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OSCILLATIONS FOR SECOND ORDER NONLINEAR IMPULSIVE DIFFERENTIAL EQUATIONS WITH DAMPING

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Abstract: In this paper, we study the oscillatory behavior of second order nonlinear impulsive differential equations with damping of the form

$$(r(t)x'(t))' + p(t)x'(t) + q(t)f(x(t)) = 0.$$

Our results are not restricted with the signs $p(t)$ and $q(t)$. Some interesting results are obtained, which illustrate that impulses play a very important role in giving rise to the oscillations of equations.

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1. Introduction

In this paper, we consider the oscillation of all solutions of second order nonlinear impulsive differential equations with damping term of the form

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$$\begin{cases} (r(t)x'(t))' + p(t)x'(t) + q(t)f(x(t)) = 0,\ t \geq t_0,\ t \neq t_k, \\ x(t_k^+) = g_k(x(t_k)), \quad x'(t_k^+) = h_k(x'(t_k)),\ t = t_k,\ k = 1, 2, 3, ..., \end{cases} \tag{1}$$

where $0 \leq t_0 < t_1 < t_2 < \cdots < t_k \cdots$ with $\lim_{k\to\infty} t_k = +\infty$.

In recent years the oscillation theory of impulsive differential equations emerging as an important area of research, since such equations have application in control theory, physics, biology, population dynamics, etc. In [1, 2, 3, 4, 5, 14, 15, 18, 19], the authors established several oscillation criteria for second order impulsive delay and neutral delay differential equations. However, to the best of our knowledge, there is little in the way of results for the oscillation of second order nonlinear impulsive differential equations with damping term[10, 13]. Motivated by this observation, in this paper, we establish some new oscillation criteria for all solutions of equation (1).

Throughout this paper, we assume that:

(H_1) $r \in C([t_0, \infty), (0, \infty))$, $p, q \in PC([t_0, \infty), R)$, $f \in C'(R, R)$ such that $xf(x) > 0$ and nondecreasing for all $x \neq 0$, $r(t) > 0$;

(H_2) $g_k(x), h_k(x)$ are continuous in R and there exist a positive constant a_k, $\overline{a_k}$, b_k, $\overline{b_k}$ such that $\overline{a_k} \leq \dfrac{g_k(x)}{x} \leq a_k$, $\quad \overline{b_k} \leq \dfrac{h_k(x)}{x} \leq b_k$ for all $x \neq 0$, $k = 1, 2, 3, \ldots$;

(H_3) $\displaystyle\lim_{t\to\infty} \int_{t_j}^{t} \prod_{s<t_k<t} \frac{\overline{b_k}}{a_k} \exp\left(-\int_s^t \frac{r'(\sigma) + p(\sigma)}{r(\sigma)} d\sigma\right) ds = +\infty.$

By a solution of (1), we mean a real valued function $x(t)$ defined on $[t_0, +\infty)$ which satisfies:

(i) for any $t \in [t_0, +\infty)$, $t \neq t_k$, $k = 1, 2, 3, \ldots$, $x(t)$ is continuously differentiable and satisfies

$(r(t)x'(t))' + p(t)x'(t) + q(t)f(x(t)) = 0;$

(ii) for any $k = 1, 2, \ldots$, $x(t_k^-)$, $x(t_k^+)$, $x'(t_k^-)$, $x'(t_k^+)$ exist and $x(t_k^-) = x(t_k)$, $x'(t_k^-) = x'(t_k)$, $x(t_k^+) = g_k(x(t_k))$, $x'(t_k^+) = h_k(x'(t_k))$.

A solution of (1) is said to be nonoscillatory, if it is eventually positive or eventually negative. Otherwise, the solution is said to be oscillatory.

The paper is organized as follows. In Section 2, we shall offer two interesting lemmas, which will be used in Section 3 to prove our main theorems. To illustrate our results, examples are provided in Section 4.

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2. Some Lemmas

Lemma 1 ([12]). *Suppose*

(i) the sequence $\{t_k\}_{k\in N}$ *satisfies* $0 \le t_0 < t_1 < t_2 < \cdots < t_k \cdots$ *with* $\lim\limits_{t\to\infty} t_k = +\infty$*;*

(ii) $m \in PC'(R_+, R)$ *is left continuous at* t_k *for* $k = 1, 2, 3, \ldots$*;*

(iii) for $k = 1, 2, 3, \ldots$ *and* $t \ne t_0$*, we have*

$$m'(t) \le p(t)m(t) + q(t), t \ne t_k, \tag{2}$$

$$m(t_k^+) \le \alpha_k m(t_k^-) + \beta_k, \tag{3}$$

where p*,* $q \in C(R_+, R)$*,* α_k *and* β_k *are real constants with* $\alpha_k \ge 0$*. Then the following inequality holds*

$$m(t) \le m(t_0) \prod_{t_0<t_k<t} \alpha_k \exp\left(\int_{t_0}^t p(s)ds\right) + \int_{t_o}^t \prod_{s<t_k<t} \exp\left(\int_s^t p(u)du\right) q(s)ds + \sum_{t_0<t_k<t} \prod_{t_k<t_j<t} \alpha_j \exp\left(\int_{t_k}^t p(s)ds\right) \beta_k,\ t \ge t_0. \tag{4}$$

Lemma 2 ([10]). *Let* $x(t)$ *be a solution of equation* (1)*. Suppose that there exist some* $T \ge t_0$ *such that* $x(t) > 0$*, for* $t \ge T$*. If* (H_1)*,* (H_2) *and* (H_3) *are satisfied, then* $x'(t_j) > 0$ *and* $x'(t) > 0$ *for* $t \in (t_k, t_{k+1}]$*, where* $t_k \ge T$*,* $k = 1, 2, \ldots$*.*

3. Main Results

In this section, we investigate the oscillatory behavior of all solutions of equation (1). We begin with the following theorem.

Theorem 1. *Suppose that* (H_1)*,* (H_2)*,* (H_3)*, and* $x(t_k^+) = x(t_k)$ *are satisfied. Assume that there exist a positive function* $g \in C'(R_+)$ *such that* $g'(t) \ge 0$ *and*

$$\lim_{t\to\infty} \int_{t_0}^t \prod_{t_0<t_k<s} \frac{1}{b_k} exp\left[\int_s^{t_0} \frac{g'(\sigma)r(\sigma) - p(\sigma)g(\sigma)}{g(\sigma)r(\sigma)} d\sigma\right] q(s)g(s)ds = +\infty, \tag{5}$$

then all solutions of equation (1) oscillate.

Proof. Let $x(t)$ be a nontrivial nonoscillatory solution of equation (1), which without loss of generality we may assume that $x(t) > 0$ on $(T_0, +\infty)$ for some large $T_0 \ge t_0$. By Lemma 2, $x'(t) \ge 0$ for all $t \ge T_0$.

Define

$$w(t)=\frac{g(t)r(t)x'(t)}{f(x(t))}, \quad t\neq t_k. \tag{6}$$

Then $w(t_k^+)\geq 0$ $(k=1,2,3,\ldots)$, $w(t)\geq 0$ for $t\geq t_0$.

Using equation (1) for $t\neq t_k$, we have

$$w'(t) = \frac{g'(t)r(t)x'(t)}{f(x(t))}-g(t)q(t)-\frac{f'(x(t))(x'(t))^2g(t)r(t)}{[f(x(t))]^2}-\frac{g(t)p(t)x'(t)}{f(x(t))},$$

Since $g(t)\geq 0$, $f'(x)\geq 0$, $x'(t)\geq 0$, the third term on the righthand side is nonnegative, and hence

$$w'(t) \leq g'(t)\frac{w(t)}{g(t)}-p(t)\frac{w(t)}{r(t)}-q(t)g(t),$$

$$w'(t) \leq \left[\frac{g'(t)r(t)-p(t)g(t)}{g(t)r(t)}\right]w(t)-q(t)g(t). \tag{7}$$

$$\begin{aligned} w(t_k^+) &= \frac{g(t_k^+)r(t_k^+)x'(t_k^+)}{f(x(t_k^+))} \\ &\leq \frac{g(t_k^+)r(t_k^+)x'(t_k)}{f(x(t_k))} \\ &\leq b_kw(t_k),\ t=t_k,\ k=1,2,\ldots. \end{aligned}$$

We now consider the following impulsive differential inequalities:

$$\begin{aligned} w'(t) &\leq \left[\frac{g'(t)r(t)-p(t)g(t)}{g(t)r(t)}\right]w(t)-q(t)g(t),\ t\neq t_k, \\ w(t_k^+) &\leq b_kw(t_k),\ t=t_k,\ k=1,2,\ldots. \end{aligned}$$

Using Lemma 1, we have,

$$\begin{aligned} w(t)\leq w(t_0)\prod_{t_0<t_k<t} b_k\, exp\left[\int_{t_0}^t \frac{g'(s)r(s)-p(s)g(s)}{g(s)r(s)}ds\right] \\ -\int_{t_0}^t \prod_{s<t_k<t} b_k\, exp\left[\int_s^t \frac{g'(\sigma)r(\sigma)-p(\sigma)g(\sigma)}{g(\sigma)r(\sigma)}d\sigma\right]g(s)q(s)ds \end{aligned}$$

$$w(t)\leq \prod_{t_0<t_k<t} b_k\, exp\left(\int_{t_0}^t \frac{g'(s)r(s)-p(s)g(s)}{g(s)r(s)}ds\right)$$

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$$\cdot \left[w(t_0) - \int_{t_0}^{t} \prod_{t_0<t_k<s} \frac{1}{b_k} exp\left(\int_{s}^{t_0} \frac{g'(\sigma)r(\sigma)-p(\sigma)g(\sigma)}{g(\sigma)r(\sigma)} d\sigma \right) g(s)q(s)ds \right]$$

By the condition (5), the above inequality is impossible. This contradiction establishes the result.

Theorem 2. *Assume that conditions* (H_1), (H_2), (H_3) *hold and* $f(ab) \geq f(a)f(b)$ *for any* $ab > 0$. *If*

$$\int_{\pm\in}^{\pm\infty} \frac{du}{f(u)} < +\infty \tag{8}$$

holds for some $\in > 0$, *and*

$$\sum_{k=0}^{+\infty} \int_{t_k^+}^{t_{k+1}} \frac{1}{r(s)g(s)} exp\left(-\int_{s}^{t} \frac{g'(u)r(u)-p(u)g(u)}{g(u)r(u)} du \right)$$
$$\cdot \left[\lim_{t\to\infty} \int_{s}^{t} \prod_{s<t_k<v} \frac{1}{c_k} exp\left(\int_{v}^{t} \frac{g'(\sigma)r(\sigma)-p(\sigma)g(\sigma)}{g(\sigma)r(\sigma)} d\sigma \right) g(v)q(v)dv \right] ds = +\infty, \tag{9}$$

where

$$c_k = \frac{b_k}{f(\overline{a}_k)}, \ \ k = 1, 2, \ldots.$$

Then all solutions of equation (1) oscillate.

Proof. Let $x(t)$ be a nonoscillatory solution of (1). Suppose that $x(t) > 0$ for $t \geq t_0$ and $k_0 = 1$. From Lemma 2, $x'(t) \geq 0$ for $t \geq t_0$. Using the function $w(t)$ defined in (6) and proceeding similarly as in the proof of Theorem 1, we have inequality (7).

$$w'(t) \leq \left[\frac{g'(t)r(t)-p(t)g(t)}{g(t)r(t)} \right] w(t) - q(t)g(t), \ t \neq t_k.$$

For $k = 1, 2, \ldots$

$$\begin{aligned} w(t_k^+) &= \frac{g(t_k^+)r(t_k^+)x'(t_k^+)}{f(x(t_k^+))} \\ &\leq \frac{b_k g(t_k^+)r(t_k^+)x'(t_k)}{f(\overline{a}_k x(t_k))} \\ &\leq \frac{b_k g(t_k^+)r(t_k^+)x'(t_k)}{f(\overline{a}_k)f(x(t_k))} \\ &\leq c_k w(t_k), \ t = t_k, \ k = 1, 2, \ldots. \end{aligned}$$

Consider the following impulsive differential inequalities:

$$w'(t) \ \ \leq \ \ \left[\frac{g'(t)r(t)-p(t)g(t)}{g(t)r(t)} \right] w(t) - q(t)g(t), \ t \neq t_k,$$

$$w(t_k^+) \quad \leq \quad c_k w(t_k),\ t = t_k,\ k = 1, 2, \ldots .$$

From Lemma 1, it follows that,

$$w(t) \leq w(s) \prod_{s<t_k<t} c_k\, exp\left(\int_s^t \frac{g'(u)r(u) - p(u)g(u)}{g(u)r(u)} du\right) - \int_s^t \prod_{v<t_k<t} c_k\, exp\left(\int_v^t \frac{g'(\sigma)r(\sigma) - p(\sigma)g(\sigma)}{g(\sigma)r(\sigma)} d\sigma\right) g(v)q(v)dv,$$

i.e.,

$$w(s) \geq \frac{w(t)}{\prod_{s<t_k<t} c_k\, exp\left(\int_s^t \frac{g'(u)r(u) - p(u)g(u)}{g(u)r(u)} du\right)} + \frac{\int_s^t \prod_{v<t_k<t} c_k\, exp\left(\int_v^t \frac{g'(\sigma)r(\sigma) - p(\sigma)g(\sigma)}{g(\sigma)r(\sigma)} d\sigma\right) g(v)q(v)dv}{\prod_{s<t_k<t} c_k\, exp\left(\int_s^t \frac{g'(u)r(u) - p(u)g(u)}{g(u)r(u)} du\right)},$$

$$w(s) \geq exp\left(-\int_s^t \frac{g'(u)r(u) - p(u)g(u)}{g(u)r(u)} du\right) \cdot \int_s^t \prod_{v<t_k<t} \frac{1}{c_k} exp\left(\int_v^t \frac{g'(\sigma)r(\sigma) - p(\sigma)g(\sigma)}{g(\sigma)r(\sigma)} d\sigma\right) g(v)q(v)dv$$

i.e.,

$$\frac{x'(s)}{f(x(s))} \geq \frac{1}{r(s)g(s)} exp\left(-\int_s^t \frac{g'(u)r(u) - p(u)g(u)}{g(u)r(u)} du\right) \cdot \int_s^t \prod_{v<t_k<t} \frac{1}{c_k} exp\left(\int_v^t \frac{g'(\sigma)r(\sigma) - p(\sigma)g(\sigma)}{g(\sigma)r(\sigma)} d\sigma\right) g(v)q(v)dv \quad (10)$$

For $s \in (t_k, t_k + 1]$, $k = 1, 2, 3, \ldots$, we obtain

$$\int_{t_k^+}^{t_k+1} \frac{x'(s)}{f(x(s))ds} = \int_{x(t_k^+)}^{x(t_k+1)} \frac{du}{f(u)} \quad (11)$$

From (10) and (11), we have

$$\sum_{k=0}^{+\infty} \int_{t_k^+}^{t_{k+1}} \frac{1}{r(s)g(s)} exp\left(-\int_s^t \frac{g'(u)r(u) - p(u)g(u)}{g(u)r(u)} du\right)$$

$$\cdot\left[\lim_{t\to\infty}\int_s^t \prod_{s<t_k<v}\frac{1}{c_k}exp\left(\int_v^t \frac{g'(\sigma)r(\sigma)-p(\sigma)g(\sigma)}{g(\sigma)r(\sigma)}d\sigma\right)g(v)q(v)dv\right]ds$$

$$\le \sum_{k=0}^{+\infty}\int_{x(t_k^+)}^{x(t_{k+1})}\frac{du}{f(u)} \le \int_{x(t_0^+)}^{+\infty}\frac{du}{f(u)},$$

which is contradiction with condition (8) and (9). Hence all solutions of equation (1) is oscillate. The proof of Theorem 2 is complete.

Corollary 3. *Assume that conditions* (H_1), (H_2), (H_3), *hold and there exists a positive integer* k_0 *and a constant* $\alpha > 0$ *such that* $\frac{1}{c_k} \ge t_{k+1}^{\alpha}$ *for* $k \ge k_0$, *where*

$$c_k = \frac{b_k}{f(\overline{a}_k)},\quad k = 1, 2, \ldots .$$

Suppose that $f(ab) \ge f(a)f(b)$ *for any* $ab > 0$ *and*

$$\int_{\pm\in}^{\pm\infty}\frac{du}{f(u)} < +\infty$$

holds for some $\in > 0$. *If*

$$\sum_{k=0}^{+\infty}[R(t_{k+1} - R(t_k^+)]$$

$$\cdot\int_{t_{k+1}^+}^{+\infty} t^{\alpha} exp\left(\int_v^t \frac{g'(\sigma)r(\sigma)-p(\sigma)g(\sigma)}{g(\sigma)r(\sigma)}d\sigma\right)g(t)q(t)dt = +\infty, \quad (12)$$

where

$$R(t) = \int_{t_0}^t \frac{1}{r(s)g(s)}\, exp\left(-\int_s^t \frac{g'(u)r(u)-p(u)g(u)}{g(u)r(u)}du\right)ds.$$

Then every solution of equation (1) oscillate.

Proof. Let $k_0 = 1$, $t_1 \ge 1$. Since $\frac{1}{c_k} \ge t_{k+1}^{\alpha}$ for $k \ge k_0$, we have

$$\frac{1}{c_{k+1}} \ge t_{k+2}^{\alpha},$$

$$\frac{1}{c_{k+1}}\frac{1}{c_{k+2}} \ge t_{k+2}^{\alpha}t_{k+3}^{\alpha} \ge t_{k+3}^{\alpha}, \ldots,$$

$$\frac{1}{c_{k+1}}\frac{1}{c_{k+2}}\cdots\frac{1}{c_{k+n}} \ge t_{k+2}^{\alpha}t_{k+3}^{\alpha}, \ldots t_{k+n+1}^{\alpha} \ge t_{k+n+1}^{\alpha}, \ldots,$$

Now,

$$\sum_{k=0}^{+\infty}\int_{t_k^+}^{t_{k+1}}\frac{1}{r(s)g(s)}\, exp\left(-\int_s^t \frac{g'(u)r(u)-p(u)g(u)}{g(u)r(u)}du\right)$$

$$\cdot\left[\lim_{t\to\infty}\int_s^t \prod_{s<t_k<v}\frac{1}{c_k}exp\left(\int_v^t \frac{g'(\sigma)r(\sigma)-p(\sigma)g(\sigma)}{g(\sigma)r(\sigma)}d\sigma\right)g(v)q(v)dv\right]ds$$

$$= \sum_{k=0}^{+\infty}\int_{t_k^+}^{t_{k+1}}\frac{1}{r(s)g(s)}\,exp\left(-\int_s^t \frac{g'(u)r(u)-p(u)g(u)}{g(u)r(u)}du\right)$$

$$.\Big[\lim_{t\to\infty}\int_s^{t_{k+1}} exp\left(\int_v^t \frac{g'(\sigma)r(\sigma)-p(\sigma)g(\sigma)}{g(\sigma)r(\sigma)}d\sigma\right)g(v)q(v)dv$$

$$+\frac{1}{c_{k+1}}\int_{t_{k+1}^+}^{t_{k+2}} exp\left(\int_v^t \frac{g'(\sigma)r(\sigma)-p(\sigma)g(\sigma)}{g(\sigma)r(\sigma)}d\sigma\right)g(v)q(v)dv+\cdots$$

$$+\frac{1}{c_{k+1}}\frac{1}{c_{k+2}}\cdots\frac{1}{c_{k+n}}\int_{t_{k+n}^+}^{t_{k+n+1}} exp\left(\int_v^t \frac{g'(\sigma)r(\sigma)-p(\sigma)g(\sigma)}{g(\sigma)r(\sigma)}d\sigma\right)$$

$$.g(v)q(v)dv\Big]ds$$

$$\geq \sum_{k=0}^{+\infty}\int_{t_k^+}^{t_{k+1}}\frac{1}{r(s)g(s)}\,exp\left(-\int_s^t \frac{g'(u)r(u)-p(u)g(u)}{g(u)r(u)}du\right)$$

$$.\Big[\lim_{t\to\infty}\int_{t_{k+1}^+}^{t_{k+n+1}} v^\alpha exp\left(-\int_s^t \frac{g'(u)r(u)-p(u)g(u)}{g(u)r(u)}du\right)g(v)q(v)dv\Big]ds$$

$$= \sum_{k=0}^{+\infty}\left(R(t_{k+1}-R(t_k^+)\right)\int_{t_{k+1}^+}^{+\infty} t^\alpha exp\left(\int_v^t \frac{g'(\sigma)r(\sigma)-p(\sigma)g(\sigma)}{g(\sigma)r(\sigma)}d\sigma\right)$$

$$.g(t)q(t)dt$$

$$= +\infty.$$

By Theorem 2, we find all solutions of equation (1) oscillate.

4. Examples

Example 1. Consider the following second order impulsive type differential equation

$$\begin{cases} x''(t)+\frac{1}{t}x'(t)+\left(1+\frac{1}{t^2}\right)x(1+x^2)=0, & t\neq 2^k,\ t\geq 1\ ,\ k=1,2,3,\dots \\ x'((2^k)^+)=\left(\frac{k}{k+1}\right)x'(2^k), \quad x((2^k)^+)=x(2^k), & k=1,2,3,\dots \end{cases} \tag{13}$$

Here $r(t)=1$, $p(t)=\frac{1}{t}$, $q(t)=1+\frac{1}{t^2}$, $f(x)=x(1+x^2)$, $a_k=\overline{a}_k=1$, $b_k=\overline{b}_k=\frac{k}{k+1}$, $t_k=2^k$. $t_0=1$. Choose $g(t)=2-\frac{1}{t}$. Obviously, the condition (H_3) is satisfied.

A straightforward calculation shows that

$$\lim_{t\to\infty}\int_{t_0}^{t}\prod_{1<t_k<s}\frac{1}{b_k}\,exp\left[\int_s^{t_0}\frac{g'(\sigma)r(\sigma)-p(\sigma)g(\sigma)}{g(\sigma)r(\sigma)}d\sigma\right]q(s)g(s)ds$$
$$=\lim_{t\to\infty}\int_1^t\prod_{1<t_k<s}\frac{k+1}{k}\left(\frac{s^2+1}{s}\right)sds=+\infty.$$

Therefore by Theorem 1, every solution of equation (13) is oscillatory.

2. Consider the following second order impulsive type differential equation

$$\begin{cases}(t^2x'(t))'+tx'(t)+t^3x^3=0, & t\neq 2^k,\ t\geq 1,\ k=1,2,3,\dots\\ x'((2^k)^+)=\left(\dfrac{k}{k+1}\right)x'(2^k), & x((2^k)^+)=x(2^k),\quad k=1,2,3,\dots\end{cases}\tag{14}$$

Here $r(t)=t^2$, $p(t)=t$, $q(t)=t^3$, $a_k=\overline{a}_k=1$, $b_k=\overline{b}_k=\dfrac{k}{k+1}$, $t_k=2^k$, $t_0=1$ and $f(x)=x^3$. Choose $g(t)=1$. Obviously, the condition (H_3) is satisfied.

A straightforward calculation shows that

$$\lim_{t\to\infty}\int_{t_0}^{t}\prod_{t_0<t_k<s}\frac{1}{b_k}\,exp\left[\int_s^{t_0}\frac{g'(\sigma)r(\sigma)-p(\sigma)g(\sigma)}{g(\sigma)r(\sigma)}d\sigma\right]q(s)g(s)ds$$
$$=\lim_{t\to\infty}\int_1^t\prod_{1<t_k<s}\frac{k+1}{k}\,exp\left(\int_s^1\frac{-1}{\sigma}d\sigma\right)s^3ds=+\infty.$$

Therefore by Theorem 1, every solution of equation (14) is oscillatory.

References

[1] L. Berenzansky and E. Braverman, On oscillation of a second order impulsive delay differential equation, *J. Math. Anal. Appl.*, **39** (2000), 217-225.

[2] E.M. Bonotto, L.P. Gimenses and M. Federson, Oscillation for a second order neutral differential equation with impulses, *Cad. Mat.* , **9** (2008), 169-190.

[3] Jin-Fa Cheng and Yu-Ming Chu, Oscillation of second order neutral impulsive differential equations,*J. Ineqs.App* (2010)Article ID:493927, 29 pages.

[4] L.P. Gimenses and Marica Federson, Oscillation by impulses for a seond order delay differential eqution, *Cad. Mat.*, **06** (2005), 181-191.

[5] K. Gopalsamy and B.G. Zhang, On delay differential equations with impulses, *J. Math. Anal. Appl*, *139* (1989), 110-122.

[6] I. Gyori and G. Ladas, *Oscillation Theory of Delay Differential Equtions With Applications*, Clarendon Press, Oxford, 1991.

[7] Chengjun Guo and Zhiting Xu, On the osillation of second order linear impulsive differential equations, *Diff. Eqns.Apps.*, 3 (2010), 319-330.

[8] Zhenlai Han, Tongxing Li, Shurong Sun and Weisong Chen, *Adv.Diff. Eqns.*, volume 2010(2010), 8 pages.

[9] Zhimin He and Weigao Ge, Oscillation of impulsive delay differential equations, *Indian. J. Pure. Appl. Math.*, **31(9)** (2000), 1089-1101.

[10] J.Jiao,L.Chen and L.Li,Asymptotic behavior of solutions of second order nonlinear impulsive differential equations, *J.Math.Anal.Appl.* doi:10.1016/j.jmaa.(2007).

[11] T.Kusano and H.Onose,Oscillation theorems for second order diffeential equations with retarded argument, *Proc.Japan Acad.,*, **50** (1974), 342-346.

[12] V. Lakshmikantham,D. Bainov and Simenov. P.S., *Theoary of Impulsive Differential Equations*, World Scientific, Singapore,1989.

[13] H.Liu and Q.Li,Asymptotic behavior of scond order impulsive differential equations *E.J.Diff.Eqns.,*, **33** (2011), 1-7.

[14] Wu Xiu-Li, Chen si-Yang and Hong Ji, Oscillation of a class of second order nonlinear ODE with impulses, *Appl. Math. Comput.*, **138** (2003), 181-188

[15] Xiaodi Li, Oscillation properties of second order delay differential -difference equations with impulses, *Adv. Appl.Math. Anal.*, **3** (2008), 55-66

[16] Qiong Meng and Jurang Yan, Bounded oscillation for second order non-linear neutral delay differential equations in critical and non-critical cases, *Nonlinear Analysis*, **64** (2006), 7543-1561.

[17] Mingshu Peng, Oscillation caused by impulses, *J. Math. Anal. Appl.*, **255** (2001), 163-176.

[18] Mingshu Peng and Weigao Ge, Oscillation criteria for second order nonlinear differential equations with impulses, *Com.Math. Appl.*, **39** (2000), 217-255.

[19] Mingshu Peng and R.P.Agarwal, Oscillation theorems of second order nonlinear neutral delay differential equations under impulsive perturbations, *Indian. J. Pure. Appl. Math*, **33** (2002), 1017-1029.

[20] Yuri.V. Rogovchenko, Oscillation of a second order nonlinear delay differential equations, *Funk.Ekv.*, **43** (2000), 1-29.

[21] Jurang Yan, Oscillation properties of a second order impulsive delay differential equation, *Com.Math. Appl.* , **47** (2004), 253-258.

[22] Y.Zhang, A.Zhao and J.Yan, Oscillation criteria for impulsive delay differential equations, *J. Math. Anal. Appl.*, **205** (1997), 461-470.

[23] A.Zhao and J.Yan, Existence of positive solutions for delay differential equations with impulses, *J. Math. Anal. Appl.*, **210** (1997), 667-678.