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QUANTITATIVE ESTIMATION OF GIBBS PHENOMENON IN DINI EXPANSION

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Abstract: Gibbs phenomena in Dini expansion are estimated analytically for moderate spatial interval and a moderate number terminated to give dependency of a peak value of overshoot on the number terminated.

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Key Words: Gibbs phenomena, Dini expansion, dependency

1. Introduction

Oscillation of the values of the function formed by truncating up to a finite number from originally infinite series for a piecewise continuous and smooth function is called a Gibbs phenomenon [1], [2]. The maximun overshoot amplitude of the oscillation at a discrete point for Fourier series expansion is analytically shown to approach a finite non-zero value as truncated number of terms tends to infinity [1], the nature of which is unique to Fourier series expansion. Numerical estimation based on Fourier-Bessel expansion for the Gibbs phenomena is treated e.g. by Cooke [3], Gray & Pinsky [4], Fay & Kloppers [5].

Hereafter characteristics of the Gibbs phenomena using Dini expansion (Bessel expansion) is analytically shown in the sense of asymptotic expansion of the truncated terms for a step change function.

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2. Analysis

The set $\{J_0(\lambda_n x/\pi)|n \geq 0\} + \{J_1(\lambda_n x/\pi)|n \geq 1\}$ is complete in $[-\pi, \pi]$, where λ_n is the positive n-th zero of $J_1(x)$ if $n \geq 1$ and $\lambda_0 = 0$. If a given function f is piecewise continuous in the said interval, then f can be expanded into Dini series such as

$$f = \sum_{n=1}^{\infty} s_n \, J_1\left(\lambda_n \frac{x}{\pi}\right) + \sum_{n=0}^{\infty} c_n \, J_0\left(\lambda_n \frac{x}{\pi}\right), \tag{1}$$

$$s_n = \int_0^{\pi} x \, J_1(\lambda_n x/\pi) \{f(x) - f(-x)\} \, dx \Big/ \left\{\pi^2 J_0^2(\lambda_n)\right\}, \quad (n \geq 1), \tag{2}$$

$$c_n = \int_0^{\pi} x \, J_0(\lambda_n x/\pi) \{f(x) + f(-x)\} \, dx \Big/ \left\{\pi^2 J_0^2(\lambda_n)\right\}, \quad (n \geq 0). \tag{3}$$

Let f be

$$f(x) = \begin{cases} 1 & (0 < x < \pi) \\ 0 & (x = 0 \ \text{or} \ x = \pm\pi) \\ -1 & (-\pi < x < 0) \end{cases} . \tag{4}$$

Then in this case all c_n's are zero, and

$$s_n = 2 \int_0^1 x \, J_1(\lambda_n x) \, dx \Big/ \left\{J_0^2(\lambda_n)\right\}, \tag{5}$$

$$\int_0^1 x \, J_1(\lambda_n x) \, dx = -\frac{1}{\lambda_n} + \frac{1}{\lambda_n^2} - \frac{1}{\lambda_n^2} \int_{\lambda_n}^{\infty} J_0(x) \, dx, \tag{6}$$

$$\lambda_n = \frac{\pi b_n}{4} \left[1 - \frac{6}{(\pi b_n)^2} + \boldsymbol{O}\left\{\left(\frac{1}{\pi b_n}\right)^4\right\}\right], \tag{7}$$

$$b_n \equiv 4n + 1 \qquad (n \geq 1),$$

$$J_0(\lambda_n) = (-1)^n \frac{2\sqrt{2}}{\pi\sqrt{b_n}} \left[1 + \frac{24}{(\pi b_n)^4} + \boldsymbol{O}\left\{\left(\frac{1}{\pi b_n}\right)^6\right\}\right]. \tag{8}$$

Thus,

$$\int_{\lambda_n}^{\infty} J_0(x) \, dx = \left(\frac{1}{\lambda_n} - \frac{3}{\lambda_n^3}\right) J_0(\lambda_n) + \boldsymbol{o}\left(\lambda_n^{-3}\right), \tag{9}$$

$$J_0(\lambda_n) \sim \boldsymbol{O}\left(\frac{1}{\sqrt{n}}\right) \ \text{as} \ n \to \infty, \tag{10}$$

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$$\int_0^1 x\mathrm{J}_1(\lambda_n x)\,dx = -\frac{1}{\lambda_n}\left\{\mathrm{J}_0(\lambda_n) + \boldsymbol{O}\left(\frac{1}{n}\right)\right\}, \tag{11}$$

$$s_n \approx -\frac{2}{\lambda_n \mathrm{J}_0(\lambda_n)} \quad (n \to \infty). \tag{12}$$

If $n \gg 1, \lambda_n x \geq 1, \pi x \ll 1$, then

$$\begin{aligned} &s_n\,\mathrm{J}_1(\lambda_n x) + s_{n+1}\,\mathrm{J}_1(\lambda_{n+1} x) \\ &\approx (-1)^{n-1}\frac{\sqrt{2}\,\pi^{3/2}}{(\lambda_n + 2\pi/3)^{3/2}}\,\mathrm{J}_1(\lambda_n x). \end{aligned} \tag{13}$$

If N is a positive integer and not small, then

$$\begin{aligned} &\sum_{n=N,N+2,N+4,\cdots}\{s_n\,\mathrm{J}_1(\lambda_n x) + s_{n+1}\,\mathrm{J}_1(\lambda_{n+1} x)\} \\ &\approx (-1)^{N-1}\frac{1}{2}\int_{N-1}^{\infty}\frac{\sqrt{2}\,\pi^{3/2}}{(\lambda_n + 2\pi/3)^{3/2}}\,\mathrm{J}_1(\lambda_n x)\,dn \\ &\approx (-1)^{N-1}\sqrt{\frac{\pi}{2}}\int_{\lambda_{N-1}}^{\infty}\frac{\mathrm{J}_1(\lambda x)}{(\lambda + 2\pi/3)^{3/2}}d\lambda. \end{aligned} \tag{14}$$

Let f_n be defined as

$$f_n \equiv \sum_{k=1}^{n} s_k\,\mathrm{J}_1(\lambda_k x). \tag{15}$$

Then,

$$f_{N-1} \approx 1 + (-1)^N\sqrt{\frac{\pi}{2}}\int_{\lambda_{N-1}}^{\infty}\frac{\mathrm{J}_1(\lambda x)}{(\lambda + 2\pi/3)^{3/2}}\,d\lambda. \tag{16}$$

To keep the character of substantially alternating series for small $x(> 0)$, focus is limited to the case that N is odd. The location points at local maximum or minimum are attained by

$$\frac{d}{dx}\int_{\lambda_{N-1}}^{\infty}\frac{\mathrm{J}_1(\lambda x)}{(\lambda + 2\pi/3)^{3/2}}\,d\lambda = 0, \tag{17}$$

which becomes

$$\begin{aligned} &\frac{-\lambda_{N-1}}{(\lambda_{N-1} + 2\pi/3)^{3/2}}\,\mathrm{J}_1(\lambda_{N-1} x) \\ &= \int_{\lambda_{N-1}}^{\infty}\mathrm{J}_1(\lambda x)\frac{d}{d\lambda}\left\{\frac{\lambda}{(\lambda + 2\pi/3)^{3/2}}\right\}d\lambda. \end{aligned} \tag{18}$$

As long as $\lambda_{N-1} \gg \pi$,

$$\lambda_{N-1}\,x - \lambda_k \approx \frac{1}{2\,\lambda_k}\left(1 - \frac{2\pi}{\lambda_{N-1}}\right) \tag{19}$$

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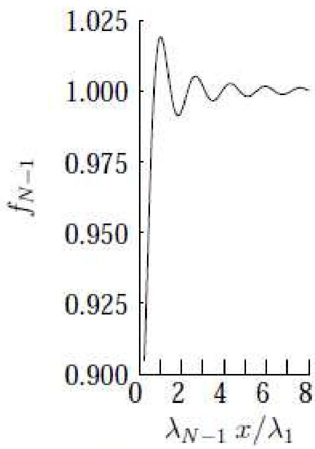

Fig. 1 Characteristics for $N = 9$

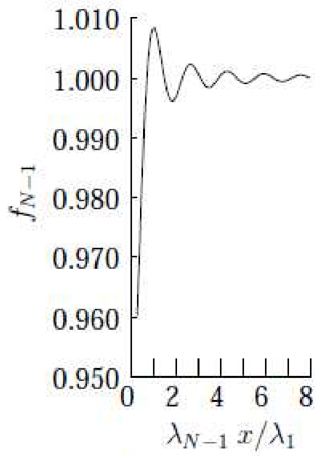

Fig. 2 Characteristics for $N = 51$

$$(k = 1, 2, \cdots, \infty).$$

For the point corresponding to $k = 1$:

$$\int_{\lambda_{N-1}}^{\infty} \frac{\mathrm{J}_1(\lambda x)}{(\lambda + 2\pi/3)^{3/2}}\, d\lambda \approx \frac{1}{\sqrt{\lambda_{N-1}}} \frac{\mathrm{J}_0(\lambda_1)}{\lambda_1}, \tag{20}$$

which means that peak deviation from the exact function value is roughly

$$-\sqrt{\frac{\pi}{2\, \lambda_{N-1}}}\, \frac{\mathrm{J}_0(\lambda_1)}{\lambda_1}$$

and that it is roughly proportional to $1/\sqrt{N-1}$, where $N-1$ stands for the number of terms summed. Actually $\lambda_1 \approx 3.8317, \mathrm{J}_0(\lambda_1) \approx -0.4028$.

3. Numerical Examples

Figures 1 and 2 show the characteristics for f_{N-1} corresponding to $N = 9$, and $N = 51$ respectively.

4. Conclusions

Gibbs phenomenon due to Dini expansion is analytically estimated for a moderate number of truncation, which shows that maximum amplitude of oscillation decreases as number of summation increases.

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