Mathematics
University of Nairobi
P.O. Box 30197, 00100, Nairobi, KENYA

Abstract: Let $B(H)$ denote the Banach algebra of bounded linear operators on a complex Hilbert space H and let $A, B \in B(H)$ satisfying the equation; $AB = \lambda BA, \lambda \in \mathbb{C}, AB \neq 0$, where $\mathbb{C}$ denotes the complex number field. In this case A and B are said to be λ-commuting operators. In this paper we investigate the conditions under which either AB and BA or B and λB have same spectrum or same essential spectrum.

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Key Words: λ-commuting operators, essential spectrum

1. Introduction

Let $B(H)$ denote the Banach algebra of bounded linear operators on a complex Hilbert space H and $A, B \in B(H)$. Then A and B are said to λ-commute non-trivially if $AB = \lambda BA$, $\lambda \in \mathbb{C}$,$AB \neq 0$.

These types of operators have been studied by a number of authors see [3], [5], [9]. It is of interest to determine for various classes of operators A and B what restriction this places on $\lambda \in \mathbb{C}$. Also note that this property of λ-commuting operators is important for the interpretation of quantum mechanical observables and the analysis of their spectra. See [13]. On the other hand, results on essential spectra aid in coming up with generalizations of Weyl's theorem. See [1], [2] and [18].

Brooke, A.J, Busch, P and Pearson, D.B. [3], proved the following results:

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Theorem A. *Let $A, B \in B(H)$ such that $AB \neq 0$ and $AB = \lambda BA$ for $\lambda \in \mathbb{C}$. Then:*

i. if A or B is self-adjoint then λ is real.

ii. if both A and B are self adjoint then $\lambda \in \{-1, 1\}$.

iii. if A and B are self adjoint and one of them is positive, then $\lambda = 1$.

Lemma B. *Let $AB = \lambda BA$, $\lambda \in \mathbb{C}$, $AB \neq 0$. Then 0 is in either both or neither of $\sigma(AB)$ and $\sigma(BA)$. Hence: $\sigma(AB) = \sigma(BA) = \lambda\sigma(AB)$.*

Proposition C. *Let $AB = \lambda BA$, $\lambda \in \mathbb{C}$, $AB \neq 0$ and assume that A has a bounded inverse. Then we have: $\sigma(B) = \lambda\sigma(B)$.*

In [5] the operator equation $AB = \lambda BA$, $\lambda \in \mathbb{C}$ was studied for normal operators A and B on a Banach space. In this paper, we will first make an improvement on Theorem A. above before we look at spectral properties including the essential spectrum of the operators AB and BA. We will make use of Putnam-Fuglede property, see [11] and [12], and the following result proved by [16].

Theorem D. *Suppose T is a pure dominant operator, K is a compact operator with dense range such that $KT = TK$. Then essential spectrum of T is equal to spectrum of T, i.e. $\sigma_e(T) = \sigma(T)$.*

2. Notation and Terminology

- Given an operator A we shall denote the spectrum, the approximate point spectrum and essential spectrum of A by $\sigma(A)$, $\sigma_\pi(A)$ and $\sigma_e(A)$ respectively. Thus we have:

$$\sigma(A) = \{\lambda \in \mathbb{C} : A - \lambda I \text{ is not invertible}\}$$

$$\sigma_\pi(A) = \{\lambda \in \mathbb{C} : A - \lambda I \text{ is not bounded below}\}$$

$$\sigma_e(A) = \{\lambda \in \mathbb{C} : A - \lambda I \text{ is not Fredholm}\}$$

- Two operators $A, B \in B(H)$ are said to be commuting operators if:

$$[A, B] = AB - BA = 0.$$

- The range of A and Kernel of A are denoted by $ranA$ and $kerA$ respectively.
- An operator A is said to be:
  - Fredholm if its range $ranA$ is closed and both $kerA$ and $kerA^*$ are finite dimensional.

- Dominant if to each $\lambda \in \mathbb{C}$ there corresponds a number $M_\lambda \geq 1$ such that:
$$||(A-\lambda)^* x|| \leq M_\lambda \, ||(A-\lambda)x|| \, \forall x \in H.$$
- M-hyponormal if $\exists$ a constant M with $M_\lambda \leq M$ for all $\lambda \in \mathbb{C}$ such that:
$$||(A-\lambda)^* x|| \leq M \, ||(A-\lambda)x|| \, \forall x \in H.$$
- Hyponormal if $A^*A \geq AA^*$.
- Normal if $A^*A = AA^*$.
- Self-adjoint if $A = A^*$.
- Pure dominant if it has no invariant subspace say N on which A/N is normal.
- Compact if it maps a unit ball of H into a relatively compact set.
- p-hypornormal, $0 < p \leq 1$ if $(A^*A)^p \geq (AA^*)^p$.
- Quasi-affinity if it is both one-one and has a dense range.
- a partial isometry if $A = AA^*A$.

- The class $\mathcal{H} \cup (p)$ denotes the class of p-hyponormal operators A for which the polar decomposition $A = U\,|A|$ is unitary where $|A| = (A^*A)^{1/2}$ and U is a partial isometry.
- Note that we have the following inclusions of classes of operators:
  - $\{self-adjoint\} \subset \{normal\} \subset \{hyponormal\} \subset \{m-hypornormal\} \subset \{dominant\}$
  - $\{normal\} \subset \{hyponormal\} \subset \{p-hyponormal\}$
  - $\{normal\} \subset \{quasinormal\} \subset \{p-hyponormal\}$
  - $\{normal\} \subset \{quasinormal\} \subset \{subnormal\} \subset \{hyponormal\} \subset \{m-hypornormal\}$

3. Results

We first show that the following result provides an alternative prove to part (ii) of Theorem A. above.

Lemma 1. *Let $AB = \lambda BA$, $\lambda \in \mathbb{C}$, $AB \neq 0$ with A and B self-adjoint. Then:*

(i) *AB and BA are normal commuting operators.*

(ii) $\lambda = \pm 1$.

Proof. Let $T = AB$. Then $T^* = BA$ for self-adjoint A and B.

(i) Thus we have $T = \lambda T^*$, i.e. $T^*T = \lambda T^{*2}$ and $TT^* = \lambda T^{*2}$, i.e. $T^*T = TT^*$. Thus T is normal and $[T, T^*] = 0$. Hence AB and BA are normal commuting operators.

(ii) Since $T = \lambda T^*$ and λ is real, taking adjoint of each side gives: $T^* = \lambda T$. Thus:

$$T = \lambda T^* \tag{1}$$

$$T^* = \lambda T \tag{2}$$

Adding (1) and (2) gives:

$$T + T^* = \lambda(T + T^*)$$

$$ReT = \lambda(ReT)$$

$$(1 - \lambda)ReT = 0 \tag{3}$$

But subtracting (2) from (1) gives:

$$T - T^* = \lambda(T^* - T)$$

$$= -\lambda(T - T^*)$$

$$i.e. \quad ImT = -\lambda ImT.$$

$$or \quad (1 + \lambda)ImT = 0 \tag{4}$$

But, since $T \neq 0$, we have that either $(ReT) \neq 0$ or $(ImT) \neq 0$ or both $(ReT) \neq 0$ and $(ImT) \neq 0$.

Hence from (3) and (4), $\lambda = \pm 1$.

Corollary 1. *Let A and B be self-adjoint operators which λ-commute non-trivially. Then we have:*

$$\sigma_\pi(AB) = \sigma_\pi(BA) = \lambda\sigma_\pi(AB).$$

Proof. Since A and B are self adjoint operators which λ-commute non-trivially, AB and BA are normal commuting operators from Lemma 1. Hence $T = AB = BA$.

Now for any normal operator T, $\sigma(T) = \sigma_\pi(T)$.

$$\sigma_\pi(BA) = \lambda\sigma_\pi(BA) \quad (from\ Lemma\ B).$$

Hence we have the end of the proof.

Theorem 1. *Let $A, B \in B(H)$ be such that $AB = \lambda BA$, $\lambda \in \mathbb{C}$, $AB \neq 0$. We have* *(i) If A is normal and B is self-adjoint then $\lambda \in \{1, -1\}$.*

(ii) If A is normal with ReA positive and B is self-adjoint then $\lambda = 1$.

Proof.

(i) Given A is normal, B is self-adjoint and $AB = \lambda BA$, we have by part (i) of Theorem A above, that λ is real. Thus by Putnam-Fuglede's theorem we have:

$$A^*B = \lambda BA^* \tag{5}$$

Now by adding the operator equation $AB = \lambda BA$ and (5) we have:

$$(A + A^*)B = \lambda B(A + A^*)$$

i.e. $\quad ReA.B = \lambda B.\, ReA.$

Since both ReA and B are self-adjoint we have by part (ii) of Theorem A above that $\lambda \in \{1, -1\}$.

(ii) From proof of part (i) above we have: $ReA.B = \lambda B.ReA$, where both ReA and B are self-adjoint. Since ReA is positive it follows from part (iii) of Theorem A. above that $\lambda = 1$.

Corollary 2. *Let A be normal with ReA positive and B be self-adjoint such that $AB = \lambda BA$, $\lambda \in \mathbb{C}$, $AB \neq 0$. Then $AB = BA$ is normal. Thus:*

$$\sigma(AB) = \sigma(BA) = \sigma_\pi(AB) = \lambda\sigma_\pi(AB).$$

Proof. We note that under the given hypothesis $\lambda = 1$ by part (ii) of Theorem 1. above. Thus $[A, B] = 0$. Letting $T = AB$ and $T^* = BA^*$, we have:

$$T^*T = BA^*AB = BAA^*B = ABA^*B = ABBA^* = TT^*.$$

Hence $T = AB$ is normal. Consequently:

$$\sigma(AB) = \sigma(BA) = \sigma_\pi(AB) = \lambda\sigma_\pi(AB)$$

Remark 1. *We now note that in Proposition C. above, if we require that one of the operators say B in the equation $AB = \lambda BA$, $\lambda \in \mathbb{C}$, $AB \neq 0$ belongs to some appropriate class of operators then we can relax the condition on the operator A. But first we have the following result:*

Let $\wp$ denote the class of all operators which satisfy the Putnam-Fuglede property. Then we have the following results:

Theorem 2. *Let $AB = \lambda BA$, $\lambda \in \mathbb{C}$, $AB \neq 0$. Then we have:*

(i) If B and $\lambda B \in \wp$ with A a quasiaffinity, then λB and B are quasisimilar.

(ii) If A and $\lambda A \in \wp$ with B a quasiaffinity, then λA and A are quasisimilar.

Proof.

(i) Given,

$$AB = \lambda BA \tag{6}$$

we have since B and λB are in $\wp$ then $AB^* = \bar{\lambda}B^*A$. Taking adjoints both sides we get:

$$BA^* = A^*\lambda B \tag{7}$$

Since A is quasi-affinity it follows that A^* is also a quasi-affinity. Now from (6) and (7) we have that the operators λB and B are quasi similar.

(ii) Similarly since A and λA are in $\wp$ then $A^*B = B\bar{\lambda}A^*$. Taking adjoints both sides we get:

$$B^*A = \lambda AB^* \tag{8}$$

Since B is quasi-affinity it follows that B^* is also a quasi-affinity. Now from (6) and (8) we have that the operators λA and A are quasi similar.

Corollary 3. *Let $AB = \lambda BA$, $\lambda \in \mathbb{C}$, $AB \neq 0$ with B and B^* p- hyponormal with A a quasiaffinity. Then we have λB and B are quasisimilar.*

Proof. Given $AB = \lambda BA$...(i), we first note that λB and B^* are p-hyponormal and by [8], $AB^* = \bar{\lambda}B^*A$ Taking adjoints on each side gives: $BA^* = A^*\lambda B$...(ii). Since A is a quasiaffintiy it follows that A^* is also a quasiaffinity. Thus form (i) and (ii) we have that λB and B are quasisimilar.

Remark 2. *We now note that equality of spectra or essential spectra for some classes of operators has been proved by a number of authors see [6],[8],[10],[14], and [15]. In our case we have the following corollary.*

Corollary 4. *Let $AB = \lambda BA$, $\lambda \in \mathbb{C}$, $AB \neq 0$ with A a quasi-affinity. Then we have:*

(i) if B is quasi normal then $\sigma_e(B) = \sigma_e(\lambda B) = \lambda\sigma_e(B)$.

(ii) if B and B^* are hyponormal and A compact, then $\sigma(B) = \lambda\sigma(B)$ and $\sigma_e(B) = \lambda\sigma_e(B)$.

(iii) if λB and $B \in \mathcal{H} \cup (p)$, then $\sigma_e(B) = \lambda\sigma_e(B)$ and $\sigma(B) = \lambda\sigma(B)$.

(iv) if B^* and λB are m-hyponormal, then $\sigma_e(B) = \sigma_e(\lambda B) = \lambda\sigma_e(B)$

Proof.

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(i) By the inclusion $\{quasinormal\} \subset \{p-hyponormal\}$, see [4], and the fact that B^* is quasinormal if B is quasinormal, we have that B and B^* are indeed also p-hypornormal. Thus applying Corollary 3, λB and B are quasisimilar. By [14] we have $\sigma_e(B) = \sigma_e(\lambda\ B) = \lambda\sigma_e(B)$. Also to authenticate our result further, [6] proved that quasisimilar subnormal operators have equal essential spectra and the inclusion $\{quasinormal\} \subset \{subnormal\}$ ensures that the result stands for quasinormal operators.

(ii) By the inclusion $\{hyponormal\} \subset \{p-hyponormal\}$ and Corollary 3, λB and B are quasisimilar. Hence by [6] and [15]: $\sigma(B) = \lambda\sigma(B)$ and $\sigma_e(B) = \lambda\sigma_e(B)$.

(iii) We only have to note that it was proved in [8] that for the operators in $\mathcal{H} \cup (p)$, they belong to $\wp$ and they have same spectrum and same essential spectrum.

(iv) From [7], λB and B belong to $\wp$ due to the inclusion $\{m-hyponormal\} \subset \{dominant\}$. Applying Theorem 2 we have that the operators λB and B are quasisimilar. By [17],we have $\sigma_e(B) = \sigma_e(\lambda B) = \lambda\sigma_e(B)$.

Remark 3. *We note that in Corollary 4 above the conditions on the operators A and B can be interchanged so that similar results can be obtained on spectrum and essential spectrum of A. We also note that in view of Theorem D. above we have the following result about essential spectra of the operators AB and BA.*

Theorem 3. *Let $AB = \lambda BA$, $\lambda \in \mathbb{C}$, $AB \neq 0$ with $\lambda \in \{1, -1\}$. If AB is pure dominant and BA is compact with dense range then: $\sigma_e(AB) = \sigma(AB) = \sigma(BA) = \lambda\sigma(AB)$.*

Proof. Let $T = AB$ and $K = BA$. Then under the conditions that $\lambda \in \{1, -1\}$, we have: $[T, K] = 0$ For if $\lambda = -1$, then $AB = -BA$. Thus: $ABBA = -BABA$ and $BAAB = -BABA$. This means AB and BA commute. Now by theorem D above: $\sigma_e(T) = \sigma(T)$. i.e $\sigma_e(AB) = \sigma(AB) = \sigma(BA) = \lambda\sigma(AB)$.

Remark 4. *Let us for convenience sake say that $T \in B(H)$ belong to a class $\mathcal{M}$ of operators if:*

(i) T is pure dominant

(ii) T is compact

(iii) T has a dense range

Then the following corollary to the theorem above is immediate.

Corollary 5. *Let $AB = \lambda BA$, $\lambda \in \mathbb{C}$, $AB \neq 0$ with $\lambda \in \{1, -1\}$. Also let $T = AB$ and $K = BA$. If $T, K \in \mathcal{M}$, then: $\sigma_e(T) = \sigma(T) = \sigma(K) = \sigma_e(K)$.*

Proof. Since the operators $T, K \in \mathcal{M}$,then from Theorem 3 above we get $\sigma_e(K) = \sigma(K) = \sigma(T) = \sigma_e(T)$. Hence $\sigma_e(AB) = \sigma_e(BA)$

Remark 5. *Note that it is a well known fact that dominant operators that are compact are quasinilpotent hence have zero as its essential spectra.*

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