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SOME USEFUL TENSORIAL IDENTIES FOR EXTENDED THERMODYNAMICS

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Abstract: The balance equations for Extended Thermodynamics with an arbitrary number of moments is here considered and a particular method is followed to obtain from them a finite number of equations; this method is based on the suggestions coming from the non relativistic limit of the corresponding relativistic model. The closure of this reduced set of equations is obtained by imposing the entropy principle and the Galilean relativity principle. To this end some tensorial properties are necessary and have already used in literature without proving them. This gap is here filled, by proving them also in the general case of the above mentioned closure method. Also other interesting consequences are outlined.

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1. Introduction

The balance equations for moments in Extended Thermodynamics have the form

$$\partial_t F^{i_1\cdots i_n} + \partial_k G^{ki_1\cdots i_n} = Q^{i_1\cdots i_n}\,, \tag{1}$$

where $F^{i_1\cdots i_n}$ are the moments, $G^{ki_1\cdots i_n}$ are their fluxes and $Q^{i_1\cdots i_n}$ their production terms.

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Now the infinite hierarchy of moments and of their corresponding balance equations has to be dropped. Initially in literature this requirement was realized letting the index n to go from 0 up to a fixed number N and leaving the moments of order greater than N and, sometimes, also some of their components (see [1]-[7] for example). But in [8]-[10] it was proved that the non relativistic limit of the relativistic model suggests another way to obtain a finite number of equations: It depends on two arbitrary numbers N and M such that $M < N$ and $M + N$ is an odd number; after that,

- The highest order for the moments to be considered is $N + M + 1$ (this fact agrees with the corresponding kinetic approach where the highest order for moments to be considered is an even number, and this is motivated by integrability reasons as it can be seen in eq. (29) of [11]).
- The moments for $n = N + 1, \cdots, N + M + 1$ intervene only through their multiple traces, in particular $F^{i_1 \cdots i_{N+M+1-n} l_1 l_1 \cdots l_{n-\frac{N+M+1}{2}} l_{n-\frac{N+M+1}{2}}}$.

In other words, the balance equations to be considered are

$$\begin{aligned} \partial_t F^{i_1 \cdots i_n} + \partial_k G^{k i_1 \cdots i_n} &= Q^{i_1 \cdots i_n} \quad \text{for}\, n = 0 \ \cdots, N \\ \partial_t F_R^{i_1 \cdots i_R} + \partial_k G_R^{k i_1 \cdots i_R} &= Q_R^{i_1 \cdots i_R} \quad \text{for}\, R = 0 \ \cdots, M , \end{aligned} \tag{2}$$

where

$$\begin{aligned} F_R^{i_1 \cdots i_R} &= F^{i_1 \cdots i_R l_1 l_1 \cdots l_{\frac{N+M+1}{2}-R} l_{\frac{N+M+1}{2}-R}} \\ G_R^{k i_1 \cdots i_R} &= G^{k i_1 \cdots i_R l_1 l_1 \cdots l_{\frac{N+M+1}{2}-R} l_{\frac{N+M+1}{2}-R}} \\ Q_R^{i_1 \cdots i_R} &= Q^{i_1 \cdots i_R l_1 l_1 \cdots l_{\frac{N+M+1}{2}-R} l_{\frac{N+M+1}{2}-R}} \end{aligned}$$

Here $F^{i_1 \cdots i_n}$, $F_R^{i_1 \cdots i_R}$, $Q^{i_1 \cdots i_n}$, $Q_R^{i_1 \cdots i_R}$ are symmetric tensors, while $G^{k i_1 \cdots i_n}$ and $G_R^{k i_1 \cdots i_R}$ are symmetric only in the case of ideal gases; in the most general case they are symmetric only with respect to the indexes $i_1 \cdots i_n$ and $i_1 \cdots i_R$. Other considerations [12], [13], lead to think that M and N are two subsequent numbers; this aspect does not affect the present results, so that we prefer to maintain their generality. Moreover, we have

- $Q = 0$, $Q^i = 0$, $Q^{ll} = 0$ so that eqs. $(2)_1$ for $n = 0$, $n = 1$, and the trace of that for $n = 2$, are the conservation laws of mass, momentum and energy, respectively.
- $F^k = G^k$ so that the flux in the conservation law of mass (i.e., eq. $(2)_1$ with $n = 0$) is the independent variable in the conservation law of momentum (i.e., eq. $(2)_1$ for $n = 1$).

The eqs. (2) become closed when the expressions of the fluxes and of the production terms are introduced; restriction on the generality of these expressions is obtained by imposing the entropy principle and the Galilean relativity principle. To this end, some tensorial properties are necessary. In the particular case without eqs. $(2)_2$, these properties are simpler and can be found in ref. [14]. When also eqs. $(2)_2$ are considered (and this is necessary, for physical grounds) they have already been used (see refs. [15]-[16]), but without proving them. This gap is filled in the present paper; in fact, in sect. 2 they are exposed and proved.

In sect. 3 these properties are applied to the moments and their fluxes as appearing in eqs. (2), so obtaining the consequences of the Galilean relativity principle.

In sect.4 they are applied in connection with the entropy principle and attention is focused on a particular partial differential equation which comes out from the above mentioned consequence, but only in the particular case of ideal gases. This equation can be found in literature [17]-[18] (eq. (1.4) and its solution exposed in sect. 2, up to eq. (2.2)) for particular cases of M and N; here we obtain its exact solution, without using Taylor' s expansions, in the general case. Finally, in sect. 5, we compare the present results with a particular decomposition of moments and fluxes into velocity dependent parts and non convective quantities; this decomposition is present in literature [14],[17] so that we cannot omit to consider it.

2. Some Useful Tensorial Properties

The first tensorial function, which we consider, is defined by

$$S^{i_1\cdots i_n}_{j_1\cdots j_s}(\vec{u}) = \begin{pmatrix} n \\ s \end{pmatrix} \delta^{(i_1}_{j_1}\cdots\delta^{i_s}_{j_s}u^{i_{s+1}}\cdots u^{i_n)} \quad \text{with } s \leq n\,, \tag{3}$$

which depends on the vectorial variable $\vec{u}$; moreover, δ^i_j is the Kronecker symbol and round brackets enclosing some indexes denote symmetrization with respect to these indexes. The following properties hold, whose proof is shifted at the end of this section

Property 1. *For every couple of integer numbers (r,s) with $r \leq [\frac{n}{2}]$, $s \leq n$, we have*

$$S^{i_1\cdots i_{n-2r}l_1l_1\cdots l_rl_r}_{j_1\cdots j_s} = \sum_{(p,q)\in S_r} \begin{pmatrix} n-2r \\ s-q-2p \end{pmatrix} 2^q \frac{r!}{p!\,q!\,(r-p-q)!}(u^2)^{r-p-q} \tag{4}$$
$$\cdot u_{(j_1}\cdots u_{j_q}\delta_{j_{q+1}j_{q+2}}\cdots\delta_{j_{q+2p-1}j_{q+2p}}\delta^{(i_1}_{j_{q+2p+1}}\cdots\delta^{i_{s-q-2p}}_{j_s)}u^{i_{s-q-2p+1}}\cdots u^{i_{n-2r})}\,,$$

where S_r is the set of the couples (p,q) of integer numbers p and q such that $p \geq 0$, $q \geq 0$, $p+q \leq r$, $s-n+2r \leq q+2p \leq s$.

Property 2. *For every $s \leq N+M+1-R$, we have*

$$S^{i_1\cdots i_R l_1 l_1 \cdots l_{\frac{N+M+1}{2}-R} l_{\frac{N+M+1}{2}-R}}_{j_1\cdots j_s} = \sum_{(p,q)\in S} \binom{R}{s-q-2p} 2^q \frac{\left(\frac{N+M+1}{2}-R\right)!}{p!\,q!\,\left(\frac{N+M+1}{2}-R-p-q\right)!} (u^2)^{\frac{N+M+1}{2}-R-p-q} \cdot u_{(j_1} \cdots u_{j_q}\delta_{j_{q+1}j_{q+2}}\cdots\delta_{j_{q+2p-1}j_{q+2p}}\delta^{(i_1}_{j_{q+2p+1}}\cdots\delta^{i_{s-q-2p}}_{j_s)} u^{i_{s-q-2p+1}}\cdots u^{i_R)}, \quad (5)$$

where S is the set of the couples (p,q) of integer numbers p and q such that $p \geq 0$, $q \geq 0$, $p+q \leq \frac{N+M+1}{2}-R$, $s-R \leq q+2p \leq s$.

Property 3. *For every $N+1 \leq s \leq N+M+1-R$, we have*

$$S^{i_1\cdots i_R l_1 l_1 \cdots l_{\frac{N+M+1}{2}-R} l_{\frac{N+M+1}{2}-R}}_{j_1\cdots j_s} = X^{i_1\cdots i_R}_{(j_1\cdots j_{N+M+1-s}}\delta_{j_{N+M+2-s}j_{N+M+3-s}}\cdots\delta_{j_{s-1}j_s)}\,, \quad (6)$$

with

$$X^{i_1\cdots i_R}_{j_1\cdots j_{N+M+1-s}} = \sum_{(p,q)\in S} \binom{R}{s-q-2p} 2^q \frac{\left(\frac{N+M+1}{2}-R\right)!}{p!\,q!\,\left(\frac{N+M+1}{2}-R-p-q\right)!}(u^2)^{\frac{N+M+1}{2}-R-p-q} u_{(j_1}\cdots u_{j_q}\delta_{j_{q+1}j_{q+2}}\cdots\delta_{j_{q+N+M+2p-2s}j_{q+N+M+2p-2s+1}}\delta^{(i_1}_{j_{q+N+M+2p-2s+2}} \cdots\delta^{i_{s-q-2p}}_{j_{N+M+1-s)}}u^{i_{s-q-2p+1}}\cdots u^{i_R)}, \quad (7)$$

where S is the set of the couples (p,q) of integer numbers p and q such that $p \geq 0$, $q \geq 0$, $p+q \leq \frac{N+M+1}{2}-R$, $s-R \leq q+2p \leq s$.

Property 4. *We have*

$$\sum_{n=s}^{p} S^{k_1\cdots k_p}_{i_1\cdots i_n}(-\vec{u})S^{i_1\cdots i_n}_{j_1\cdots j_s}(\vec{u}) = \begin{cases} 0 & \text{if} \quad s<p \\ \delta^{(k_1}_{j_1}\cdots\delta^{k_p)}_{j_1} & \text{if} \quad s=p\,.\text{”} \end{cases} \quad (8)$$

Property 5. *We have*

$$\sum_{n=s}^{N+M+1-R} X^{k_1\cdots k_R}_{i_1\cdots i_{N+M+1-n}}(-\vec{u})X^{i_1\cdots i_{N+M+1-n}}_{(j_1\cdots j_{N+M+1-s}}(\vec{u})\delta_{j_{N+M+2-s}j_{N+M+3-s}}\cdots\delta_{j_{s-1}j_s)} = \quad (9)$$

$$= \begin{cases} 0 & \text{if} \quad N+1 \leq s < N+M+1-R \\ \delta^{k_1}_{(j_1}\cdots\delta^{k_R}_{j_R}\delta_{j_{R+1}j_{R+2}}\cdots\delta_{j_{N+M-R}j_{N+M+1-R})} & \text{if} \quad s = N+M+1-R\,.\text{”} \end{cases}$$

Before introducing the other properties, the following Lemma is necessary

Lemma. *For every symmetric tensor* $\phi_{i_1\cdots i_p}$ *we have*

$$\phi_{(i_1\cdots i_p}\delta_{i_{p+1}i_{p+2}}\cdots\delta_{i_{p+2k-1}i_{p+2k})} = 0 \quad \textit{if and only if} \quad \phi_{i_1\cdots i_p} = 0\,.\text{”} \tag{10}$$

We can now continue with the other properties.

Property 6. *We have*

$$\sum_{n=s}^{N+M+1-R} X^{k_1\cdots k_R}_{i_1\cdots i_{N+M+1-n}}(-\vec{u})X^{i_1\cdots i_{N+M+1-n}}_{j_1\cdots j_{N+M+1-s}}(\vec{u}) = \tag{11}$$

$$= \begin{cases} 0 & \textit{if} \quad N+1 \le s < N+M+1-R \\ \delta^{k_1}_{(j_1}\cdots\delta^{k_R}_{j_R)} & \textit{if} \quad s = N+M+1-R\,.\text{”} \end{cases}$$

Property 7. *The following identity holds*

$$S^{i_1\cdots i_{n+1}}_{j_1\cdots j_s} = \begin{cases} S^{i_1\cdots i_n}_{j_1\cdots j_s}u^{i_{n+1}} + S^{i_1\cdots i_n}_{(j_1\cdots j_{s-1}}\delta^{i_{n+1}}_{j_s)} & \textit{for} \quad s = 1, \cdots, n \\ S^{i_1\cdots i_n}u^{i_{n+1}} & \textit{for} \quad s = 0 \\ S^{i_1\cdots i_n}_{(j_1\cdots j_n}\delta^{i_{n+1}}_{j_{n+1})} & \textit{for} \quad s = n+1\,. \end{cases} \tag{12}$$

Property 8. *We have*

$$\delta_{i_{R+1}i_{R+2}}\cdots\delta_{i_{M+N-R}i_{M+N+1-R}}S^{i_1\cdots i_{N+M+2-R}}_{j_1\cdots j_s} = \tag{13}$$

$$= \begin{cases} \delta_{i_{R+1}i_{R+2}}\cdots\delta_{i_{M+N-R}i_{M+N+1-R}}\Big[S^{i_1\cdots i_{N+M+1-R}}_{j_1\cdots j_s}u^{i_{N+M+2-R}} + \\ \qquad + S^{i_1\cdots i_{N+M+1-R}}_{(j_1\cdots j_{s-1}}\delta^{i_{N+M+2-R}}_{j_s)}\Big] \\ \qquad\qquad \textit{for} \quad s = 1, \cdots, N+M+1-R \\ \delta_{i_{R+1}i_{R+2}}\cdots\delta_{i_{M+N-R}i_{M+N+1-R}}S^{i_1\cdots i_{N+M+1-R}}u^{i_{N+M+2-R}} \\ \qquad\qquad \textit{for} \quad s = 0 \\ \delta_{i_{R+1}i_{R+2}}\cdots\delta_{i_{M+N-R}i_{M+N+1-R}}S^{i_1\cdots i_{N+M+1-R}}_{(j_1\cdots j_{N+M+1-R}}\delta^{i_{N+M+2-R}}_{j_{N+M+2-R+1})}, \\ \qquad\qquad \textit{for} \quad s = N+M+2-R\,. \end{cases}$$

Property 9. *The following identity holds*

$$\delta_{i_{R+1}i_{R+2}}\cdots\delta_{i_{M+N-R}i_{M+N+1-R}}S^{i_1\cdots i_{N+M+2-R}}_{j_1\cdots j_s} = \tag{14}$$

$$= \begin{cases} X^{i_1\cdots i_R}_{(j_1\cdots j_{N+M+1-s}}\delta_{j_{N+M+2-s}j_{N+M+3-s}}\cdots\delta_{j_{s-1}j_s)}u^{i_{N+M+2-R}} + \\ \quad + X^{i_1\cdots i_R}_{(j_1\cdots j_{N+M+2-s}}\delta_{j_{N+M+3-s}j_{N+M+4-s}}\cdots\delta_{j_{s-2}j_{s-1}}\delta^{i_{N+M+2-R}}_{j_s)} \\ \qquad\qquad \textit{for} \quad s = N+1, \cdots, N+M+1-R \\ \delta^{i_1}_{(j_1}\cdots\delta^{i_R}_{j_R}\delta_{j_{R+1}j_{R+2}}\cdots\delta_{j_{N+M-R}j_{N+M+1-R}}\delta^{i_{N+M+2-R}}_{j_{N+M+2-R})} \\ \qquad\qquad \textit{for} \quad s = N+M+2-R\,. \end{cases}$$

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Property 10. *The following identity is satisfied*

$$\frac{\partial}{\partial u^j} S^{i_1\cdots i_n}_{j_1\cdots j_s} = \begin{cases} n S^{(i_1\cdots i_{n-1}}_{j_1\cdots j_s}\delta^{i_n)j} & \text{for} \quad s=0,\cdots,n-1 \\ 0 & \text{for} \quad s=n\,. \end{cases} \tag{15}$$

Let us now define the tensor

$$Y^{i_1\cdots i_R}_{j_1\cdots j_s} = \delta_{i_{R+1}i_{R+2}}\cdots\delta_{i_{M+N-R}i_{M+N+1-R}} S^{i_1\cdots i_{N+M+1-R}}_{j_1\cdots j_s} \quad \text{for} \quad s\le N\,. \tag{16}$$

It follows

Property 11. *We have*

$$\frac{\partial}{\partial u^j} Y^{i_1\cdots i_R}_{j_1\cdots j_s} = \left[(N+M+1-2R)\delta^j_{j_R}\delta^{(i_1}_{j_{R+1}} + R\delta^{j(i_1}\delta_{j_R j_{R+1}}\right] Y^{i_2\cdots i_R)j_R j_{R+1}}_{j_1\cdots j_s},$$

for $s=0,\cdots,N$.

$$\frac{\partial}{\partial u^j} X^{i_1\cdots i_R}_{j_1\cdots j_s} = \begin{cases} \left[(N+M+1-2R)\delta^j_{j_R}\delta^{(i_1}_{j_{R+1}} + R\delta^{j(i_1}\delta_{j_R j_{R+1}}\right] X^{i_2\cdots i_R)j_R j_{R+1}}_{j_1\cdots j_s}, \\ \qquad\qquad\qquad\qquad \text{for}\, s=R+1,\cdots,M, \\ 0 \qquad \text{for}\, s=R\,, \end{cases} \tag{17}$$

where it is understood that, for $R=0$, *only the contribute of the first term in square brackets has to be considered; moreover, the contraction with* $\delta^{i_1}_{j_{R+1}}$ *has to be executed and also the subsequent simmetrization, without taking into account the fact that, before of these passages, there is apparently a negative number of indexes of the type i.... More precisely, we have*

$$\frac{\partial}{\partial u^j} Y_{j_1\cdots j_s} = (N+M+1)Y^j_{j_1\cdots j_s} \qquad \text{for}\, s=0,\cdots,N.$$

$$\frac{\partial}{\partial u^j} X_{j_1\cdots j_s} = \begin{cases} (N+M+1)X^j_{j_1\cdots j_s} & \text{for}\, s=1,\cdots,M. \\ 0 & \text{for}\, s=0\,. \end{cases}$$

Property 12. *For every* $s\le N+M+2-R$, *we have*

$$S^{i_1\cdots i_{R+1} l_1 l_1 \cdots l_{\frac{N+M+1}{2}-R} l_{\frac{N+M+1}{2}-R}}_{j_1\cdots j_s}$$

$$= \sum_{(p,q)\in S^*} \binom{R+1}{s-q-2p} 2^q \frac{\left(\frac{N+M+1}{2}-R\right)!}{p!\,q!\,\left(\frac{N+M+1}{2}-R-p-q\right)!}$$

$$\cdot (u^2)^{\frac{N+M+1}{2}-R-p-q} \cdot u_{(j_1}\cdots u_{j_q}\delta_{j_{q+1}j_{q+2}}$$

$$\cdots\delta_{j_{q+2p-1}j_{q+2p}}\delta^{(i_1}_{j_{q+2p+1}}\cdots\delta^{i_{s-q-2p}}_{j_s)} u^{i_{s-q-2p+1}}\cdots u^{i_{R+1})}, \tag{18}$$

where S^* *is the set of the couples* (p,q) *of integer numbers* p *and* q *such that* $p \geq 0$, $q \geq 0$, $p+q \leq \frac{N+M+1}{2} - R$, $s-R-1 \leq q+2p \leq s$.

Property 13. *For every* $N+2 \leq s \leq N+M+2-R$, *we have*

$$S^{i_1 \cdots i_{R+1} l_1 l_1 \cdots l_{\frac{N+M+1}{2}-R} l_{\frac{N+M+1}{2}-R}}_{j_1 \cdots j_s} = Z^{i_1 \cdots i_{R+1}}_{(j_1 \cdots j_{N+M+3-s}} \delta_{j_{N+M+4-s} j_{N+M+5-s}} \cdots \delta_{j_{s-1} j_s)} \,, \quad (19)$$

with

$$Z^{i_1 \cdots i_{R+1}}_{j_1 \cdots j_{N+M+3-s}} = \sum_{(p,q) \in S^*} \binom{R+1}{s-q-2p} 2^q \frac{\left(\frac{N+M+1}{2} - R\right)!}{p!\, q!\, \left(\frac{N+M+1}{2} - R - p - q\right)!} (u^2)^{\frac{N+M+1}{2} - R - p - q}$$
$$u_{(j_1} \cdots u_{j_q} \delta_{j_{q+1} j_{q+2}} \cdots \delta_{j_{q+N+M+2+2p-2s} j_{q+N+M+2p-2s+3}} \delta^{(i_1}_{j_{q+N+M+2p-2s+4}} \cdots \delta^{i_{s-q-2p}}_{j_{N+M+3-s})} u^{i_{s-q-2p+1}} \cdots u^{i_{R+1})}, \quad (20)$$

where S^* *is the same set defined in eq.* $(18)_2$.

Property 14. *For every* $N+2 \leq s \leq N+M+2-R$ *we have*

$$Z^{i_1 \cdots i_{R+1}}_{j_1 \cdots j_{N+M+3-s}} = \begin{cases} X^{i_1 \cdots i_R}_{(j_1 \cdots j_{N+M+1-s}} \delta_{j_{N+M+2-s} j_{N+M+3-s})} u^{i_{R+1}} + X^{i_1 \cdots i_R}_{(j_1 \cdots j_{N+M+2-s}} \delta^{i_{R+1}}_{j_{N+M+3-s})} & \\ \qquad \text{for} \quad s = N+2, \cdots, N+M+1-R \\ \delta^{i_1}_{(j_1} \cdots \delta^{i_{R+1}}_{j_{R+1})} \qquad \text{for} \quad s = N+M+2-R \,. \end{cases} \quad (21)$$

Property 15. *For every enumerable family* $K^{j_1 \cdots j_s}$ *of symmetric tensors, with* $0 \leq s \leq n+1$, *we have*

$$\sum_{s=0}^{n+1} S^{i_1 \cdots i_{n+1}}_{j_1 \cdots j_s} K^{j_1 \cdots j_s} - u^{i_{n+1}} \sum_{s=0}^{n} S^{i_1 \cdots i_n}_{j_1 \cdots j_s} K^{j_1 \cdots j_s} = \sum_{S=0}^{n} S^{i_1 \cdots i_n}_{j_1 \cdots j_S} K^{j_1 \cdots j_S i_{n+1}} \,. \quad (22)$$

Property 16. *For every* $R \geq 1$, *we have*

$$Y^{i_1 \cdots i_{R-1} ll}_{j_1 \cdots j_N} = T^{i_1 \cdots i_{R-1}}_{(j_1 \cdots j_{M-1}} \delta_{j_M j_{M+1}} \cdots \delta_{j_{N-1} j_N)} \,, \quad (23)$$

with

$$T^{i_1 \cdots i_{R-1}}_{j_1 \cdots j_{M-1}}$$

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$$= \sum_{(p,q)\in\tilde{S}} \begin{pmatrix} R-1 \\ N-q-2p \end{pmatrix} 2^q \frac{\left(\frac{N+M+1}{2}-R\right)!}{p!\,q!\,\left(\frac{N+M+1}{2}-R-p-q\right)!}(u^2)^{\frac{N+M+1}{2}-R-p-q}$$
$$u_{(j_1}\cdots u_{j_q}\delta_{j_{q+1}j_{q+2}}\cdots\delta_{j_{q+2p-N+M-2}j_{q+2p-N+M-1}}\delta^{(i_1}_{j_{q+2p-N+M}}$$
$$\cdots\delta^{i_{N-q-2p}}_{j_{M-1})}u^{i_{N-q-2p+1}}\cdots u^{i_{R-1})},$$

where $\tilde{S}$ is the set of the couples (p,q) of integer numbers p and q such that $p \geq \frac{N-M+1}{2}$, $q \geq 0$, $p+q \leq \frac{N+M+1}{2}-R$, $N-R+1 \leq q+2p \leq N$.

Property 17. *For every $R \geq 2$, we have*

$$X^{i_1\cdots i_{R-1}ll}_{j_1\cdots j_S} = \eta^{i_1\cdots i_{R-1}}_{(j_1\cdots j_{S-2}}\delta_{j_{S-1}j_S)}\,, \tag{24}$$

with

$$\eta^{i_1\cdots i_{R-1}}_{j_1\cdots j_{S-2}}$$
$$= \sum_{(p,q)\in S^*} \begin{pmatrix} R-1 \\ N+M+1-S-q-2p \end{pmatrix} 2^q \frac{\left(\frac{N+M+1}{2}-R\right)!}{p!\,q!\,\left(\frac{N+M+1}{2}-R-p-q\right)!}$$
$$(u^2)^{\frac{N+M+1}{2}-R-p-q}$$
$$u_{(j_1}\cdots u_{j_q}\delta_{j_{q+1}j_{q+2}}\cdots\delta_{j_{q+2p+2S-N-M-4}j_{q+2p+2S-N-M-3}}\delta^{(i_1}_{j_{q+2p+2S-N-M-2}}$$
$$\cdots\delta^{i_{N+M+1-S-q-2p}}_{j_{S-2})}u^{i_{N+M+2-S-q-2p}}\cdots u^{i_{R-1})},$$

where S^* is the set of the couples (p,q) of integer numbers p and q such that $p \geq 0$, $q \geq 0$, $p+q \leq \frac{N+M+1}{2}-R$, $N+M+2-S-R \leq q+2p \leq N+M+1-S$.

Property 18. *A consequence of the last property is*

$$\eta^{i_1\cdots i_{R-1}}_{j_1\cdots j_{R-1}} = \delta^{i_1}_{(j_1}\cdots\delta^{i_{R-1}}_{j_{R-1})}\,. \tag{25}$$

Property 19. *For $s = 1, \cdots, N$ the following identity holds*

$$Y^{ki_1\cdots i_{R-1}}_{j_1\cdots j_s} = u^k Y^{i_1\cdots i_{R-1}ll}_{j_1\cdots j_s} + Y^{i_1\cdots i_{R-1}ll}_{(j_1\cdots j_{s-1}}\delta^k_{j_s)}\,. \tag{26}$$

Property 20. *The following identities hold*

$$Y^{ki_1\cdots i_{R-1}} = u^k u^{i_1}\cdots u^{i_{R-1}}(u^2)^{\frac{N+M+1}{2}-R}, \quad Y^{i_1\cdots i_{R-1}ll} = u^{i_1}\cdots u^{i_{R-1}}(u^2)^{\frac{N+M+1}{2}-R}. \tag{27}$$

Property 21. *For $S = R+1, \cdots, M$ we have*

$$X^{ki_1\cdots i_{R-1}}_{(j_1\cdots j_S}\delta_{j_{S+1}j_{S+2}}\cdots\delta_{j_{N+M-S}j_{N+M+1-S})} = u^k S^{i_1\cdots i_{R-1}l_1l_1\cdots l_{\frac{N+M+1}{2}-R}l_{\frac{N+M+1}{2}-R}}_{j_1\cdots j_{N+M+1-S}} + S^{i_1\cdots i_{R-1}l_1l_1\cdots l_{\frac{N+M+1}{2}-R}l_{\frac{N+M+1}{2}-R}}_{(j_1\cdots j_{N+M-S}}\delta^k_{j_{N+M+1-S})}. \quad (28)$$

Property 22. *The following identity holds*

$$X^{ki_1\cdots i_{R-1}}_{j_1\cdots j_M} = T^{i_1\cdots i_{R-1}}_{(j_1\cdots j_{M-1}}\delta^k_{j_M)} + u^k\eta^{i_1\cdots i_{R-1}}_{(j_1\cdots j_{M-2}}\delta_{j_{M-1}j_M)}\,. \quad (29)$$

Property 23. *For $S = R+1, \cdots, M-1$ we have that*

$$X^{ki_1\cdots i_{R-1}}_{j_1\cdots j_S} = \eta^{i_1\cdots i_{R-1}}_{(j_1\cdots j_{S-1}}\delta^k_{j_S)} + u^k\eta^{i_1\cdots i_{R-1}}_{(j_1\cdots j_{S-2}}\delta_{j_{S-1}j_S)}\,. \quad (30)$$

Property 24. *For $S = R+1, \cdots, M+1$ we have that*

$$Z^{i_1\cdots i_R ll}_{j_1\cdots j_{S+1}} = X^{i_1\cdots i_R}_{(j_1\cdots j_{S-1}}\delta_{j_S j_{S+1})}\,, \quad (31)$$

Property 25. *We have*

$$u^k Y^{i_1\cdots i_R}_{j_1\cdots j_s} + Y^{i_1\cdots i_R}_{(j_1\cdots j_{s-1}}\delta^k_{j_s)} = S^{ki_1\cdots i_R l_1l_1\cdots l_{\frac{N+M+1}{2}-R}l_{\frac{N+M+1}{2}-R}}_{j_1\cdots j_s} \quad \textit{for} \quad s = 1, \cdots, N, \quad (32)$$

$$u^k X^{i_1\cdots i_R}_{(j_1\cdots j_M}\delta_{j_{M+1}j_{M+2}}\cdots\delta_{j_N j_{N+1})} + Y^{i_1\cdots i_R}_{(j_1\cdots j_N}\delta^k_{j_{N+1})} = S^{ki_1\cdots i_R l_1l_1\cdots l_{\frac{N+M+1}{2}-R}l_{\frac{N+M+1}{2}-R}}_{j_1\cdots j_{N+1}}.$$

Property 26. *The following identity holds*

$$u^k X^{i_1\cdots i_R}_{(j_1\cdots j_{S-1}}\delta_{j_S j_{S+1})} + X^{i_1\cdots i_R}_{(j_1\cdots j_S}\delta^k_{j_{S+1})} = Z^{i_1\cdots i_R k}_{j_1\cdots j_{S+1}} \quad \textit{for} \quad s = R+1, \cdots, M \quad (33)$$

We report now the proofs of these properties. A reader not interested in the details may overcome them and go on in the next section.

Proof of Property 1. Let us prove it with the iterative procedure with respect to the integer r. The property holds for $r = 0$ because, in this case, it is nothing more than eq. (3). Let us suppose that it holds up to the index r and prove that it holds also with $r+1$ instead of r. Let us contract eq. (4) with $\delta_{i_{n-2r-1} i_{n-2r}}$ by explicitating in the first passage the symmetrization in (4) with respect to the indexes $i_{n-2r-1} i_{n-2r}$, that is

$$\delta_{i_{n-2r-1} i_{n-2r}} S^{i_1 \cdots i_{n-2r} l_1 l_1 \cdots l_r l_r}_{j_1 \cdots j_s} = \delta_{i_{n-2r-1} i_{n-2r}} \sum_{(p,q) \in S_r} \binom{n-2r}{s-q-2p} 2^q \frac{r!}{p!\, q!\, (r-p-q)!} (u^2)^{r-p-q} \cdot$$
$$\cdot \Big[\frac{s-q-2p}{n-2r} u_{(j_1} \cdots u_{j_q} \delta_{j_{q+1} j_{q+2}} \cdots \delta_{j_{q+2p-1} j_{q+2p}} \delta^{i_{n-2r}}_{j_{q+2p+1}} \delta^{(i_1}_{j_{q+2p+2}} \cdots \delta^{i_{s-q-2p-1}}_{j_s)} u^{i_{s-q-2p}} \cdots u^{i_{n-2r-1})} +$$
$$+ \frac{n-2r-s+q+2p}{n-2r} u_{(j_1} \cdots u_{j_q} \delta_{j_{q+1} j_{q+2}} \cdots \delta_{j_{q+2p-1} j_{q+2p}} \cdot$$

$$\cdot \delta^{(i_1}_{j_{q+2p+1}} \cdots \delta^{i_{s-q-2p}}_{j_s)} u^{i_{s-q-2p+1}} \cdots u^{i_{n-2r-1})} u^{i_{n-2r}} \Big] = \tag{34}$$
$$= \delta_{i_{n-2r-1} i_{n-2r}} \sum_{(p,q) \in S_r} \binom{n-2r}{s-q-2p} 2^q \frac{r!}{p!\, q!\, (r-p-q)!} (u^2)^{r-p-q} \Big[\frac{s-q-2p}{n-2r} \frac{s-q-2p-1}{n-2r-1} \cdot$$
$$\cdot u_{(j_1} \cdots u_{j_q} \delta_{j_{q+1} j_{q+2}} \cdots \delta_{j_{q+2p-1} j_{q+2p}} \delta^{i_{n-2r}}_{j_{q+2p+1}} \delta^{i_{n-2r-1}}_{j_{q+2p+2}} \delta^{(i_1}_{j_{q+2p+3}} \cdots \delta^{i_{s-q-2p-2}}_{j_s)} u^{i_{s-q-2p-1}} \cdots u^{i_{n-2r-2})} +$$

$$+2 \frac{s-q-2p}{n-2r} \frac{n-2r-s+q+2p}{n-2r-1} u_{(j_1} \cdots u_{j_q} \delta_{j_{q+1} j_{q+2}} \cdots \delta_{j_{q+2p-1} j_{q+2p}} \delta^{i_{n-2r}}_{j_{q+2p+1}} \delta^{(i_1}_{j_{q+2p+2}} \cdots \delta^{i_{s-q-2p-1}}_{j_s)} \cdot$$
$$\cdot u^{i_{s-q-2p}} \cdots u^{i_{n-2r-2})} u^{i_{n-2r-1}} \Big] \text{ (see the motivation in the following note *) } +$$
$$+ \frac{n-2r-s+q+2p}{n-2r} \frac{n-2r-s+q+2p-1}{n-2r-1} u_{(j_1} \cdots u_{j_q} \delta_{j_{q+1} j_{q+2}} \cdots \delta_{j_{q+2p-1} j_{q+2p}} \delta^{(i_1}_{j_{q+2p+1}} \cdots \delta^{i_{s-q-2p}}_{j_s)} \cdot$$

$$\cdot u^{i_{s-q-2p+1}} \cdots u^{i_{n-2r-2})} u^{i_{n-2r-1}} u^{i_{n-2r}} \Big] =$$
$$= \sum_{(p,q) \in S_r} \binom{n-2r}{s-q-2p} 2^q \frac{r!}{p!\, q!\, (r-p-q)!} (u^2)^{r-p-q} \Big[\frac{s-q-2p}{n-2r} \frac{s-q-2p-1}{n-2r-1} \cdot$$

$$\cdot u_{(j_1} \cdots u_{j_q} \delta_{j_{q+1} j_{q+2}} \cdots \delta_{j_{q+2p+1} j_{q+2p+2}} \delta^{(i_1}_{j_{q+2p+3}} \cdots \delta^{i_{s-q-2p-2}}_{j_s)} u^{i_{s-q-2p-1}} \cdots u^{i_{n-2r-2})} +$$
$$+2 \frac{s-q-2p}{n-2r} \frac{n-2r-s+q+2p}{n-2r-1} u_{(j_1} \cdots u_{j_{q+1}} \delta_{j_{q+2} j_{q+3}} \cdots \delta_{j_{q+2p} j_{q+2p+1}} \delta^{(i_1}_{j_{q+2p+2}} \cdots \delta^{i_{s-q-2p-1}}_{j_s)} \cdot$$
$$\cdot u^{i_{s-q-2p}} \cdots u^{i_{n-2r-2})} \Big] +$$

$$+ \frac{n-2r-s+q+2p}{n-2r} \frac{n-2r-s+q+2p-1}{n-2r-1} u_{(j_1} \cdots u_{j_q} \delta_{j_{q+1} j_{q+2}} \cdots \delta_{j_{q+2p-1} j_{q+2p}} \delta^{(i_1}_{j_{q+2p+1}} \cdots \delta^{i_{s-q-2p}}_{j_s)} \cdot$$
$$\cdot u^{i_{s-q-2p+1}} \cdots u^{i_{n-2r-2})} u^2 \Big] .$$

Note *: In this passage we have taken into account that, by exchanging the indexes i_{n-2r} and i_{n-2r-1}, we obtain the same result thanks to the contraction with $\delta_{i_{n-2r-1} i_{n-2r}}$.

Now,

- For the first one of these terms, we can omit the values with $q+2p=s$ and with $q+2p=s-1$; this is equivalent to replace $\sum_{(p,q)\in S_r}$ with $\sum_{(p,q)\in S_r^*}$ where S_r^* is the set of the couples (p,q) of integer numbers p and q such that $p \geq 0$, $q \geq 0$, $p+q \leq r$, $s-n+2r \leq q+2p \leq s-2$.

After that, we substitute

$$\begin{pmatrix} n-2r \\ s-q-2p \end{pmatrix} \frac{1}{p!} \frac{s-q-2p}{n-2r} \frac{s-q-2p-1}{n-2r-1}$$

with

$$\begin{pmatrix} n-2r-2 \\ s-q-2p-2 \end{pmatrix} \frac{p+1}{(p+1)!}.$$

Finally, we change index according to $p = P - 1$ and note that $\sum_{(p,q)\in S_r^*}$ becomes $\sum_{(P,q)\in S_{r+1}}$; it is true that in this case we enclose the new term with $P = 0$ but this is zero, thanks to the factor $p + 1$.

- For the second term in (34), we can omit the values with $q + 2p = s$ and with $q + 2p = s + 2r - n$; this is equivalent to replace $\sum_{(p,q)\in S_r}$ with $\sum_{(p,q)\in S_r^{**}}$ where S_r^{**} is the set of the couples (p, q) of integer numbers p and q such that $p \geq 0$, $q \geq 0$, $p + q \leq r$, $s - n + 2r + 1 \leq q + 2p \leq s - 1$.

  After that, we substitute

$$\begin{pmatrix} n-2r \\ s-q-2p \end{pmatrix} \frac{1}{q!} \frac{s-q-2p}{n-2r} \frac{n-2r-s+q+2p}{n-2r-1}$$

  with

$$\begin{pmatrix} n-2r-2 \\ s-q-2p-1 \end{pmatrix} \frac{q+1}{(q+1)!}.$$

  Finally, we change index according to $q = Q - 1$ and note that $\sum_{(p,q)\in S_r^{**}}$ becomes $\sum_{(p,Q)\in S_{r+1}}$; it is true that in this case we enclose the new term with $Q = 0$ but this is zero, thanks to the factor $q + 1$.

- For the third term in (34), we can omit the values with $q + 2p = -n + 2r + s$ and with $q + 2p = -n + 2r + s + 1$; this is equivalent to replace $\sum_{(p,q)\in S_r}$ with $\sum_{(p,q)\in S_r^{***}}$ where S_r^{***} is the set of the couples (p, q) of integer numbers p and q such that $p \geq 0$, $q \geq 0$, $p + q \leq r$, $s - n + 2r + 2 \leq q + 2p \leq s$.

  After that, we substitute $\begin{pmatrix} n-2r \\ s-q-2p \end{pmatrix} \frac{1}{(r-p-q)!} \frac{n-2r-s+q+2p}{n-2r} \frac{n-2r-s+q+2p-1}{n-2r-1}$ with $\begin{pmatrix} n-2r-2 \\ s-q-2p \end{pmatrix} \frac{r+1-p-q}{(r+1-p-q)!}$. Finally, we note that $\sum_{(p,q)\in S_r^{***}}$ can be expressed as $\sum_{(p,q)\in S_{r+1}}$; it is true that in this case we enclose the new term with $p+q = r + 1$ but this is zero, thanks to the factor $r + 1 - p - q$.

With the above described three changes, the expression (34) becomes

$$\sum_{(p,q)\in S_{r+1}} \begin{pmatrix} n-2r-2 \\ s-q-2p \end{pmatrix} 2^q \frac{p\, r!}{p!\, q!\, (r-p+1-q)!} (u^2)^{r-p+1-q}.$$

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$$\cdot u_{(j_1}\cdots u_{j_q}\delta_{j_{q+1}j_{q+2}}\cdots\delta_{j_{q+2p-1}j_{q+2p}}\delta^{(i_1}_{j_{q+2p+1}}\cdots\delta^{i_{s-q-2p}}_{j_s)}u^{i_{s-q-2p+1}}\cdots u^{i_{n-2r-2})}+$$

$$+\sum_{(p,Q)\in S_{r+1}}2^Q\begin{pmatrix} n-2r-2\\ s-Q-2p\end{pmatrix}\frac{Q\,r!}{p!\,Q!\,(r-p-Q+1)!}(u^2)^{r-p-Q+1}\cdot$$

$$\cdot u_{(j_1}\cdots u_{j_Q}\delta_{j_{Q+1}j_{Q+2}}\cdots\delta_{j_{Q+2p-1}j_{Q+2p}}\delta^{(i_1}_{j_{Q+2p+1}}\cdots\delta^{i_{s-Q-2p}}_{j_s)}u^{i_{s-Q-2p+1}}\cdots u^{i_{n-2r-2})}+$$

$$+\sum_{(p,q)\in S_{r+1}}2^q\begin{pmatrix} n-2r-2\\ s-q-2p\end{pmatrix}\frac{(r+1-p-q)\,r!}{p!\,q!\,(r+1-p-q)!}(u^2)^{r-p-q+1}\cdot$$

$$\cdot u_{(j_1}\cdots u_{j_q}\delta_{j_{q+1}j_{q+2}}\cdots\delta_{j_{q+2p-1}j_{q+2p}}\delta^{(i_1}_{j_{q+2p+1}}\cdots\delta^{i_{s-q-2p}}_{j_s)}u^{i_{s-q-2p+1}}\cdots u^{i_{n-2r-2})}$$

and this is composed by three terms similar one to the other; the sum is exactly eq. (4) with $r+1$ instead of r, because $(p)+(q)+(r+1-p-q)=r+1$. This completes our proof.

Proof of Property 2. It is just eq. (4) with $n=N+M+1-R$ and $r=\frac{N+M+1}{2}-R$. The condition $s\leq N+M+1-R$ is a consequence of that for which (4) holds, that is $s\leq n$. The other condition $r\leq[\frac{n}{2}]$ is surely satisfied.

Proof of Property 3. It is just another way to write the Property 2, that is eq. (5). The condition $s\leq N+M+1-R$ already appears in (5). The other condition $s\geq N+1$ is important because has the consequence $2s-(N+M+1)\geq N+1-M$; but $N+1-M\geq 0$, so that we are assured that in eq. (6) there is not a negative number of symbols $\delta_{..}$. Also in eq. (7) there is not a negative number of symbols $\delta_{..}$; in fact, from the definition of the set S reported after eq. (7), it follows $s-R\leq q+2p=(q+p)+p\leq\frac{N+M+1}{2}-R+p$ where the property $p+q\leq\frac{N+M+1}{2}-R$ has been used; it follows $p\geq s-\frac{N+M+1}{2}$, or $2p\geq 2s-(N+M+1)$.

Proof of Property 4. Thanks to eq. (3), we have

$$\sum_{n=s}^{p}S^{k_1\cdots k_p}_{i_1\cdots i_n}(-\vec{u})S^{i_1\cdots i_n}_{j_1\cdots j_s}(\vec{u})=$$

$$\sum_{n=s}^{p}\begin{pmatrix} p\\ n\end{pmatrix}\delta^{(k_1}_{i_1}\cdots\delta^{k_n}_{i_n}u^{k_{n+1}}\cdots u^{k_p)}(-1)^{p-n}\begin{pmatrix} n\\ s\end{pmatrix}\delta^{(i_1}_{j_1}\cdots\delta^{i_s}_{j_s}u^{i_{s+1}}\cdots u^{i_n)}.$$

Now we eliminate in the second factor the symmetrization with respect to $i_1\cdots i_n$; this is possible thanks to the contraction with the first factor which is symmetric with respect to these indexes. So our expression becomes

$$\sum_{n=s}^{p}\begin{pmatrix} p\\ n\end{pmatrix}\begin{pmatrix} n\\ s\end{pmatrix}(-1)^{p-n}\delta^{(k_1}_{j_1}\cdots\delta^{k_s}_{j_s}u^{k_{s+1}}\cdots u^{k_p)}=$$

$$=\left(\sum_{n=s}^{p}\frac{p!}{(p-n)!\,s!\,(n-s)!}(-1)^{p-n}\right)\delta^{(k_1}_{j_1}\cdots\delta^{k_s}_{j_s}u^{k_{s+1}}\cdots u^{k_p)}=$$

$$= \left(\sum_{\eta=0}^{p-s} \binom{p-s}{\eta} (-1)^{\eta} \right) \binom{p}{s} (-1)^{p-s} \delta_{j_1}^{(k_1} \cdots \delta_{j_s}^{k_s} u^{k_{s+1}} \cdots u^{k_p)} \,, \quad (35)$$

where in the last passage we have changed index according to $n = s + \eta$. Now, if $s < p$, we have $\sum_{\eta=0}^{p-s} \binom{p-s}{\eta} (-1)^{\eta} = [(-1)+1]^{p-s} = 0$ (for the Newtonian Binomial Rule) so that the expression (35) is zero. Instead of this, if $s = p$, the expression (35) is $\delta_{j_1}^{(k_1} \cdots \delta_{j_p}^{k_p)}$ and this completes the proof of (8).

Proof of Property 5. Let us write eq. (8) in the case $s \geq N+1$, with $p = N+M+1-R$; after that, we contract it with $\delta_{k_{R+1}k_{R+2}} \cdots \delta_{k_{N+M-R}k_{N+M+1-R}}$ and use eq. (6). In this way we obtain

$$\sum_{n=s}^{N+M+1-R} X_{i_1 \cdots i_{N+M+1-n}}^{k_1 \cdots k_R}(-\vec{u}) S_{j_1 \cdots j_s}^{i_1 \cdots i_n}(\vec{u}) \delta_{i_{N+M+2-n} i_{N+M+3-n}} \cdots \delta_{i_{n-1} i_n}$$
$$= \begin{cases} 0 & \text{if} \quad s < N+M+1-R \\ \psi_{j_1 \cdots j_{N+M+1-R}}^{k_1 \cdots k_R} & \text{if} \quad s = N+M+1-R \,, \end{cases} \quad (36)$$

where

$$\psi_{j_1 \cdots j_{N+M+1-R}}^{k_1 \cdots k_R} = \delta_{j_1}^{(k_1} \cdots \delta_{j_{N+M+1-R}}^{k_{N+M+1-R})} \delta_{k_{R+1}k_{R+2}} \cdots \delta_{k_{N+M-R}k_{N+M+1-R}}$$
$$= \delta_{(j_1}^{k_1} \cdots \delta_{j_{N+M+1-R})}^{k_{N+M+1-R}} \delta_{k_{R+1}k_{R+2}} \cdots \delta_{k_{N+M-R}k_{N+M+1-R}}$$
$$= \delta_{(j_1}^{k_1} \cdots \delta_{j_R}^{k_R} \delta_{j_{R+1}j_{R+2}} \cdots \delta_{j_{N+M-R}j_{N+M+1-R})} \,.$$

Now, for the left hand side of eq. (36) we apply again eq. (6) so obtaining eq. (9).

Proof of the Lemma. It is obvious that $\phi_{(i_1 \cdots i_p} \delta_{i_{p+1} i_{p+2}} \cdots \delta_{i_{p+2k-1} i_{p+2k})} = 0$ is a consequence of $\phi_{i_1 \cdots i_p} = 0$.

Let us prove the vice versa trough the iterative procedure with respect to k. It surely holds when $k = 0$. Let us suppose that it holds up to a given value of k and prove it for the case with $k+1$ instead of k. Let us write $(10)_1$ with $k+1$ instead of k and contract it with $\delta^{i_{p+2k+1} i_{p+2k+2}}$; we obtain

$$0 = \phi_{(i_1 \cdots i_p} \delta_{i_{p+1} i_{p+2}} \cdots \delta_{i_{p+2k+1} i_{p+2k+2})} \delta^{i_{p+2k+1} i_{p+2k+2}} =$$
$$= \frac{1}{p+2k+2} \Big[(2k+2) \phi_{(i_1 \cdots i_p} \delta_{i_{p+1} i_{p+2}} \cdots \delta_{i_{p+2k+1}) i_{p+2k+2}} +$$
$$+ p \phi_{i_{p+2k+2} (i_1 \cdots i_{p-1}} \delta_{i_p i_{p+1}} \cdots \delta_{i_{p+2k} i_{p+2k+1})} \Big] \delta^{i_{p+2k+1} i_{p+2k+2}} =$$
$$= \frac{1}{(p+2k+2)(p+2k+1)} \big[(2k+2)(2k+3+2p) \phi_{(i_1 \cdots i_p} +$$
$$+ p(p-1) \phi_{ll(i_1 \cdots i_{p-2}} \delta_{i_{p-1} i_p} \big] \delta_{i_{p+1} i_{p+2}} \cdots \delta_{i_{p+2k-1} i_{p+2k})} \,.$$

For the iterative hypothesis applied to the tensor in square brackets instead of $\phi_{i_1\cdots i_p}$, it follows that

$$(2k+2)(2k+3+2p)\phi_{i_1\cdots i_p} + p(p-1)\phi_{ll(i_1\cdots i_{p-2}}\delta_{i_{p-1}i_p)} = 0\,. \tag{37}$$

Obviously, if $p=0$ or $p=1$, from (37) we deduce $\phi_{i_1\cdots i_p}=0$; let us prove now that this result holds also for $p\geq 2$. To this end, let us distinguish two cases.

- If p is even, let us contract eq. (37) with $\delta^{i_1 i_2}\cdots\delta^{i_{p-1}i_p}$ so obtaining

$$\begin{aligned}
0 &= (2k+2)(2k+3+2p)\phi_{l_1 l_1\cdots l_{\frac{p}{2}} l_{\frac{p}{2}}} + p(p-1)\phi_{lli_1\cdots i_{p-2}}\delta_{i_{p-1}i_p}\delta^{(i_1 i_2}\cdots\delta^{i_{p-1}i_p)} = \\
&= (2k+2)(2k+3+2p)\phi_{l_1 l_1\cdots l_{\frac{p}{2}} l_{\frac{p}{2}}} + p(p-1)\phi_{lli_1\cdots i_{p-2}}\delta_{i_{p-1}i_p}\delta^{(i_1 i_2}\cdots\delta^{i_{p-1})i_p} = \\
&= (2k+2)(2k+3+2p)\phi_{l_1 l_1\cdots l_{\frac{p}{2}} l_{\frac{p}{2}}} + p\phi_{lli_1\cdots i_{p-2}}\delta_{i_{p-1}i_p}\Big[\delta^{(i_1 i_2}\cdots\delta^{i_{p-3}i_{p-2})}\delta^{i_{p-1}i_p} + \\
&\qquad +(p-2)\delta^{i_{p-1}(i_1}\cdots\delta^{i_{p-2})i_p}\Big] = \\
&= [(2k+2)(2k+3+2p)+p(p+1)]\phi_{l_1 l_1\cdots l_{\frac{p}{2}} l_{\frac{p}{2}}} \quad \text{from which it follows}
\end{aligned}$$

$$\phi_{l_1 l_1\cdots l_{\frac{p}{2}} l_{\frac{p}{2}}} = 0\,. \tag{38}$$

- If p is odd, let us contract eq. (37) with $\delta^{i_2 i_3}\cdots\delta^{i_{p-1}i_p}$ so obtaining

$$\begin{aligned}
0 &= (2k+2)(2k+3+2p)\phi_{i_1 l_1 l_1\cdots l_{\frac{p-1}{2}} l_{\frac{p-1}{2}}} + [(p-1)(p-2)\phi_{lli_1\cdots i_{p-2}}\delta_{i_{p-1}i_p} \\
&+2(p-1)\phi_{lli_2\cdots i_{p-1}}\delta_{i_p i_1}]\delta^{(i_2 i_3}\cdots\delta^{i_{p-1}i_p)} = (2k+2)(2k+3+2p)\phi_{i_1 l_1 l_1\cdots l_{\frac{p-1}{2}} l_{\frac{p-1}{2}}} \\
&+[(p-1)(p-2)\phi_{lli_1\cdots i_{p-2}}\delta_{i_{p-1}i_p} \\
&+2(p-1)\phi_{lli_2\cdots i_{p-1}}\delta_{i_p i_1}]\delta^{(i_2 i_3}\cdots\delta^{i_{p-1})i_p} \\
&= (2k+2)(2k+3+2p)\phi_{i_1 l_1 l_1\cdots l_{\frac{p-1}{2}} l_{\frac{p-1}{2}}} \\
&+(p-2)\phi_{lli_1\cdots i_{p-2}}\delta_{i_{p-1}i_p}\Big[\delta^{(i_2 i_3}\cdots\delta^{i_{p-3}i_{p-2})}\delta^{i_{p-1}i_p} + \\
&\quad +(p-2)\delta^{i_{p-1}(i_2}\cdots\delta^{i_{p-2})i_p}\Big] \\
&+2(p-1)\phi_{i_1 l_1 l_1\cdots l_{\frac{p-1}{2}} l_{\frac{p-1}{2}}} = \\
&= [(2k+2)(2k+3+2p)+(p-2)(p+1)+2(p-1)]\phi_{i_1 l_1 l_1\cdots l_{\frac{p-1}{2}} l_{\frac{p-1}{2}}}
\end{aligned}$$

from which it follows $$\phi_{i_1 l_1 l_1\cdots l_{\frac{p-1}{2}} l_{\frac{p-1}{2}}} = 0\,. \tag{39}$$

Let us prove now that

$$\forall\, R: \quad 0\leq R\leq \left[\frac{p}{2}\right], \quad \text{we have} \quad \phi_{i_1\cdots i_{p-2R} l_1 l_1\cdots l_R l_R} = 0\,. \tag{40}$$

Also in this case, we use the iterative procedure (which is inside the previous one) with respect to R. Eq. (40) holds for $R=\left[\frac{p}{2}\right]$ because in this case it is nothing else than eq. (38) if p is even, and eq. (39) if p is odd.

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Let us suppose that (40) holds up to the index R and let us prove that it holds also with $R-1$ instead of R.

To this end, let us contract eq. (37) with $\delta^{i_{p+1-2r}i_{p+2-2r}} \cdots \delta^{i_{p-1}i_p}$ where $r = R-1$; in this way we obtain

$$\begin{aligned}
&0 = (2k+2)(2k+3+2p)\phi_{i_1\cdots i_{p-2r}l_1l_1\cdots l_rl_r} + \\
&+p(p-1)\phi_{llj_1\cdots j_{p-2}}\delta_{j_{p-1}j_p}\delta^{(j_1}_{i_1}\cdots\delta^{j_{p-2}}_{i_{p-2}}\delta^{j_{p-1}}_{i_{p-1}}\delta^{j_p)}_{i_p}\delta^{i_{p+1-2r}i_{p+2-2r}}\cdots\delta^{i_{p-1}i_p} = \\
&= (2k+2)(2k+3+2p)\phi_{i_1\cdots i_{p-2r}l_1l_1\cdots l_rl_r} + \\
&+p(p-1)\phi_{llj_1\cdots j_{p-2}}\delta_{j_{p-1}j_p}\delta^{(j_1}_{i_1}\cdots\delta^{j_{p-2r}}_{i_{p-2r}}\delta^{j_{p+1-2r}j_{p+2-2r}}\cdots\delta^{j_{p-1}j_p)} = \\
&= (2k+2)(2k+3+2p)\phi_{i_1\cdots i_{p-2r}l_1l_1\cdots l_rl_r} + \\
&+(p-1)\phi_{llj_1\cdots j_{p-2}}\delta_{j_{p-1}j_p}\Big[(p-2r)\delta^{j_p}_{(i_1}\delta^{(j_1}_{i_2}\cdots\delta^{j_{p-2r-1}}_{i_{p-2r})}\delta^{j_{p-2r}j_{p+1-2r}}\cdots\delta^{j_{p-2}j_{p-1})} + \\
&+2r\delta^{(j_1}_{i_1}\cdots\delta^{j_{p-2r}}_{i_{p-2r}}\delta^{j_{p+1-2r}j_{p+2-2r}}\cdots\delta^{j_{p-1})j_p}\Big] = \\
&= (2k+2)(2k+3+2p)\phi_{i_1\cdots i_{p-2r}l_1l_1\cdots l_rl_r} + \\
&+\phi_{llj_1\cdots j_{p-2}}\delta_{j_{p-1}j_p}\Big[(p-2r)(p-2r-1)\delta^{j_p}_{(i_1}\delta^{j_{p-1}}_{i_2}\delta^{(j_1}_{i_3}\cdots\delta^{j_{p-2r-2}}_{i_{p-2r})}\delta^{j_{p-2r-1}j_{p-2r}}\cdots\delta^{j_{p-3}j_{p-2})} + \\
&+(p-2r)2r\delta^{j_p}_{(i_1}\delta^{(j_1}_{i_2}\cdots\delta^{j_{p-2r-1}}_{i_{p-2r})}\delta^{j_{p-2r}j_{p+1-2r}}\cdots\delta^{j_{p-2})j_{p-1}} + \\
&+2r(p-2r)\delta^{j_{p-1}}_{(i_1}\delta^{(j_1}_{i_2}\cdots\delta^{j_{p-2r-1}}_{i_{p-2r})}\delta^{j_{p-2r}j_{p+1-2r}}\cdots\delta^{j_{p-2})j_p} + \\
&+2r(2r-2)\delta^{(j_1}_{(i_1}\cdots\delta^{j_{p-2r}}_{i_{p-2r})}\delta^{j_{p+1-2r}j_{p+2-2r}}\cdots\delta^{j_{p-3}}_{j_{p-1}}\delta^{j_{p-2})j_p} + \\
&+2r\delta^{(j_1}_{(i_1}\cdots\delta^{j_{p-2r}}_{i_{p-2r})}\delta^{j_{p+1-2r}j_{p+2-2r}}\cdots\delta^{j_{p-3}j_{p-2})}\delta^{j_{p-1}j_p}\Big] = \\
&= [(2k+2)(2k+3+2p)+(p-2r)2r+2r(p-2r)+2r(2r-2)+6r]\phi_{i_1\cdots i_{p-2r}l_1l_1\cdots l_rl_r} + \\
&+(p-2r)(p-2r-1)\delta_{(i_1i_2}\phi_{i_3\cdots i_{p-2r})l_0l_0l_1l_1\cdots l_rl_r}\,.
\end{aligned}$$

But the second one of these terms is zero for the iterative hypothesis according to which eq. (40) holds up to the index R. There remains $\phi_{i_1\cdots i_{p-2R+2}l_1l_1\cdots l_{R-1}l_{R-1}} = 0$, that is eq. (40) with $R-1$ instead of R. This completes the proof of (40). Now (40) written with $R=0$ gives $\phi_{i_1\cdots i_p} = 0$ and this completes the proof of the Lemma.

Proof of Property 6. To prove this property it suffices to start from the property 5 and to apply the above Lemma with $\phi_{\dots} = \sum_{n=s}^{N+M+1-R} X^{k_1\cdots k_R}_{i_1\cdots i_{N+M+1-n}}(-\vec{u})X^{i_1\cdots i_{N+M+1-n}}_{j_1\cdots j_{N+M+1-s}}(\vec{u})$ in the case $N+1 \le s < N+M+1-R$ and with $\phi_{\dots} = \sum_{n=s}^{N+M+1-R} X^{k_1\cdots k_R}_{i_1\cdots i_{N+M+1-n}}(-\vec{u})X^{i_1\cdots i_{N+M+1-n}}_{j_1\cdots j_{N+M+1-s}}(\vec{u}) - \delta^{k_1}_{(j_1}\cdots\delta^{k_R}_{j_R)}$ in the case $s = N+M+1-R$.

Proof of Property 7. It is an easy consequence of eq. (3) written with $n+1$ instead of n, and making explicit the sites where in index i_{n+1} appears.

Proof of Property 8. It suffices to write eq. (12) with $n = N+M+1-R$ and, after that, to contract it with $\delta_{i_{R+1}i_{R+2}}\cdots\delta_{i_{M+N-R}i_{M+N+1-R}}$.

Proof of Property 9, It suffices to use eq. (13) for $s = N+1, \cdots, N+M+2-R$ and, after that, eq. (6) and the property $S^{i_1\cdots i_{N+M+1-R}}_{j_1\cdots j_{N+M+1-R}} = \delta^{(i_1}_{j_1}\cdots\delta^{i_{N+M+1-R})}_{j_{N+M+1-r}}$.

Proof of Property 10. It is an easy consequence of eq. (3) and of the identity $\binom{n}{s}(n-s) = n\binom{n-1}{s}$ which holds for $s < n$.

Proof of Property 11. Let us take the derivative of eq. (16) with respect to u^j and use eq. (15) taking into account that, for $s = 0, \cdots, N$, we have $s \leq N+M-R$. We obtain that this derivative is equal to $\delta_{i_{R+1}i_{R+2}} \cdots \delta_{i_{M+N-R}i_{M+N+1-R}}(N+M+1-R)S^{(i_1\cdots i_{N+M-R}}_{j_1\cdots j_s}\delta^{i_{N+M+1-R})j}$.

Let us now explicitate what index appears near j in $\delta^{\cdot j}$ distinguishing the cases where it is one among $i_{R+1}\cdots i_{M+N+1-R}$ from those where it is different from them; in this way our expression takes the form $\delta_{i_{R+1}i_{R+2}} \cdots \delta_{i_{M+N-2-R}i_{M+N-1-R}}(N+M+1-2R)S^{i_1\cdots i_{N+M-1-R}j}_{j_1\cdots j_s}+$

$$+ R\delta_{i_{R+1}i_{R+2}} \cdots \delta_{i_{M+N-R}i_{M+N+1-R}}S^{i_{R+1}\cdots i_{N+M+1-R}(i_1\cdots i_{R-1}}_{j_1\cdots j_s}\delta^{i_R)j}.$$

By using now eq. (16) with $R+1$ instead of R, we obtain eq. $(17)_1$.

Let us take now the derivative of (6) with respect to u^j, by using eq. (15).

For $N+1 \leq s \leq N+M+1-R$, we have

$$\begin{aligned}
&\frac{\partial}{\partial u^j}X^{i_1\cdots i_R}_{(j_1\cdots j_{N+M+1-s}}\delta_{j_{N+M+2-s}j_{N+M+3-s}}\cdots\delta_{j_{s-1}j_s)} = \qquad (41)\\
&= \left(\frac{\partial}{\partial u^j}S^{i_1\cdots i_{N+M+1-R}}_{j_1\cdots j_s}\right)\delta_{i_{R+1}i_{R+2}}\cdots\delta_{i_{M+N-R}i_{M+N+1-R}}\\
&= \delta_{i_{R+1}i_{R+2}}\cdots\delta_{i_{M+N-R}i_{M+N+1-R}}\cdot\\
&\cdot\begin{cases} (N+M+1-R)S^{(i_1\cdots i_{N+M-R}}_{j_1\cdots j_s}\delta^{i_{N+M+1-R})j} & \text{for } s = 0,\cdots, N+M-R\,.\\ 0 & \text{for } s = N+M+1-R\,.\end{cases}
\end{aligned}$$

In particular, for $s = N+M+1-R$, thanks to the eq. (10) of the above Lemma, we obtain eq. $(17)_2$ for $r = R$. For the other values of s, let us explicitate what index appears near j in $\delta^{\cdot j}$ distinguishing the cases where it is one among $i_{R+1}\cdots i_{M+N+1-R}$ from those where it is different from them; in this way (41) takes the form

$$\begin{aligned}
&(N+M+1-2R)\delta_{i_{R+1}i_{R+2}}\cdots\delta_{i_{M+N-2-R}i_{M+N-1-R}}S^{i_1\cdots i_{N+M-1-R}j}_{j_1\cdots j_s}+\\
&\qquad + R\delta_{i_{R+1}i_{R+2}}\cdots\delta_{i_{M+N-R}i_{M+N+1-R}}S^{i_{R+1}\cdots i_{N+M+1-R}(i_1\cdots i_{R-1}}_{j_1\cdots j_s}\delta^{i_R)j} =\\
&= \left[(N+M+1-2R)\delta^j_{j_R}\delta^{(i_1}_{j_{R+1}} + R\delta^{j(i_1}\delta_{j_Rj_{R+1}}\right]\cdot\\
&\qquad \cdot S^{i_2\cdots i_R)j_Rj_{R+1}i_{R+1}\cdots i_{N+M-1-R}}_{j_1\cdots j_s}\delta_{i_{R+1}i_{R+2}}\cdots\delta_{i_{M+N-2-R}i_{M+N-1-R}}.
\end{aligned}$$

We note that in the last tensor $S^{\cdots}_{\cdots}$ the upper free indexes are $R+1$ and the total number of upper indexes is $N+M-R$; consequently, we can apply eq. (6) with $R+1$ instead of R so that our expression takes the form

$$\begin{aligned}
&\left[(N+M+1-2R)\delta^j_{j_R}\delta^{(i_1}_{j_{R+1}} + R\delta^{j(i_1}\delta_{j_Rj_{R+1}}\right]\cdot\\
&\qquad \cdot X^{i_2\cdots i_R)j_Rj_{R+1}}_{(j_1\cdots j_{N+M+1-s}}\delta_{j_{N+M+2-s}j_{N+M+3-s}}\cdots\delta_{j_{s-1}j_s)}.
\end{aligned}$$

By comparing with the initial left hand side of (41) and by using eq. (10) of the above Lemma, we obtain eq. $(17)_2$ for the other values of r.

Proof of Property 12. It is another particular and interesting case of eq. (4) with $n = N + M + 2 - R$ and $r = \frac{N+M+1}{2} - R$.

Proof of Property 13. It is a consequence of eq. (18). From the hypothesis $s \geq N + 2$ it follows $2s - (N + M + 3) \geq N + 1 - M \geq 0$ and this fact assures that the number of $\delta_{\cdots}$ appearing in (19) is not negative. Also the number of $\delta_{\cdots}$ appearing in (20) is not negative. In fact, from the definition $(18)_2$ of S^*, it follows $s - R - 1 \leq q + 2p = (q + p) + p \leq \frac{N+M+1}{2} - R + p$ where we have used the fact that $p + q \leq \frac{N+M+1}{2} - R$; it follows $p \geq s - 1 - \frac{N+M+1}{2}$, that is, $2p \geq 2s - 3 - N - M$.

Proof of Property 14. It suffices to substitute (19) in the left hand side of (14) and use the eq. (10) of the above Lemma.

Proof of Property 15. By using the property 7, we have that the left hand side of eq. (22) is equal to

$$S^{i_1\cdots i_{n+1}}K + \sum_{s=1}^{n}\left(S^{i_1\cdots i_n}_{j_1\cdots j_s}u^{i_{n+1}} + S^{i_1\cdots i_n}_{(j_1\cdots j_{s-1}}\delta^{i_{n+1}}_{j_s)}\right)K^{j_1\cdots j_s} +$$
$$+\delta^{i_1}_{(j_1}\cdots\delta^{i_{n+1}}_{j_{n+1})}K^{j_1\cdots j_{n+1}} - u^{i_{n+1}}\left(S^{i_1\cdots i_n}K + \sum_{s=1}^{n}S^{i_1\cdots i_n}_{j_1\cdots j_s}K^{j_1\cdots j_s}\right)$$

Now, we see that the coefficient of K in the above expression is zero; after that, also the coefficient of $u^{i_{n+1}}$ is identically zero and in the remaining summation we may change index according to the law $s = 1 + S$. In this way our expression becomes $\sum_{S=0}^{n-1} S^{i_1\cdots i_n}_{j_1\cdots j_S}K^{j_1\cdots j_S i_{n+1}} + K^{i_1\cdots i_{n+1}}$ which is equal to the right hand side of eq. (22).

Proof of Property 16. Let us consider eq. (16) with $R+1$ instead of R and contract it with $\delta_{i_R i_{R+1}}$, that is $Y^{i_1\cdots i_{R-1}ll}_{j_1\cdots j_s} = \delta_{i_R i_{R+1}}\cdots\delta_{i_{M+N-R-1}i_{M+N-R}}S^{i_1\cdots i_{N+M-R}}_{j_1\cdots j_s}$. Now we apply the property 1 with $n = N + M - R$ and $r = \frac{N+M+1}{2} - R$; this is possible only if the condition $R + 1 \geq 2$ is satisfied and this is the case because $R \geq 1$. In this way we obtain

$$Y^{i_1\cdots i_{R-1}ll}_{j_1\cdots j_s} =$$
$$\sum_{(p,q)\in\tilde{S}}\binom{R-1}{s-q-2p}2^q\frac{\left(\frac{N+M+1}{2}-R\right)!}{p!\,q!\,\left(\frac{N+M+1}{2}-R-p-q\right)!}(u^2)^{\left(\frac{N+M+1}{2}-R-p-q\right)}$$
$$\cdot\, u_{(j_1}\cdots u_{j_q}\delta_{j_{q+1}j_{q+2}}\cdots\delta_{j_{q+2p-1}j_{q+2p}}\delta^{(i_1}_{j_{q+2p+1}}\cdots\delta^{i_{s-q-2p}}_{j_s)}u^{i_{s-q-2p+1}}\cdots u^{i_{R-1})},$$

where $\tilde{S}$ is the set of the couples (p, q) of integer numbers p and q such that $p \geq 0$, $q \geq 0$, $p + q \leq \frac{N+M+1}{2} - R$, $s - R + 1 \leq q + 2p \leq s$.

Now the compatibility between the last two inequalities in the definition of $\tilde{S}$ is $s - R + 1 - p \leq q + p \leq \frac{N+M+1}{2} - R$ which implies $p \geq s + 1 - \frac{N+M+1}{2}$; when $s = N$

this inequality becomes $p \geq \frac{N-M+1}{2}$ of which $p \geq 0$ is a consequence. This completes the proof of our property because it says that there is at least a number $N-M+1$ of indexes belonging to $\delta_{j_{q+1}j_{q+2}} \cdots \delta_{j_{q+2p-1}j_{q+2p}}$ in each term of $Y^{i_1 \cdots i_{R-1} ll}_{j_1 \cdots j_N}$.

Proof of Property 17. Let us consider eq. (6) with $R+1$ instead of R and contract it with $\delta_{i_R i_{R+1}}$; we find that for $N+1 \leq s \leq N+M-R$ we have

$$S^{i_1 \cdots i_{R-1} l_1 l_1 \cdots l_{\frac{N+M+1}{2}-R} l_{\frac{N+M+1}{2}-R}}_{j_1 \cdots j_s} = X^{i_1 \cdots i_{R-1} ll}_{(j_1 \cdots j_{N+M+1-s}} \delta_{j_{N+M+2-s} j_{N+M+3-s}} \cdots \delta_{j_{s-1} j_s)} . \quad (42)$$

For the left hand side of this relation we use the property 1 with $n = N+M-R$ and $r = \frac{N+M+1}{2} - R$, noting that the condition $r \leq [\frac{n}{2}]$ is satisfied because $R \geq 2$ (otherwise we could not contract with $\delta_{i_R i_{R+1}}$). In this way it becomes

$$\sum_{(p,q) \in S^*} \binom{R-1}{s-q-2p} 2^q \frac{\left(\frac{N+M+1}{2} - R\right)!}{p!\, q!\, \left(\frac{N+M+1}{2} - R - p - q\right)!} (u^2)^{\left(\frac{N+M+1}{2} - R - p - q\right)}$$
$$\cdot u_{(j_1} \cdots u_{j_q} \delta_{j_{q+1} j_{q+2}} \cdots \delta_{j_{q+2p-1} j_{q+2p}} \delta^{(i_1}_{j_{q+2p+1}} \cdots \delta^{i_{s-q-2p}}_{j_s)} u^{i_{s-q-2p+1}} \cdots u^{i_{R-1})} ,$$

where S^* is the set of the couples (p,q) of integer numbers p and q such that $p \geq 0$, $q \geq 0$, $p+q \leq \frac{N+M+1}{2} - R$, $s-R+1 \leq q+2p \leq s$.

Now the compatibility condition between the last two inequalities in the definition of S^* is $s-R+1-p \leq q+p \leq \frac{N+M+1}{2} - R$ which implies $2p \geq 2s - N - M + 1$; so there is at least a number $2s-N-M+1$ of indexes belonging to $\delta_{j_{q+1}j_{q+2}} \cdots \delta_{j_{q+2p-1}j_{q+2p}}$ in each term of the left hand side of eq. (42) while there is a number $2s-N-M-1$ of indexes belonging to $\delta_{j_{N+M+2-s} j_{N+M+3-s}} \cdots \delta_{j_{s-1} j_s}$ in the right hand side of (42). In other words, in each term of the left hand side of (42) there is at least a symbol $\delta_{\dots}$ more than in its right hand side. By comparing both sides and by using the Lemma, our property follows as a consequence (The index s has been changed according to the law $s = N+M+1-S$).

Proof of Property 18. It is a simply consequence of (24) written in the particular case $S = R+1$, of the Lemma and of the property $X^{i_1 \cdots i_{R+1}}_{j_1 \cdots j_{R+1}} = \delta^{i_1}_{(j_1} \cdots \delta^{i_{R+1}}_{j_{R+1})}$.

Proof of Property 19. From eq. (16), by using (12) with $n = N+M-R$, we obtain

$$Y^{k i_1 \cdots i_{R-1}}_{j_1 \cdots j_s} = u^k S^{i_1 \cdots i_{R-1} l_1 l_1 \cdots l_{\frac{N+M+1}{2}-R} l_{\frac{N+M+1}{2}-R}}_{j_1 \cdots j_s} + S^{i_1 \cdots i_{R-1} l_1 l_1 \cdots l_{\frac{N+M+1}{2}-R} l_{\frac{N+M+1}{2}-R}}_{(j_1 \cdots j_{s-1}} \delta^k_{j_s)} ,$$

from which our property follows (We are in the range of validity of (12) because for this equation we need $s = 1, \cdots, n = N+M-R$ and this range is respected because $N \leq N+M-R$).

Proof of Property 20. It is a consequence of eq. (16) and (3).

Proof of Property 21. From eq. (6) with $s = N + M + 1 - S$ we obtain

$$X^{ki_1\cdots i_{R-1}}_{(j_1\cdots j_S}\delta_{j_{S+1}j_{S+2}}\cdots\delta_{j_{N+M-S}j_{N+M+1-S})} = S^{ki_1\cdots i_{R-1}l_1l_1\cdots l_{\frac{N+M+1}{2}-R}l_{\frac{N+M+1}{2}-R}}_{j_1\cdots j_{N+M+1-S}}.$$

By applying eq. (12) with $n = N + M - R$ our property follows. We note that (12) holds for $s = 1, \cdots, n$ and this range of validity is respected because S goes from $R + 1$ to M.

Proof of Property 22. Eq. (28) with $S = M$ becomes

$$\begin{aligned}
&X^{ki_1\cdots i_{R-1}}_{(j_1\cdots j_M}\delta_{j_{M+1}j_{M+2}}\cdots\delta_{j_Nj_{N+1})} = \\
&= u^kS^{i_1\cdots i_{R-1}l_1l_1\cdots l_{\frac{N+M+1}{2}-R}l_{\frac{N+M+1}{2}-R}}_{j_1\cdots j_{N+1}} + S^{i_1\cdots i_{R-1}l_1l_1\cdots l_{\frac{N+M+1}{2}-R}l_{\frac{N+M+1}{2}-R}}_{(j_1\cdots j_N}\delta^k_{j_{N+1})} = \\
&= u^k\delta_{i_Ri_{R+1}}S^{i_1\cdots i_{R+1}l_1l_1\cdots l_{\frac{N+M-1}{2}-R}l_{\frac{N+M-1}{2}-R}}_{j_1\cdots j_{N+1}} \\
&+\delta_{i_Ri_{R+1}}S^{i_1\cdots i_{R+1}l_1l_1\cdots l_{\frac{N+M-1}{2}-R}l_{\frac{N+M-1}{2}-R}}_{(j_1\cdots j_N}\delta^k_{j_{N+1})} = \\
&\stackrel{1}{=} u^k\delta_{i_Ri_{R+1}}X^{i_1\cdots i_{R+1}}_{(j_1\cdots j_M}\delta_{j_{M+1}j_{M+2}}\cdots\delta_{j_Nj_{N+1})} + Y^{i_1\cdots i_{R-1}ll}_{(j_1\cdots j_N}\delta^k_{j_{N+1})} = \\
&\stackrel{2}{=} u^k\eta^{i_1\cdots i_{R-1}}_{(j_1\cdots j_{M-2}}\delta_{j_{M-1}j_M}\cdots\delta_{j_Nj_{N+1})} + T^{i_1\cdots i_{R-1}}_{(j_1\cdots j_{M-1}}\delta_{j_Mj_{M+1}}\cdots\delta_{j_{N-1}j_N}\delta^k_{j_{N+1})},
\end{aligned}$$

where in the passage denoted with $\stackrel{1}{=}$ we have used eqs. (16) and (6), while in the passage denoted with $\stackrel{2}{=}$ we have used eqs. (23) and (24). By comparing the initial expression in this list of equalities, with the last one and by using the Lemma, we obtain (29).

Proof of Property 23. Eq. (28) with $S = R + 1, \cdots, M - 1$ becomes

$$\begin{aligned}
&X^{ki_1\cdots i_{R-1}}_{(j_1\cdots j_S}\delta_{j_{S+1}j_{S+2}}\cdots\delta_{j_{N+M-S}j_{N+M+1-S})} \\
&= u^k\delta_{i_Ri_{R+1}}S^{i_1\cdots i_{R+1}l_1l_1\cdots l_{\frac{N+M-1}{2}-R}l_{\frac{N+M-1}{2}-R}}_{j_1\cdots j_{N+M+1-S}} \\
&+\delta_{i_Ri_{R+1}}S^{i_1\cdots i_{R+1}l_1l_1\cdots l_{\frac{N+M-1}{2}-R}l_{\frac{N+M-1}{2}-R}}_{(j_1\cdots j_{N+M-S}}\delta^k_{j_{N+M+1-S})} \\
&= u^kX^{i_1\cdots i_{R-1}ll}_{(j_1\cdots j_S}\delta_{j_{S+1}j_{S+2}}\cdots\delta_{j_{N+M-S}j_{N+M+1-S})} \\
&+X^{i_1\cdots i_{R-1}ll}_{(j_1\cdots j_{S+1}}\delta_{j_{S+2}j_{S+3}}\cdots\delta_{j_{N+M-S-1}j_{N+M-S}}\delta^k_{j_{N+M-S+1})},
\end{aligned}$$

where in the last passage, eq. (6) has been used.

By substituting in this expression from eq. (24), by comparing the initial expression in this list of equalities, with the last one and by using the Lemma, we obtain (30).

Proof of Property 24. Let us write eq. (19) with $R+1$ instead of R and let us take its trace; we find

$$S_{j_1\cdots j_s}^{i_1\cdots i_R l_1 l_1\cdots l_{\frac{N+M+1}{2}-R} l_{\frac{N+M+1}{2}-R}} = Z^{i_1\cdots i_R ll}_{(j_1\cdots j_{N+M+3-s}}\delta_{j_{N+M+4-s}j_{N+M+5-s}}\cdots\delta_{j_{s-1}j_s)}.$$

For the left hand side of this equation, we may use eq. (6) (but remembering that it holds only for $N+1\le s\le N+M+1-R$) and also the Lemma. So we find $X^{i_1\cdots i_R}_{(j_1\cdots j_{N+M+1-s}}\delta_{j_{N+M+2-s}j_{N+M+3-s})}=Z^{i_1\cdots i_R ll}_{j_1\cdots j_{N+M+3-s}}$.

By changing index according to $s=N+M+2-S$ we find our thesis.

Proof of Property 25. To prove $(32)_1$ it suffices to consider eq. (13) for $s=1,\cdots,N$ and apply eq. (16). Similarly, eq. (13) for $s=N+1$and by using eqs. (16) and (6) gives $(32)_2$.

Proof of Property 26. Let us consider eq. (14) and transform its left hand side by using eq. (19); after that, we use the Lemma and obtain

$$Z^{i_1\cdots i_{R+1}}_{j_1\cdots j_{N+M+3-s}} = u^{i_{R+1}}X^{i_1\cdots i_R}_{(j_1\cdots j_{N+M+1-s}}\delta_{j_{N+M+2-s}j_{N+M+3-s})} + X^{i_1\cdots i_R}_{(j_1\cdots j_{N+M+2-s}}\delta^{i_{R+1}}_{j_{N+M+3-s})},$$

for $s=N+1,\cdots,N+M+1-R$. By changing index according to the law $s=N+M+2-S$, we obtain our thesis.

3. Application of the above Properties to the Moments and the Fluxes

A first application of the above properties arises when we impose the Galilean relativity principle to our eqs. (2); in particular, let us see how the moments and their fluxes transform under a change of frames moving, one with respect to the other, with a translational rectilinear uniformly motion with velocity $\vec{u}$. For the variables $F^{i_1\cdots i_n}$ and $G^{ki_1\cdots i_n}$ it can be found in [19] , [20] and reads

$$F^{i_1\cdots i_n} = \sum_{s=0}^{n} S^{i_1\cdots i_n}_{j_1\cdots j_s}(\vec{u})F^{Ij_1\cdots j_s}, \tag{43}$$
$$G^{ki_1\cdots i_n} - \frac{F^k}{F}F^{i_1\cdots i_n} = \sum_{s=0}^{n} S^{i_1\cdots i_n}_{j_1\cdots j_s}(\vec{u})\left(G^{Ikj_1\cdots j_s} - \frac{F^{Ik}}{F^I}F^{Ij_1\cdots j_s}\right),$$

where $F^{Ij_1\cdots j_s}$, $G^{Ikj_1\cdots j_s}$ are the counterparts of $F^{j_1\cdots j_s}$, $G^{kj_1\cdots j_s}$ in the other reference frame.

Regarding the other variables, we use the eqs. (6) and (7) of property 3 and the definition (16) of $Y^{i_1\cdots i_R}_{j_1\cdots j_s}$. Now we can extend the eqs. (43) also for the value

$n = N + M + 1 - R$ and contract the result with $\delta_{i_{R+1} i_{R+2}} \cdots \delta_{i_{N+M-R} i_{N+M+1-R}}$; by using eqs. (6) and (16) we obtain

$$
\begin{aligned}
F_R^{i_1 \cdots i_R} &= \sum_{s=0}^{N+M+1-R} S_{j_1 \cdots j_s}^{i_1 \cdots i_R l_1 l_1 l_{\frac{N+M+1}{2}-R} l_{\frac{N+M+1}{2}-R}} F^{I j_1 \cdots j_s} = \qquad (44)\\
&= \delta_{i_{R+1} i_{R+2}} \cdots \delta_{i_{M+N-R} i_{M+N+1-R}} \sum_{s=0}^{N} S_{j_1 \cdots j_s}^{i_1 \cdots i_{M+N+1-R}} F^{I j_1 \cdots j_s} + \\
&+ \sum_{s=N+1}^{M+N+1-R} X_{j_1 \cdots j_{N+M+1-s}}^{i_1 \cdots i_R} F^{I j_1 \cdots j_{N+M+1-s} l_1 l_1 \cdots l_{s-\frac{N+M+1}{2}} l_{s-\frac{N+M+1}{2}}} = \\
&= \sum_{s=0}^{N} Y_{j_1 \cdots j_s}^{i_1 \cdots i_R} F^{I j_1 \cdots j_s} + \sum_{S=R}^{M} X_{j_1 \cdots j_S}^{i_1 \cdots i_R} F_S^{I j_1 \cdots j_S} ,
\end{aligned}
$$

$$
\begin{aligned}
G_R^{k i_1 \cdots i_R} - \frac{F^k}{F} F_R^{i_1 \cdots i_R} &= \sum_{s=0}^{N} Y_{j_1 \cdots j_s}^{i_1 \cdots i_R} \left(G^{I k j_1 \cdots j_s} - \frac{F^{Ik}}{F^I} F^{I j_1 \cdots j_s} \right) + \\
&+ \sum_{S=R}^{M} X_{j_1 \cdots j_S}^{i_1 \cdots i_R} \left(G_S^{I k j_1 \cdots j_S} - \frac{F^{Ik}}{F^I} F_S^{I j_1 \cdots j_S} \right) ,
\end{aligned}
$$

where in the second summation we have changed index according to $S = N + M + 1 - s$.

The eqs. $(43)_1$ and $(44)_1$ give the relation between the moments $F^{i_1 \cdots i_n}$, $F_R^{i_1 \cdots i_R}$ and their counterparts in the other reference frame. This correspondence is not casual, but is due to the good choice of the field eqs. (2). In fact, if we would take

$$
\partial_t F + \partial_k G^k = 0 \, , \; \partial_t F^i + \partial_k G^{ki} = 0 \, , \; \partial_t F^{ll} + \partial_k G^{kll} = 0 \, , \; \partial_t F^{ill} + \partial_k G^{ki} = Q^i \, , \qquad (45)
$$

then the decomposition for the independent variables would be

$$
\begin{aligned}
&F = F^I \, , \qquad (46)\\
&F^i = F^{Ii} + F^I u^i \, , \\
&F^{ll} = F^{Ill} + 2F^{Il} u_l + F^I u^2 \, , \\
&F^{ill} = F^{Iill} + 2F^{Iil} u_l + F^{Ill} u^i + F^{Ii} u^2 + 2F^{Il} u_l u^i + F^I u^i u^2 \, ,
\end{aligned}
$$

and, in this last equation appears F^{Iil} while there is no F^{il} between the independent variables! We have seen above that this problem does not arise with our field equations, neither with the independent variables nor with the fluxes whose transformation law is given by $(43)_2$ and $(44)_2$.

If we indicate with F^A the generic independent variable between $F^{i_1\cdots i_n}$ and $F_R^{i_1\cdots i_R}$, eqs. (43) and (44) can be written in a more compact form as

$$F^A = X^A{}_B(\vec{u})F^{IB} \quad , \quad G^{kA} - \frac{F^k}{F}F^A = X^A{}_B(\vec{u})\left(G^{IkB} - \frac{F^{Ik}}{F^I}F^{IB}\right), \tag{47}$$

with obvious meaning of G^{kA} and of $X^A{}_B(\vec{u})$. This last matrix satisfies the following properties

$$1) \quad X^C{}_A(-\vec{u})X^A{}_B(\vec{u}) = \delta^C{}_A\,, \tag{48}$$

$$2) \quad \frac{\partial X^A{}_C(\vec{u})}{\partial u^j} = M^{jA}{}_B X^B{}_C(\vec{u})\,, \tag{49}$$

where $M^{jA}{}_B$ is the constant matrix defined in the following way

- If the multindex A is the same appearing in $F^{i_1\cdots i_n}$ with $n=0$, then $M^{j0}{}_B = 0$;
- If the multindex A is the same appearing in $F^{i_1\cdots i_n}$ with $1 \le n \le N$, then

$$M^{ji_1\cdots i_n}{}_B = \begin{cases} n\,\delta^{(i_1}_{j_1}\cdots\delta^{i_{n-1}}_{j_{n-1}}\delta^{i_n)j} & \text{if} \quad B = j_1\cdots j_{n-1} \\ 0 & \text{if} \quad B \neq j_1\cdots j_{n-1} \end{cases} ;$$

- If the multindex A is the same appearing in $F_R^{i_1\cdots i_R}$ with $R = M$, then

$$M^{ji_1\cdots i_M}{}_B = \begin{cases} \left[(N-M+1)\delta^j_{(j_{M+1}}\delta^{(i_1}_{j_1} + M\delta^{j(i_1}\delta_{(j_{M+1}j_1}\right]\delta^{i_2}_{j_2}\cdots\delta^{i_M)}_{j_M}\cdot \\ \qquad\qquad \cdot\,\delta_{j_{M+2}j_{M+3}}\cdots\delta_{j_{N-1}j_N)} \quad \text{if } B = j_1\cdots j_N \\ 0 \qquad\qquad\qquad\qquad\qquad\qquad \text{if } B \neq j_1\cdots j_N \end{cases} ;$$

- If the multindex A is the same appearing in $F_R^{i_1\cdots i_R}$ with $0 \le R \le M-1$, then

$$M^{ji_1\cdots i_R}{}_B = \begin{cases} \left[(N+M+1-2R)\delta^j_{(j_{R+1}}\delta^{(i_1}_{j_1} + R\delta^{j(i_1}\delta_{(j_{R+1}j_1}\right] \\ \qquad\qquad \cdot\,\delta^{i_2}_{j_2}\cdots\delta^{i_R)}_{j_R)} \quad \text{if} B = j_1\cdots j_{R+1}\,,\ R\neq 0 \\ (N+M+1)\delta^j_{j_1} \qquad \text{if} B = j_1\cdots j_{R+1}\,,\ R = 0 \\ 0 \qquad\qquad\qquad\qquad \text{if } B \neq j_1\cdots j_{R+1} \end{cases} ;$$

Another property of the matrix $X^A{}_B$ is

$$3) \quad X^A{}_B(\vec{u})M^{jB}{}_C = M^{jA}{}_B X^B{}_C(\vec{u})\,. \tag{50}$$

The counterparts of these properties when only eqs. $(2)_1$ are considered and eqs. $(2)_2$ are omitted, have been already found in [14]; now we have found that they hold

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also for the complete system (2). In [16] we have already used them, but without proving them. We report now here their proofs.

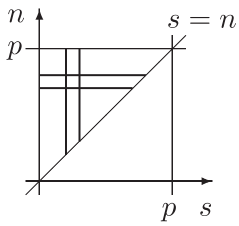
Proof of eq. (48). This equation means that $X^C{}_A(-\vec{u})$ is the inverse matrix of $X^A{}_B(\vec{u})$. We will prove it if we show that from $(47)_1$ it follows

$$F^{IC} = X^C{}_A(-\vec{u})F^A \,. \tag{51}$$

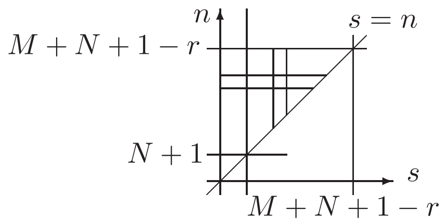
To this end, let us perform the following calculations

$$\cdot)\sum_{n=0}^{p} S^{k_1\cdots k_p}_{i_1\cdots i_n}(-\vec{u})F^{i_1\cdots i_n} = \sum_{n=0}^{p} S^{k_1\cdots k_p}_{i_1\cdots i_n}(-\vec{u})\sum_{s=0}^{n} S^{i_1\cdots i_n}_{j_1\cdots j_s}(\vec{u})F^{Ij_1\cdots j_s}$$
$$= \sum_{s=0}^{p}\sum_{n=s}^{p} S^{k_1\cdots k_p}_{i_1\cdots i_n}(-\vec{u})S^{i_1\cdots i_n}_{j_1\cdots j_s}(\vec{u})F^{Ij_1\cdots j_s} = F^{Ik_1\cdots k_p}\,, \tag{52}$$

where in the first passage we have applied eq. (6) and in the second passage we have changed the order of the two summations (something like what we can do for a double integral extended to a domain which is normal with respect to both the axis), as explicated by the following Picture n. 1.



Picture n. 1



Picture n. 2

Finally, the Property 4 has been applied.

·) Similarly, let us write the right hand side of eq. $(44)_1$ with $(r\,, n\,, -\vec{u})$ instead of $(R\,, s\,, -\vec{u})$ respectively, and F^B instead of F^{IB}; in this way we obtain

$$\delta_{i_{r+1}i_{r+2}}\cdots\delta_{i_{M+N-r}i_{M+N+1-r}}\sum_{n=0}^{N} S^{i_1\cdots i_{M+N+1-r}}_{j_1\cdots j_n}(-\vec{u})F^{j_1\cdots j_n} \tag{53}$$
$$+\sum_{n=N+1}^{M+N+1-r} X^{i_1\cdots i_r}_{j_1\cdots j_{N+M+1-n}}(-\vec{u})F^{j_1\cdots j_{N+M+1-n}l_1 l_1\cdots l_{n-\frac{N+M+1}{2}}l_{n-\frac{N+M+1}{2}}}$$
$$= \delta_{i_{r+1}i_{r+2}}\cdots\delta_{i_{M+N-r}i_{M+N+1-r}}\sum_{n=0}^{N} S^{i_1\cdots i_{M+N+1-r}}_{j_1\cdots j_n}(-\vec{u})\sum_{s=0}^{n} S^{j_1\cdots j_n}_{k_1\cdots k_s}(\vec{u})F^{Ik_1\cdots k_s}$$
$$+\sum_{n=N+1}^{M+N+1-r} X^{i_1\cdots i_r}_{j_1\cdots j_{N+M+1-n}}(-\vec{u})\Bigg[\delta_{j_{N+M+2-n}j_{N+M+3-n}}\cdots\delta_{j_{n-1}j_n}\sum_{s=0}^{N} S^{j_1\cdots j_n}_{k_1\cdots k_s}(\vec{u})F^{Ik_1\cdots k_s}$$

$$+ \sum_{s=N+1}^{n} X^{j_1\cdots j_{N+M+1-n}}_{k_1\cdots k_{N+M+1-s}}(\vec{u}) F^{Ik_1\cdots k_{N+M+1-s} l_1 l_1 \cdots l_{s-\frac{N+M+1}{2}} l_{s-\frac{N+M+1}{2}}} \Bigg] ,$$

where in the first passage we have used eq. $(43)_1$ with k instead of j, and the intermediate side of $(44)_1$ with $R = N + M + 1 - n$ and with (j, k) instead of (ij), respectively. Now, we can use the following changes of order of summations, that is $\sum_{n=0}^{N} \sum_{s=0}^{n} \cdots = \sum_{s=0}^{N} \sum_{n=s}^{N} \cdots$ and $\sum_{n=N+1}^{M+N+1-r} \sum_{s=0}^{N} \cdots = \sum_{s=0}^{N} \sum_{n=N+1}^{M+N+1-r} \cdots$ (the first one is based on Picture n. 1 with N instead of p, the second one is based on the fact that the summations on indexes n and s are independent one of the other; in other words, the couple (n, s) belongs to a rectangular domain); in this way, the terms containing $F^{Ik_1\cdots k_s}$ can now be written as

$$\sum_{s=0}^{N} \sum_{n=s}^{N} \delta_{i_{r+1} i_{r+2}} \cdots \delta_{i_{M+N-r} i_{M+N+1-r}} S^{i_1\cdots i_{M+N+1-r}}_{j_1\cdots j_n}(-\vec{u}) S^{j_1\cdots j_n}_{k_1\cdots k_s}(\vec{u}) F^{Ik_1\cdots k_s}$$
$$+ \sum_{s=0}^{N} \sum_{n=N+1}^{M+N+1-r} X^{i_1\cdots i_r}_{(j_1\cdots j_{N+M+1-n}}(-\vec{u}) \delta_{j_{N+M+2-n} j_{N+M+3-n}} \cdots \delta_{j_{n-1} j_n)} S^{j_1\cdots j_n}_{k_1\cdots k_s}(\vec{u}) F^{Ik_1\cdots k_s}$$
$$= \sum_{s=0}^{N} \sum_{n=s}^{N} \delta_{i_{r+1} i_{r+2}} \cdots \delta_{i_{M+N-r} i_{M+N+1-r}} S^{i_1\cdots i_{M+N+1-r}}_{j_1\cdots j_n}(-\vec{u}) S^{j_1\cdots j_n}_{k_1\cdots k_s}(\vec{u}) F^{Ik_1\cdots k_s}$$
$$+ \sum_{s=0}^{N} \sum_{n=N+1}^{M+N+1-r} S^{i_1\cdots i_r l_1 l_1 \cdots l_{\frac{N+M+1}{2}-r} l_{\frac{N+M+1}{2}-r}}_{j_1\cdots j_n}(-\vec{u}) S^{j_1\cdots j_n}_{k_1\cdots k_s}(\vec{u}) F^{Ik_1\cdots k_s}$$
$$= \sum_{s=0}^{N} \left(\sum_{n=s}^{M+N+1-r} S^{i_1\cdots i_r l_1 l_1 \cdots l_{\frac{N+M+1}{2}-r} l_{\frac{N+M+1}{2}-r}}_{j_1\cdots j_n}(-\vec{u}) S^{j_1\cdots j_n}_{k_1\cdots k_s}(\vec{u}) \right) F^{Ik_1\cdots k_s} = 0 ,$$

where, in the second line, we were allowed to insert the symmetrization with respect to $j_1 \cdots j_n$ because they are contracted with those of $S^{j_1\cdots j_n}_{k_1\cdots k_s}$ which is symmetric; after that, in the subsequent passage, we have used (6). Finally, we have used the property 4 contracted with some $\delta_{\cdots}$ and taken into account that $s \leq N < N + M + 1 - R$.

After that, from eq. (53) there remains

$$\sum_{n=N+1}^{M+N+1-r} X^{i_1\cdots i_r}_{j_1\cdots j_{N+M+1-n}}(-\vec{u})$$
$$\sum_{s=N+1}^{n} X^{j_1\cdots j_{N+M+1-n}}_{k_1\cdots k_{N+M+1-s}}(\vec{u}) F^{Ik_1\cdots k_{N+M+1-s} l_1 l_1 \cdots l_{s-\frac{N+M+1}{2}} l_{s-\frac{N+M+1}{2}}} =$$
$$\sum_{s=N+1}^{M+N+1-r} \sum_{n=s}^{M+N+1-r} X^{i_1\cdots i_r}_{j_1\cdots j_{N+M+1-n}}(-\vec{u}) X^{j_1\cdots j_{N+M+1-n}}_{k_1\cdots k_{N+M+1-s}}(\vec{u})$$
$$F^{Ik_1\cdots k_{N+M+1-s} l_1 l_1 \cdots l_{s-\frac{N+M+1}{2}} l_{s-\frac{N+M+1}{2}}} =$$

$$= F^{Ik_1\cdots k_r l_1 l_1 \cdots l_{\frac{N+M+1}{2}-r} l_{\frac{N+M+1}{2}-r}},$$

where in the first passage we have changed the order of the two summations according to the Picture n. 2 and, in the final passage, we have used the property 6.

This completes the proof of (51). In fact, we started from the variables F^{IA} and we have contracted them with $X^B{}_A(-\vec{u})$ calling F^B the result; finally we have contracted this with $X^C{}_B(\vec{u})$ and found F^{IC} as result. This proves (48) and, consequently (51).

Proof of eq. (49). The derivative of eq. $(43)_1$ with respect to u^j, by applying also eq. (15) is

$$\frac{\partial}{\partial u^j} F^{i_1\cdots i_n} = \sum_{s=0}^{n-1} n S^{(i_1\cdots i_{n-1}}_{j_1\cdots j_s} \delta^{i_n)j} F^{Ij_1\cdots j_s},$$

from which, by applying again $(43)_1$, we deduce

$$\frac{\partial}{\partial u^j} F^{i_1\cdots i_n} = n F^{(i_1\cdots i_{n-1}} \delta^{i_n)j} \quad \text{for} \quad n = 1, \cdots, N. \tag{54}$$

Obviously, we have also $\frac{\partial}{\partial u^j} F = 0$.

The derivative of eq. $(44)_1$ with respect to u^j, by applying also eq. (17) is

$$\frac{\partial}{\partial u^j} F_R^{i_1\cdots i_R} =$$
$$\sum_{s=0}^{N} \left[(N+M+1-2R)\delta^j_{j_R} \delta^{(i_1}_{j_{R+1}} + R\delta^{j(i_1}\delta_{j_R j_{R+1}} \right] Y^{i_2\cdots i_R) j_R j_{R+1}}_{j_1\cdots j_s} F^{Ij_1\cdots j_s} +$$
$$+ \sum_{S=R+1}^{M} \left[(N+M+1-2R)\delta^j_{j_R} \delta^{(i_1}_{j_{R+1}} + R\delta^{j(i_1}\delta_{j_R j_{R+1}} \right] X^{i_2\cdots i_R) j_R j_{R+1}}_{j_1\cdots j_S} F_S^{Ij_1\cdots j_S},$$

from which, by applying again $(44)_1$, we deduce

$$\frac{\partial}{\partial u^j} F_R^{i_1\cdots i_R} = \left[(N+M+1-2R)\delta^j_{j_R} \delta^{(i_1}_{j_{R+1}} + R\delta^{j(i_1}\delta_{j_R j_{R+1}} \right] F_{R+1}^{i_2\cdots i_R) j_R j_{R+1}}$$
$$\text{for } R = 0, \cdots, M-1, \tag{55}$$

$$\frac{\partial}{\partial u^j} F_M^{i_1\cdots i_M}$$
$$= \left[(N-M+1)\delta^j_{j_M} \delta^{(i_1}_{j_{M+1}} + M\delta^{j(i_1}\delta_{j_M j_{M+1}} \right] F^{i_2\cdots i_M) j_M j_{M+1} l_1 l_1 \cdots l_{\frac{N-M-1}{2}} l_{\frac{N-M-1}{2}}}.$$

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(For the second line we have used eq. (16)). The eqs. (54) and (55) show that $\frac{\partial F^A}{\partial u^j}$ is a linear and homogeneous combination of F^B with constant coefficients; by calling $M^{jA}{}_B$ the constant matrix of coefficients, these equations can be written in compact form as

$$\frac{\partial}{\partial u^j} F^A = M^{jA}{}_B F^B \, . \tag{56}$$

Moreover, from (54) and (55) we deduce the expression of $M^{jA}{}_B$ and it is that described after eq. (49). By substituting from eq. $(47)_1$ in (56), we obtain (49). If we calculate this in $\vec{u} = \vec{0}$ and take into account the fact that $M^{jA}{}_B$ is constant and the property $X^B{}_C(\vec{0}) = \delta^B_C$, we deduce also that

$$M^{jA}{}_B = \left[\frac{\partial}{\partial u^j} X^A{}_B(\vec{u}) \right]_{\vec{u}=\vec{0}} \, . \tag{57}$$

Also this relation extends to our set of field equations (2) a result already obtained by Prof. Ruggeri only for the eqs. $(2)_1$, as it can be seen in [14] and [19]. Thanks to (57), we can consider only the linear and homogeneous part in u^j of the matrix $X^A{}_B$ (which can be deduced from eqs. $(47)_1$, $(43)_1$ and $(44)_1$) ; its derivative with respect to u^j gives the matrix $M^{jA}{}_B$.

Proof of eq. (50). Let us take the derivative with respect to u^j of the relation $X^A{}_B(-\vec{u}) X^B{}_C(\vec{u}) = \delta^A{}_C$ (which comes from (48)) and take into account, for the first factor, of the derivation rule for composite functions; it follows

$$- \left(\frac{\partial}{\partial u^j} X^A{}_B \right)_{(-\vec{u})} X^B{}_C(\vec{u}) + X^A{}_B(-\vec{u}) \frac{\partial}{\partial u^j} X^B{}_C(\vec{u}) = \vec{0} \, ;$$

from this relation, for (49), it follows

$$-M^{jA}{}_D X^D{}_B(-\vec{u}) X^B{}_C(\vec{u}) + X^A{}_B(-\vec{u}) M^{jB}{}_D X^D{}_C(\vec{u}) = \vec{0} \, , \quad \text{from which}$$

$$M^{jA}{}_C = X^A{}_B(-\vec{u}) M^{jB}{}_D X^D{}_C(\vec{u}) \, . \tag{58}$$

By contracting this with $X^E{}_A(\vec{u})$, we obtain (50). This last equation expresses the following fact: For every fixed j, the quantity $M^{jB}{}_C$ is a matrix; well, its product with the other matrix $X^A{}_B$ is commutative.

Let us conclude this part by finding the derivatives with respect to u^j of the fluxes, obtaining for them the counterpart of eq. (56) which holds for the moments. To this end we note that from $(47)_2$ it follows

$$\frac{\partial}{\partial u^j} \left(G^{kA} - \frac{F^k}{F} F^A \right)$$

$$= M^{jA}{}_C X^C{}_B(\vec{u}) \left(G^{IkB} - \frac{F^{Ik}}{F^I} F^{IB} \right) = M^{jA}{}_C \left(G^{kC} - \frac{F^k}{F} F^C \right) , \quad (59)$$

where in the first passage we have used eq. (49) and in the second one again $(47)_2$. From this result we obtain

$$\frac{\partial}{\partial u^j} G^{kA}$$
$$= \delta^{kj} F^A + \frac{F^k}{F} M^{jA}{}_B F^B + M^{jA}{}_C \left(G^{kC} - \frac{F^k}{F} F^C \right) = \delta^{kj} F^A + M^{jA}{}_B G^{kB} , \quad (60)$$

where in the first passage we have used eq. (56). To write explicitly eq. (60), let us now compare the last term of eq. (60) with the right hand side of eq. (56) and note that now we have only the additional index k upon G^B; now eq. (56) was a compact form for eqs. (54) and (55); consequently, to write explicitly eq. (60) it will suffice to rewrite eqs. (54) and (55), substituting the F^A with G^{kA} and adding in the right hand side the term $\delta^{kj} F^A$. So we find

$$\frac{\partial}{\partial u^j} G^{ki_1 \cdots i_n} = \delta^{kj} F^{i_1 \cdots i_n} + n G^{k(i_1 \cdots i_{n-1}} \delta^{i_n)j} \quad \text{for} \quad n = 1, \cdots , N ,$$

$$\frac{\partial}{\partial u^j} G_R^{ki_1 \cdots i_R} = \delta^{kj} F_R^{i_1 \cdots i_R} + G_{R+1}^{kj_1 j_2 (i_2 \cdots i_R} \left[(N + M + 1 - 2R) \delta_{j_1}^{i_1)} \delta_{j_2}^{j} + R \delta^{i_1)j} \delta_{j_1 j_2} \right]$$
$$\text{for} \quad R = 1, \cdots , M - 1 , \quad (61)$$

$$\frac{\partial}{\partial u^j} G_M^{ki_1 \cdots i_M} = \delta^{kj} F_M^{i_1 \cdots i_M}$$
$$+ G^{kj_1 j_2 l_1 l_1 \cdots l_{\frac{N-M-1}{2}} l_{\frac{N-M-1}{2}} (i_2 \cdots i_M} \left[(N - M + 1) \delta_{j_1}^{i_1)} \delta_{j_2}^{j} + M \delta^{i_1)j} \delta_{j_1 j_2} \right] .$$

3.1. Application of the Properties to the Fluxes

Let us focus now our attention to the decompositions $(43)_2$ and $(44)_2$ for the fluxes. We firstly note that $(43)_2$ is an identity when $n = 0$ because $G^{ki_1 \cdots i_n} - \frac{F^k}{F} F^{i_1 \cdots i_n}$ is zero for this value of n because $G^k = F^k$ for physical reasons. Similarly, $G^{Ikj_1 \cdots j_s} - \frac{F^{Ik}}{F^I} F^{Ij_1 \cdots j_s}$ is zero for $s = 0$.

For the other values of n, the decompositions $(43)_2$ and $(44)_2$ are assumed as hypothesis. But, in the particular case of ideal gases, other symmetry conditions holds besides those already indicated above; they are

$$F^{i_1 \cdots i_{n+1}} = G^{i_1 \cdots i_{n+1}} \quad \text{for}\, n = 0, \cdots , N - 1 , \quad (62)$$
$$F_M^{i_1 \cdots i_M} = G^{i_1 \cdots i_{N+1}} \delta_{i_{M+1} i_{M+2}} \cdots \delta_{i_N i_{N+1}} ,$$

$$F_R^{i_1\cdots i_R} = G^{i_1\cdots i_{N+M+1-R}}\delta_{i_{R+1}i_{R+2}}\cdots\delta_{i_{N+M-R}i_{N+M+1-R}} = G_{R+1}^{i_1\cdots i_R ll}$$
for $R = 0, \cdots, M-1$;

moreover, the fluxes G^{kA} must be symmetric with respect to all pair of indexes.

The first one of these relations expresses the fact that the independent variable in the eq. $(2)_1$ is the flux in the equation with the previous value of n. For $n = N$ we would have $F^{i_1\cdots i_{N+1}}$ which appears in $(2)_2$ but only through its contraction with a number $\frac{N-M+1}{2}$ of symbols $\delta_{\dots}$; this is the reason for eq. $(62)_2$. Similarly, for $n = N+M-R$ and $R = 0, \cdots, M-1$ we would have $F^{i_1\cdots i_{N+M+1-R}}$ which appears in $(2)_2$ but only through its contraction with a number $\frac{N+M+1}{2} - R$ of symbols $\delta_{\dots}$; this is the reason for eq. $(62)_3$. These equations may be interpreted by saying that the moments for $n \geq 1$ are expressed in terms of the fluxes. Because of them, we cannot simply say that the decompositions $(43)_2$ and $(44)_2$ are assumptions, but something needs to be proved. More precisely, different interpretations are possible:

1. A first possible consideration which comes out eqs. (62) is that also the fluxes $G^{i_1\cdots i_{n+1}}$ are moments; it is true that this can be said only for $n = 0, \cdots, N-1$ but N is arbitrary so that they would be moments if we had considered from the beginning an higher value of N. Consequently, also the fluxes change according to the law $(43)_1$, that is, with

$$G^{ki_1\cdots i_n} = \sum_{s=0}^{n+1} S^{ki_1\cdots i_n}_{j_1\cdots j_s}(\vec{u})G^{Ij_1\cdots j_s}\,, \tag{63}$$

   with $G = F = G^I = F^I$.

   In this case, we have to prove that (63) is equivalent to $(43)_2$ and that from (63) and $(43)_1$ for $n = 0, \cdots, N+M+1$ we can deduce eqs. (44), (62) and the symmetry of G^{kA}.

2. We may observe that the in the previous approach are present also moments and fluxes which do not intervene in the balance equations except that trough their traces. Consequently, it is more correct to use $(43)_1$ and $(44)_1$ as transformation law for the moments; while the transformation law for the fluxes is given by (63) for $n = 0, \cdots, N$ and by its implication for $G_R^{ki_1\cdots i_R}$ which is given by the subsequent eq. (65).

   Starting only from these laws, we have to prove $(44)_2$, (62) and the symmetry of G^{kA} in the sense that they hold in the generic reference frame if and only if they hold in the frame denoted by the apex I.

3. We may observe that the in the previous approach we have used eq. (65) which was obtained by initially considering also components of fluxes which do not intervene in the balance equations, after that letting them disappear

when the pertinent traces are taken. Therefore, it is more self consistent to use eqs. $(43)_1$ and $(44)_1$ as transformation law for the moments, $(43)_2$ and $(44)_2$ as transformation law for the fluxes; after that we have to prove that (62) and the symmetry of G^{kA} hold in the generic reference frame if and only if they hold in the frame denoted by the apex I.

Proofs for the item 1). Let us firstly note that the symmetry of G^{kA} is evident from (63) and the symmetry of $S^{ki_1\cdots i_n}_{j_1\cdots j_s}$. Moreover, from (63) and $(43)_1$ it is evident that $(62)_1$ holds also for $n = 0, \cdots, N+M$. After that, $(62)_{2,3}$ are simply the definition of $F^{i_1\cdots i_R}_R$ for $R = 0, \cdots, M$ as given at the beginning of sect. 3.

Let us now prove that (63) is equivalent to $(43)_2$. To this end, let us use eq. (22) with $K^{j_1\cdots j_s} = G^{Ij_1\cdots j_s}$; it follows

$$G^{i_1\cdots i_{n+1}} - u^{i_{n+1}} F^{i_1\cdots i_n} = \sum_{s=0}^{n} S^{i_1\cdots i_n}_{j_1\cdots j_s} G^{Ii_{n+1}j_1\cdots j_s}, \tag{64}$$

where $F^{i_1\cdots i_n}$ has substituted $G^{i_1\cdots i_n}$ thanks to $(62)_1$. The sum of this relation and of $(43)_1$ premultiplied by $-\frac{F^{Ik}}{F^I}$ gives $(43)_2$. Vice versa, the sum of $(43)_2$ and of $(43)_1$ premultiplied by $\frac{F^{Ik}}{F^I}$ gives (64); this, thanks to eq. (22), leads to eq. (63).

There remains to prove eqs. (44), but this is exactly the way in which they were introduced at the beginning of sect. 3, so that there is nothing more to prove.

Proofs for the item 2). Let us begin by finding the implication of (63) for $G^{ki_1\cdots i_R}_R$. Eq. (63) with $n = N+M+1-R$, written with $k = i_{M+N+2-R}$ and after that contracted with $\delta_{i_{R+2}i_{R+3}} \cdots \delta_{i_{M+N+1-R}i_{M+N+2-R}}$, gives

$$G^{i_1\cdots i_{R+1}l_1l_1\cdots l_{\frac{N+M+1}{2}-R}l_{\frac{N+M+1}{2}-R}} = \sum_{s=0}^{N+1} S^{i_1\cdots i_{R+1}l_1l_1\cdots l_{\frac{N+M+1}{2}-R}l_{\frac{N+M+1}{2}-R}}_{j_1\cdots j_s} G^{Ij_1\cdots j_s}$$
$$+ \sum_{s=N+2}^{N+M+2-R} Z^{i_1\cdots i_{R+1}}_{j_1\cdots j_{N+M+3-s}} G^{Ij_1\cdots j_{N+M+3-s}l_1l_1\cdots l_{s-\frac{N+M+3}{2}}l_{s-\frac{N+M+3}{2}}},$$

where we have used (19). By changing index in the second summation, according to $s = N+M+2-S$, we obtain

$$G^{i_1\cdots i_{R+1}l_1l_1\cdots l_{\frac{N+M+1}{2}-R}l_{\frac{N+M+1}{2}-R}} = \sum_{s=0}^{N+1} S^{i_1\cdots i_{R+1}l_1l_1\cdots l_{\frac{N+M+1}{2}-R}l_{\frac{N+M+1}{2}-R}}_{j_1\cdots j_s} G^{Ij_1\cdots j_s}$$
$$+ \sum_{S=R}^{M} Z^{i_1\cdots i_{R+1}}_{j_1\cdots j_{S+1}} G^{Ij_1\cdots j_{S+1}l_1l_1\cdots l_{\frac{N+M+1}{2}-S}l_{\frac{N+M+1}{2}-S}}, \tag{65}$$

that is

$$G^{i_1\cdots i_{R+1}}_R = \sum_{s=0}^{N+1} S^{i_1\cdots i_{R+1}l_1l_1\cdots l_{\frac{N+M+1}{2}-R}l_{\frac{N+M+1}{2}-R}}_{j_1\cdots j_s} G^{Ij_1\cdots j_s} + \sum_{S=R}^{M} Z^{i_1\cdots i_{R+1}}_{j_1\cdots j_{S+1}} G^{Ij_1\cdots j_{S+1}}_S.$$

Also here we appreciate the correspondence between the fluxes in a given reference frame

$$\begin{cases} G^{i_1\cdots i_n} & \text{for}\, n=1,\cdots,N+1\,, \\ G_R^{i_1\cdots i_{R+1}} & \text{for}\, R=0,\cdots,M \end{cases}$$

and their counterparts in the other reference frame.

Moreover, we see that eq. (65) for the fluxes is the counterpart of $(44)_1$ for the moments.

We note now that the symmetry of G^{kA} is evident from eq. (63) for $n = 0, \cdots, N$, eq. (65), the symmetry of $S^{k\cdots}_{\cdots}$ and the symmetry of $Z^{\cdots}_{\cdots}$.

Moreover, from (63) for $n = 0, \cdots, N-1$ and $(43)_1$ it is evident that $(62)_1$ holds. Regarding $F_M^{i_1\cdots i_M}$ it was defined at the beginning of sect. 3 as $F_M^{i_1\cdots i_M} = \sum_{s=0}^{N+1} S_{j_1\cdots j_s}^{i_1\cdots i_M l_1 l_1 l_{\frac{N-M+1}{2}} l_{\frac{N-M+1}{2}}} F^{Ij_1\cdots j_s}$ (see the first line of eq. $(44)_1$ with$R = M$); by using (63) for $n = N$ it is so evident that $(62)_2$ holds.

Regarding $(62)_3$, let us write eq. $(65)_2$ with $R+1$ instead of R and let us take the trace of the result; so we obtain

$$G_{R+1}^{i_1\cdots i_R ll} = \sum_{s=0}^{N+1} S_{j_1\cdots j_s}^{i_1\cdots i_R l_1 l_1\cdots l_{\frac{N+M+1}{2}-R} l_{\frac{N+M+1}{2}-R}} G^{Ij_1\cdots j_s} + \sum_{S=R+1}^{M} Z_{j_1\cdots j_{S+1}}^{i_1\cdots i_R ll} G_S^{Ij_1\cdots j_{S+1}}$$
$$= \sum_{s=0}^{N} S_{j_1\cdots j_s}^{i_1\cdots i_R l_1 l_1\cdots l_{\frac{N+M+1}{2}-R} l_{\frac{N+M+1}{2}-R}} G^{Ij_1\cdots j_s} + S_{j_1\cdots j_{N+1}}^{i_1\cdots i_R l_1 l_1\cdots l_{\frac{N+M+1}{2}-R} l_{\frac{N+M+1}{2}-R}} G^{Ij_1\cdots j_{N+1}}$$
$$+ \sum_{S=R+1}^{M} X_{j_1\cdots j_{S-1}}^{i_1\cdots i_R} G_S^{Ij_1\cdots j_{S-1} ll}\,, \quad (66)$$

where from the first summation we have written explicitly the value for $s = N+1$ and for the second summation we have used the Property 24, that is eq. (31). Now, for the first term of the above result we use eq. $(62)_1$, for the second term we use (6) with $s = N+1$ and $(62)_2$, for the third we change index according to $S = 1+s$; so we obtain

$$G_{R+1}^{i_1\cdots i_R ll} = \sum_{s=0}^{N} S_{j_1\cdots j_s}^{i_1\cdots i_R l_1 l_1\cdots l_{\frac{N+M+1}{2}-R} l_{\frac{N+M+1}{2}-R}} F^{Ij_1\cdots j_s} + X_{j_1\cdots j_M}^{i_1\cdots i_R} F_M^{Ij_1\cdots j_M}$$
$$+ \sum_{s=R}^{M-1} X_{j_1\cdots j_s}^{i_1\cdots i_R} G_{s+1}^{Ij_1\cdots j_s ll}\,.$$

Now the last two terms may be put together in a unique $\sum_{s=R}^{M}$, while the other one thanks to eq. (16) and $(44)_1$ is equal to $F_R^{i_1\cdots i_R} - \sum_{s=R}^{M} X_{j_1\cdots j_s}^{i_1\cdots i_R} F_s^{Ij_1\cdots j_s}$. In other words, we have obtained

$$G_{R+1}^{i_1\cdots i_R ll} - F_R^{i_1\cdots i_R} = \sum_{s=R}^{M} X_{j_1\cdots j_s}^{i_1\cdots i_R}(G_{s+1}^{Ij_1\cdots j_s ll} - F_s^{Ij_1\cdots j_s}) \,. \tag{67}$$

It follows that, if $(62)_3$ is satisfied in the reference frame denoted with the apex I, then it is satisfied also in the other one; the vice versa follows from Property 6, that is eq. (11).

There remains now to prove $(44)_2$. To this end, let us begin by using eq. (13) for the coefficient of $G^{Ij_1\cdots j_s}$ in $(65)_2$ so transforming it into

$$S^{i_1\cdots i_R l_1 l_1\cdots l_{\frac{N+M+1}{2}-R} l_{\frac{N+M+1}{2}-R}} u^{i_{N+M+2-R}} G^I + \sum_{s=1}^{N+1} S_{j_1\cdots j_s}^{i_1\cdots i_R l_1 l_1\cdots l_{\frac{N+M+1}{2}-R} l_{\frac{N+M+1}{2}-R}} \cdot$$
$$\cdot\, u^{i_{N+M+2-R}} G^{Ij_1\cdots j_s} + \sum_{s=1}^{N+1} S_{(j_1\cdots j_{s-1}}^{i_1\cdots i_R l_1 l_1\cdots l_{\frac{N+M+1}{2}-R} l_{\frac{N+M+1}{2}-R}} \delta_{j_s)}^{i_{N+M+2-R}} G^{Ij_1\cdots j_s}$$

(Obviously, we have here $i_{N+M+2-R}$ instead of i_{R+1}). Now, the first one of these terms may be included in the second one for $s=0$, from the second term we may exclude the value for $s=N+1$ and write it explicitly as a supplementary term; finally, in the third term we change index according to the rule $s=1+S$. In this way the above expression becomes

$$\sum_{s=0}^{N} S_{j_1\cdots j_s}^{i_1\cdots i_R l_1 l_1\cdots l_{\frac{N+M+1}{2}-R} l_{\frac{N+M+1}{2}-R}} u^{i_{N+M+2-R}} G^{Ij_1\cdots j_s} + \tag{68}$$
$$+ S_{j_1\cdots j_{N+1}}^{i_1\cdots i_R l_1 l_1\cdots l_{\frac{N+M+1}{2}-R} l_{\frac{N+M+1}{2}-R}} u^{i_{N+M+2-R}} G^{Ij_1\cdots j_{N+1}} +$$
$$+ \sum_{S=0}^{N} S_{j_1\cdots j_S}^{i_1\cdots i_R l_1 l_1\cdots l_{\frac{N+M+1}{2}-R} l_{\frac{N+M+1}{2}-R}} G^{Ij_1\cdots j_S i_{N+M+2-R}} \,.$$

We transform now the last term on the right hand side of eq. $(65)_2$ by using (21) with $s=N+M+2-S$, $i_{R+1}=i_{N+M+2-R}$; in this way it becomes

$$\sum_{S=R+1}^{M} \left(X_{(j_1\cdots j_{S-1}}^{i_1\cdots i_R} \delta_{j_S j_{S+1})} u^{i_{N+M+2-R}} + X_{(j_1\cdots j_S}^{i_1\cdots i_R} \delta_{j_{S+1})}^{i_{N+M+2-R}} \right) G_S^{Ij_1\cdots j_{S+1}} +$$
$$+ \delta_{(j_1}^{i_1} \cdots \delta_{j_R}^{i_R} \delta_{j_{R+1})}^{i_{N+M+2-R}} G_R^{Ij_1\cdots j_{R+1}} \,. \tag{69}$$

So we have obtained that $G_R^{i_1\cdots i_{R+1}}$ is sum of (68) and (69); in this sum, the coefficient of $u^{i_{N+M+2-R}}$ is

$$\sum_{S=0}^{N} S_{j_1\cdots j_S}^{i_1\cdots i_R l_1 l_1\cdots l_{\frac{N+M+1}{2}-R} l_{\frac{N+M+1}{2}-R}} G^{Ij_1\cdots j_S} + S_{j_1\cdots j_{N+1}}^{i_1\cdots i_R l_1 l_1\cdots l_{\frac{N+M+1}{2}-R} l_{\frac{N+M+1}{2}-R}} G^{Ij_1\cdots j_{N+1}}$$

$$+ \sum_{S=R+1}^{M} X_{j_1\cdots j_{S-1}}^{i_1\cdots i_R} \delta_{j_S j_{S+1}} G_S^{Ij_1\cdots j_{S+1}}$$

$$= \sum_{S=0}^{N} S_{j_1\cdots j_S}^{i_1\cdots i_R l_1 l_1\cdots l_{\frac{N+M+1}{2}-R} l_{\frac{N+M+1}{2}-R}} G^{Ij_1\cdots j_S} + X_{(j_1\cdots j_M}^{i_1\cdots i_R} \delta_{j_{M+1} j_{M+2}} \cdots \delta_{j_N j_{N+1})} G^{Ij_1\cdots j_{N+1}}$$

$$+ \sum_{s=R}^{M-1} X_{j_1\cdots j_s}^{i_1\cdots i_R} G_{s+1}^{Ij_1\cdots j_s ll}$$

where in the first passage for the second term we have used eq. (6) with $s = N+1$, and for the third term we have changed index according to $S = 1+s$; in the second passage we have noted that the second term may be included in the last one for $s = M$. Now, by using (62) our expression becomes

$$\sum_{S=0}^{N} S_{j_1\cdots j_S}^{i_1\cdots i_R l_1 l_1\cdots l_{\frac{N+M+1}{2}-R} l_{\frac{N+M+1}{2}-R}} F^{Ij_1\cdots j_S} + X_{j_1\cdots j_M}^{i_1\cdots i_R} F_M^{Ij_1\cdots j_M}$$

$$+ \sum_{s=R}^{M-1} X_{j_1\cdots j_s}^{i_1\cdots i_R} F_s^{Ij_1\cdots j_s} = F_R^{i_1\cdots i_R} ,$$

where we have used $(44)_1$. So, until now we have obtained that $G_R^{i_1\cdots i_R i_{N+M+2-R}} - u^{i_{N+M+2-R}} F_R^{i_1\cdots i_R}$ is sum of the remaining terms of (68) and (69), that is

$$\sum_{S=0}^{N} S_{j_1\cdots j_S}^{i_1\cdots i_R l_1 l_1\cdots l_{\frac{N+M+1}{2}-R} l_{\frac{N+M+1}{2}-R}} G^{Ij_1\cdots j_S i_{N+M+2-R}}$$

$$+ \sum_{S=R+1}^{M} X_{j_1\cdots j_S}^{i_1\cdots i_R} G_S^{Ij_1\cdots j_S i_{N+M+2-R}} + G_R^{Ii_1\cdots i_R i_{N+M+2-R}} . \quad (70)$$

Now the last one of these terms may be included in the previous one for $S = R$, while for the first term eq. (16) may be used. In this way we find

$$G_R^{ki_1\cdots i_R} - u^k F_R^{i_1\cdots i_R} = \sum_{s=0}^{N} Y_{j_1\cdots j_s}^{i_1\cdots i_R} G^{Ikj_1\cdots j_s} + \sum_{S=R}^{M} X_{j_1\cdots j_S}^{i_1\cdots i_R} G_S^{Ikj_1\cdots j_S} .$$

The sum of this equation and of eq. $(44)_1$ premultiplied by $-\frac{F^{Ik}}{F}$ gives eq. $(44)_2$, if we take into account that $\frac{F^k}{F} = \frac{F^{Ik}+F^I u^k}{F^I} = \frac{F^{Ik}}{F^I} + u^k$. This completes the proof of the item 2).

Proofs for the item 3). Let us assume that (62) and the symmetry of G^{kA} hold in the reference frame denoted by the apex I and prove that this is true also in the generic reference frame. Starting from $(43)_2$ and using $(43)_1$ we obtain

$$G^{ki_1\cdots i_n} = \sum_{s=0}^{n} S^{i_1\cdots i_n}_{j_1\cdots j_s}(G^{Ikj_1\cdots j_s} + u^k F^{Ij_1\cdots j_s})$$
$$= \sum_{s=0}^{n} S^{i_1\cdots i_n}_{j_1\cdots j_s}(G^{Ikj_1\cdots j_s} + u^k G^{Ij_1\cdots j_s})\,, \quad (71)$$

where in the second passage we have substituted $F^{Ij_1\cdots j_s}$ with $G^{Ij_1\cdots j_s}$ because they are equal in the reference frame denoted by the apex I. Now we use the Property 15 with $K^{j_1\cdots j_s} = G^{Ij_1\cdots j_s}$ (These are symmetric because belong to the reference frame denoted by the apex I) and obtain

$$\sum_{s=0}^{n+1} S^{i_1\cdots i_n k}_{j_1\cdots j_s} G^{Ij_1\cdots j_s} = u^k \sum_{s=0}^{n} S^{i_1\cdots i_n}_{j_1\cdots j_s} G^{Ij_1\cdots j_s} + \sum_{s=0}^{n} S^{i_1\cdots i_n}_{j_1\cdots j_s} G^{Ij_1\cdots j_s k}\,. \quad (72)$$

By comparing with (71) we obtain

$$G^{ki_1\cdots i_n} = \sum_{s=0}^{n+1} S^{i_1\cdots i_n k}_{j_1\cdots j_s} G^{Ij_1\cdots j_s}\,, \quad (73)$$

from which the symmetry of $G^{ki_1\cdots i_n}$ follows. If $n \leq N-1$ we may substitute again $G^{Ij_1\cdots j_s}$ with $F^{Ij_1\cdots j_s}$ from which $(62)_1$ easily follows. Instead of this, if $n = N$, we firstly isolate from (73) the term with $s = N+1$ and in the others we substitute $G^{Ij_1\cdots j_s}$ with $F^{Ij_1\cdots j_s}$; so it becomes

$$G^{ki_1\cdots i_N} = \sum_{s=0}^{N} S^{i_1\cdots i_N k}_{j_1\cdots j_s} F^{Ij_1\cdots j_s} + G^{Ii_1\cdots i_N k}\,;$$

After that, we contract this last equation with $\delta_{i_M i_{M+1}} \cdots \delta_{i_{N-1} i_N}$ so obtaining

$$G^{ki_1\cdots i_N}\delta_{i_M i_{M+1}} \cdots \delta_{i_{N-1} i_N} = F^{i_1\cdots i_M}_M - F^{Ii_1\cdots i_M}_M + G^{Ii_1\cdots i_N k}\delta_{i_M i_{M+1}} \cdots \delta_{i_{N-1} i_N}\,,$$

where we have used $(44)_1$ with $R = M$ and the property $X^{i_1\cdots i_M}_{j_1\cdots j_M} = \delta^{i_1}_{(j_1} \cdots \delta^{i_M}_{j_M)}$ which comes out from eq. (6) with $R = M$, $s = N+1$ and from the Lemma. The resulting equation shows that eq. $(62)_2$ holds in the generic reference frame because it holds in that denoted with the apex I.

Let us prove now eq. $(62)_3$.

From eq. $(44)_2$, taking $-\frac{F^k}{F} F^{i_1\cdots i_R}_R$ in the right hand side and using eq. $(44)_1$, we find

$$G_R^{ki_1\cdots i_R} = \sum_{s=0}^{N} Y^{i_1\cdots i_R}_{j_1\cdots j_s}\left(G^{Ikj_1\cdots j_s} + u^k F^{Ij_1\cdots j_s}\right) + \sum_{s=R}^{M} X^{i_1\cdots i_R}_{j_1\cdots j_s}\left(G_s^{Ikj_1\cdots j_s} + u^k F_s^{Ij_1\cdots j_s}\right) . \quad (74)$$

We now write this equation with $R+1$ instead of R and take the trace of the result, so finding

$$G_{R+1}^{ki_1\cdots i_{R-1}ll} = \sum_{s=0}^{N} Y^{i_1\cdots i_{R-1}ll}_{j_1\cdots j_s}\left(G^{Ikj_1\cdots j_s} + u^k F^{Ij_1\cdots j_s}\right) + \sum_{s=R+1}^{M} X^{i_1\cdots i_{R-1}ll}_{j_1\cdots j_s}\left(G_s^{Ikj_1\cdots j_s} + u^k F_s^{Ij_1\cdots j_s}\right) .$$

This equation, thanks to eqs. (23), (24) and $(62)_{1,2}$ becomes

$$\begin{aligned} G_{R+1}^{ki_1\cdots i_{R-1}ll} &= \sum_{s=0}^{N-1} Y^{i_1\cdots i_{R-1}ll}_{j_1\cdots j_s}\left(F^{Ikj_1\cdots j_s} + u^k F^{Ij_1\cdots j_s}\right) + \\ &+ T^{i_1\cdots i_{R-1}}_{j_1\cdots j_{M-1}}\left(F_M^{Ikj_1\cdots j_{M-1}} + u^k F^{Ij_1\cdots j_{M-1}l_1l_1\cdots l_{\frac{N-M+1}{2}}l_{\frac{N-M+1}{2}}}\right) + \\ &+ \sum_{s=R+1}^{M} \eta^{i_1\cdots i_{R-1}}_{j_1\cdots j_{s-2}}\left(G_s^{Ikj_1\cdots j_{s-2}ll} + u^k F_s^{Ij_1\cdots j_{s-2}ll}\right) . \end{aligned}$$

Adding to both sides of this equality the quantity

$$-F_R^{ki_1\cdots i_{R-1}} - \sum_{s=R+1}^{M} \eta^{i_1\cdots i_{R-1}}_{j_1\cdots j_{s-2}}\left(G_s^{Ikj_1\cdots j_{s-2}ll} - F_{s-1}^{Ikj_1\cdots j_{s-2}}\right)$$

and using eq. $(44)_1$, we find

$$\begin{aligned} &G_{R+1}^{ki_1\cdots i_{R-1}ll} - F_R^{ki_1\cdots i_{R-1}} - \sum_{s=R+1}^{M} \eta^{i_1\cdots i_{R-1}}_{j_1\cdots j_{s-2}}\left(G_s^{Ikj_1\cdots j_{s-2}ll} - F_{s-1}^{Ikj_1\cdots j_{s-2}}\right) = \\ &= \sum_{s=0}^{N-1} Y^{i_1\cdots i_{R-1}ll}_{j_1\cdots j_s}\left(F^{Ikj_1\cdots j_s} + u^k F^{Ij_1\cdots j_s}\right) + \\ &+ T^{i_1\cdots i_{R-1}}_{j_1\cdots j_{M-1}}\left(F_M^{Ikj_1\cdots j_{M-1}} + u^k F^{Ij_1\cdots j_{M-1}l_1l_1\cdots l_{\frac{N-M+1}{2}}l_{\frac{N-M+1}{2}}}\right) + \\ &+ \sum_{s=R+1}^{M} \eta^{i_1\cdots i_{R-1}}_{j_1\cdots j_{s-2}}\left(F_{s-1}^{Ikj_1\cdots j_{s-2}} + u^k F_s^{Ij_1\cdots j_{s-2}ll}\right) + \end{aligned}$$

$$-\sum_{s=0}^{N} Y^{ki_1\cdots i_{R-1}}_{j_1\cdots j_s} F^{Ij_1\cdots j_s} - \sum_{s=R}^{M} X^{ki_1\cdots i_{R-1}}_{j_1\cdots j_s} F^{Ij_1\cdots j_s}_s \,.$$

For the term $-\sum_{s=0}^{N} Y^{ki_1\cdots i_{R-1}}_{j_1\cdots j_s} F^{Ij_1\cdots j_s}$ let us write firstly the value for $s=0$ and, after that, let use eq. (26); so it becomes

$$-Y^{ki_1\cdots i_{R-1}} F^I - \sum_{s=1}^{N} u^k Y^{i_1\cdots i_{R-1}ll}_{j_1\cdots j_{s-1}} F^{Ij_1\cdots j_s} - \sum_{s=1}^{N} Y^{i_1\cdots i_{R-1}ll}_{j_1\cdots j_{s-1}} F^{Ikj_1\cdots j_{s-1}}$$
$$= -\sum_{s=0}^{N} u^k Y^{i_1\cdots i_{R-1}ll}_{j_1\cdots j_s} F^{Ij_1\cdots j_s} - \sum_{S=0}^{N-1} Y^{i_1\cdots i_{R-1}ll}_{j_1\cdots j_S} F^{Ikj_1\cdots j_S} \,,$$

where we have included the first term in the second one thanks to eq. (27), while for the third term we have changed index according to the law $s = 1 + S$. By using this partial result, our expression becomes

$$G^{ki_1\cdots i_{R-1}ll}_{R+1} - F^{ki_1\cdots i_{R-1}}_R - \sum_{s=R+1}^{M} \eta^{i_1\cdots i_{R-1}}_{j_1\cdots j_{s-2}} \left(G^{Ikj_1\cdots j_{s-2}ll}_s - F^{Ikj_1\cdots j_{s-2}}_{s-1} \right)$$
$$= \underline{-u^k Y^{i_1\cdots i_{R-1}ll}_{j_1\cdots j_N} F^{Ij_1\cdots j_N}} + T^{i_1\cdots i_{R-1}}_{j_1\cdots j_{M-1}} \left(F^{Ikj_1\cdots j_{M-1}}_M + \underline{u^k F^{Ij_1\cdots j_{M-1}l_1l_1\cdots l_{\frac{N-M+1}{2}} l_{\frac{N-M+1}{2}}}} \right)$$
$$+ \sum_{s=R+1}^{M} \eta^{i_1\cdots i_{R-1}}_{j_1\cdots j_{s-2}} \left(F^{Ikj_1\cdots j_{s-2}}_{s-1} + u^k F^{Ij_1\cdots j_{s-2}ll}_s \right) - \sum_{s=R}^{M} X^{ki_1\cdots i_{R-1}}_{j_1\cdots j_s} F^{Ij_1\cdots j_s}_s \,.$$

Now the underlined terms elide one another thanks to eq. (23), the coefficient of $F^{Ij_1\cdots j_M}_M$ in the right hand side is zero thanks to eq. (29), the coefficient of $F^{Ij_1\cdots j_R}_R$ is $\eta^{i_1\cdots i_{R-1}}_{(j_1\cdots j_{R-1}} \delta^k_{j_R)} - X^{ki_1\cdots i_{R-1}}_{j_1\cdots j_R}$ which is zero thanks to eq. (25), the coefficient of $F^{Ij_1\cdots j_s}_s$ for $s = R+1, \cdots, M-1$ is zero thanks to eq. (30). There remains

$$G^{ki_1\cdots i_{R-1}ll}_{R+1} - F^{ki_1\cdots i_{R-1}}_R - \sum_{s=R+1}^{M} \eta^{i_1\cdots i_{R-1}}_{j_1\cdots j_{s-2}} \left(G^{Ikj_1\cdots j_{s-2}ll}_s - F^{Ikj_1\cdots j_{s-2}}_{s-1} \right) = 0 \,.$$

This allows to conclude that eq.$(62)_3$ is satisfied in the generic reference frame if and only if it holds in the frame denoted by the apex I.

Let us conclude by proving that the symmetry of $G^{ki_1\cdots i_R}_R$ holds if it holds in the frame denoted by the apex I.

To this end we can rewrite eq. (74) using the symmetry conditions in the frame denoted by the apex I. We obtain

$$G^{ki_1\cdots i_R}_R = \sum_{s=0}^{N} Y^{i_1\cdots i_R}_{(j_1\cdots j_s} \delta^k_{j_{s+1})} G^{Ij_1\cdots j_s j_{s+1}} + \sum_{s=0}^{N} Y^{i_1\cdots i_R}_{j_1\cdots j_s} u^k F^{Ij_1\cdots j_s} +$$

$$+ \sum_{s=R}^{M} X^{i_1\cdots i_R}_{(j_1\cdots j_s}\delta^k_{j_{s+1})} G_s^{Ij_1\cdots j_s j_{s+1}} + \sum_{s=R}^{M} X^{i_1\cdots i_R}_{j_1\cdots j_s} u^k F_s^{Ij_1\cdots j_s} .$$

In this equation we substitute the moments obtained from eqs. (62) as functions of the fluxes; so it becomes

$$G_R^{ki_1\cdots i_R} = Y^{i_1\cdots i_R}_{(j_1\cdots j_N}\delta^k_{j_{N+1})} G^{Ij_1\cdots j_{N+1}} + \sum_{s=0}^{N-1} Y^{i_1\cdots i_R}_{(j_1\cdots j_s}\delta^k_{j_{s+1})} G^{Ij_1\cdots j_s j_{s+1}} + Y^{i_1\cdots i_R} u^k F^I +$$
$$+ \sum_{s=1}^{N} Y^{i_1\cdots i_R}_{j_1\cdots j_s} u^k G^{Ij_1\cdots j_s} + X^{i_1\cdots i_R}_{(j_1\cdots j_M}\delta^k_{j_{M+1})} G_M^{Ij_1\cdots j_{M+1}} + X^{i_1\cdots i_R}_{(j_1\cdots j_R}\delta^k_{j_{R+1})} G_s^{Ij_1\cdots j_R j_{R+1}} +$$
$$+ \sum_{s=R+1}^{M-1} X^{i_1\cdots i_R}_{(j_1\cdots j_s}\delta^k_{j_{s+1})} G_s^{Ij_1\cdots j_s j_{s+1}} + \sum_{s=R}^{M-2} X^{i_1\cdots i_R}_{j_1\cdots j_s} u^k G_{s+1}^{Ij_1\cdots j_s ll} +$$
$$+ X^{i_1\cdots i_R}_{j_1\cdots j_M} u^k G^{Ij_1\cdots j_{N+1}} \delta_{j_{M+1}j_{M+2}}\cdots\delta_{j_N j_{N+1}} + X^{i_1\cdots i_R}_{j_1\cdots j_{M-1}} u^k G_M^{Ij_1\cdots j_{M-1}ll} ,$$

where we have firstly isolated the value for $s = N$ of the first summation, the value with $s = 0$ of the second summation, the values with $s = M$ and $s = R$ of the third summation, and the values for $s = M$ and $s = M - 1$ of the last summation. Now in the first and third summation we change index according to the law $s + 1 = S$ and in the last summation according to $s = S - 2$; so our expression becomes

$$G_R^{ki_1\cdots i_R} =$$
$$(Y^{i_1\cdots i_R}_{(j_1\cdots j_N}\delta^k_{j_{N+1})} + X^{i_1\cdots i_R}_{(j_1\cdots j_M} u^k \delta_{j_{M+1}j_{M+2}}\cdots\delta_{j_N j_{N+1})}) G^{Ij_1\cdots j_{N+1}} + Y^{i_1\cdots i_R} u^k F^I +$$
$$+ \sum_{S=1}^{N} \left(Y^{i_1\cdots i_R}_{(j_1\cdots j_{S-1}}\delta^k_{j_S)} + Y^{i_1\cdots i_R}_{j_1\cdots j_s} u^k \right) G^{Ij_1\cdots j_s} + X^{i_1\cdots i_R}_{(j_1\cdots j_M}\delta^k_{j_{M+1})} G_M^{Ij_1\cdots j_{M+1}} +$$
$$+ X^{i_1\cdots i_R}_{(j_1\cdots j_R}\delta^k_{j_{R+1})} G_s^{Ij_1\cdots j_R j_{R+1}} + \sum_{S=R+2}^{M} X^{i_1\cdots i_R}_{(j_1\cdots j_{S-1}}\delta^k_{j_S)} G_{S-1}^{Ij_1\cdots j_S} +$$
$$+ \sum_{S=R+2}^{M} X^{i_1\cdots i_R}_{j_1\cdots j_{S-2}} u^k G_{S-1}^{Ij_1\cdots j_{S-2}ll} + X^{i_1\cdots i_R}_{j_1\cdots j_{M-1}} u^k G_M^{Ij_1\cdots j_{M-1}ll} .$$

This equation can be rearranged as

$$G_R^{ki_1\cdots i_R} = \left(Y^{i_1\cdots i_R}_{(j_1\cdots j_N}\delta^k_{j_{N+1})} + X^{i_1\cdots i_R}_{(j_1\cdots j_M} u^k \delta_{j_{M+1}j_{M+2}}\cdots\delta_{j_N j_{N+1})} \right) G^{Ij_1\cdots j_{N+1}}$$
$$+ Y^{i_1\cdots i_R} u^k F^I + + \sum_{s=1}^{N} \left(Y^{i_1\cdots i_R}_{(j_1\cdots j_{s-1}}\delta^k_{j_s)} + Y^{i_1\cdots i_R}_{j_1\cdots j_s} u^k \right) G^{Ij_1\cdots j_s} + G_s^{Ii_1\cdots i_R k}$$
$$+ \sum_{S=R+2}^{M} \left(X^{i_1\cdots i_R}_{(j_1\cdots j_{S-1}}\delta^k_{j_S)} + X^{i_1\cdots i_R}_{(j_1\cdots j_{S-2}}\delta_{j_{S-1}j_S)} u^k \right) G_{S-1}^{Ij_1\cdots j_S}$$

$$+\left(X^{i_1\cdots i_R}_{j_1\cdots j_{M-1}}\delta_{j_M j_{M+1}}u^k + X^{i_1\cdots i_R}_{(j_1\cdots j_M}\delta^k_{j_{M+1})}\right)G^{Ij_1\cdots j_{M+1}}_M .$$

and finally,

$$G^{ki_1\cdots i_R}_R = S^{ki_1\cdots i_R l_1 l_1\cdots l_{\frac{N+M+1}{2}-R} l_{\frac{N+M+1}{2}-R}}_{j_1\cdots j_{N+1}} G^{Ij_1\cdots j_{N+1}} + u^{i_1}\cdots u^{i_R}(u^2)^{\frac{N+M+1}{2}-R}u^k F^I$$
$$+\sum_{s=1}^{N} S^{ki_1\cdots i_R l_1 l_1\cdots l_{\frac{N+M+1}{2}-R} l_{\frac{N+M+1}{2}-R}}_{j_1\cdots j_s} G^{Ij_1\cdots j_s} + G^{Ii_1\cdots i_R k}_s$$
$$+\sum_{S=R+2}^{M} Z^{i_1\cdots i_R k}_{j_1\cdots j_S} G^{Ij_1\cdots j_S}_{S-1} + Z^{i_1\cdots i_R k}_{j_1\cdots j_{M+1}} G^{Ij_1\cdots j_{M+1}}_M .$$

where eqs. $(32)_2$, $(27)_2$, $(32)_1$, (33) have been used. We appreciate that the right hand side of this equation is symmetric and this completes our proof.

4. Application of the above Properties in Connection with the Entropy Principle

The entropy principle states that the supplementary law

$$\partial_t h + \partial_k h^k = \sigma \geq 0 , \tag{75}$$

holds for every solution of eqs. (2), where h is the entropy density and h^k its flux. Now the Galilean relativity principle, besides the above seen laws of transformation of moments and fluxes, imposes that h and $h^k - h\frac{F^k}{F}$, composite functions through eq. $(47)_1$, are non convective quantities in the sense that they do not depend on u^j. In other words, they do not depend on the reference frame. To impose this restriction we have simply to say that their derivatives with respect to u^j are zero, that is

$$\frac{\partial h}{\partial F^A} M^{jA}{}_B F^B = 0 \quad , \quad \frac{\partial h^k}{\partial F^A} M^{jA}{}_B F^B - h\delta^{kj} = 0 , \tag{76}$$

where we have used (56).

The equations (76) do not depend on the reference frame; in fact, by substituting F^B from $(47)_1$ and using the property (50), they become

$$0 = \frac{\partial h}{\partial F^A} M^{jA}{}_B X^B{}_C F^{IC} = \frac{\partial h}{\partial F^A} X^A{}_B M^{jB}{}_C F^{IC} = \frac{\partial h}{\partial F^{IB}} M^{jB}{}_C F^{IC} ,$$

$$0 = \frac{\partial h^k}{\partial F^A} M^{jA}{}_B X^B{}_C F^{IC} - h\delta^{kj} = \frac{\partial h^k}{\partial F^A} X^A{}_B M^{jB}{}_C F^{IC} - h\delta^{kj}$$

$$= \frac{\partial h^k}{\partial F^{IB}} M^{jB}{}_C F^{IC} - h\delta^{kj} \,,$$

because, for the derivation rule of composite functions, we have $\frac{\partial \cdots}{\partial F^{IB}} = \frac{\partial \cdots}{\partial F^A} \frac{\partial F^A}{\partial F^{IB}} = \frac{\partial \cdots}{\partial F^A} X^A{}_B$.

For what regards the general solution of (76), without use of approximations, we remand to the paper [16].

Let us now come back to the entropy principle which states that the supplementary law (75) holds for every solution of eqs. (2) which can be written in the compact form $\partial_t F^A + \partial_k G^{kA} = Q^A$. For Liu' s Theorem [21] this is equivalent to assume the existence of Lagrange Multipliers λ_A such that

$$\partial_t h + \partial_k h^k - \sigma - \lambda_A \left(\partial_t F^A + \partial_k G^{kA} - Q^A \right) = 0 \,,$$

holds whatever value of the independent variables, so that

$$d\,h - \lambda_A \, d\,F^A = 0 \,,\; d\,h^k - \lambda_A \, d\,G^{kA} = 0 \,, -\sigma + \lambda_A Q^A = 0 \,, \tag{77}$$

which are equivalent to

$$\lambda_A = \frac{\partial h}{\partial F^A} \quad , \quad \frac{\partial h^k}{\partial F^B} = \lambda_A \frac{\partial G^{kA}}{\partial F^B} \,,\; \sigma = \lambda_A Q^A \geq 0 \,. \tag{78}$$

By substituting F^A from $(47)_1$ into $(77)_1$ it follows

$$\lambda_A X^A{}_B(\vec{u}) d\,F^{IB} + \lambda_A M^{jA}{}_C X^C{}_B(\vec{u}) F^{IB} d\,u^j = \frac{\partial h}{\partial F^{IB}} d\,F^{IB} \,. \tag{79}$$

This suggests to define

$$\lambda_A X^A{}_B(\vec{u}) = \lambda^I_B \,, \tag{80}$$

so that (79) is equivalent to

$$\lambda^I_B = \frac{\partial h}{\partial F^{IB}} \quad , \quad \lambda_A M^{jA}{}_C F^C = 0 \,. \tag{81}$$

The first one of these is the counterpart of (78) in the other reference frame; also the second one is independent on the frame, because we have

$$\lambda_A M^{jA}{}_C F^C = \lambda_E X^E{}_B(\vec{u}) X^B{}_A(-\vec{u}) M^{jA}{}_C F^C = \lambda^I_B M^{jB}{}_A X^A{}_C(-\vec{u}) F^C$$
$$= \lambda^I_B M^{jB}{}_A F^{IA} \,,$$

so that $(81)_2$ holds iff the following relation is satisfied

$$\lambda^I_B M^{jB}{}_A F^{IA} = 0 \,. \tag{82}$$

Now this last relation, thanks to $(81)_1$ can be written as

$$\frac{\partial h}{\partial F^{IB}} M^{jB}{}_A F^{IA} = 0 \, ,$$

which is the counterpart of $(76)_1$ in the other frame. We observe also that eq. (80) contracted with $X^B{}_C(-\vec{u})$ becomes

$$\lambda_C = X^B{}_C(-\vec{u}) \lambda^I_B \, . \tag{83}$$

The equation (81) suggests to call λ^I_B the non convective Lagrange multipliers; after that, (83) gives the decomposition of the Lagrange multipliers into non convective and velocity dependent parts.

In [22] the idea has been conceived to define the 4-potentials h' , h'^k as

$$h' = -h + \lambda_A F^A \, , \; h'^k = -h^k + \lambda_A G^{kA} \, , \tag{84}$$

so that eqs. (77) become

$$d\, h' = F^A d\, \lambda_A \, , \; d\, h'^k = G^{kA} d\, \lambda_A \, , \; \sigma = \lambda_A Q^A \, . \tag{85}$$

Another idea exposed in [22] is to take the Lagrange Multipliers as independent variables and call them "main field"; in this way (85) become

$$F^A = \frac{\partial h'}{\partial \lambda_A} \quad , \quad G^{kA} = \frac{\partial h'^k}{\partial \lambda_A} \, , \tag{86}$$

so that everything is determined in terms of h' and h'^k; moreover, by substituting eqs. (86) in (2) we obtain a symmetric system of evolution equations which is hyperbolic if h' is a convex function of the main field.

The equation $(76)_1$, thanks to $(78)_1$ and $(86)_1$, becomes

$$\lambda_A M^{jA}{}_B \frac{\partial h'}{\partial \lambda_B} = 0 \, . \tag{87}$$

For what concerns $(76)_2$, let us firstly transform it still remaining with the independent variables F^A. By substituting $\frac{\partial h^k}{\partial F^A}$ from $(78)_2$, it becomes $\lambda_C \frac{\partial G^{kC}}{\partial F^A} M^{jA}{}_B F^B - h\delta^{kj} = 0$; this, thanks to (56), becomes

$$\begin{aligned} &\lambda_C \frac{\partial G^{kC}}{\partial u^j} - h\delta^{kj} = 0 \, , \\ \text{or} \quad &\lambda_A M^{jA}{}_B G^{kB} + \left(\lambda_A F^A - h\right) \delta^{kj} = 0 \, , \end{aligned} \tag{88}$$

where, for the second one of these, we have used (60). Finally, taking the Lagrange Multipliers as independent variables and using $(86)_2$ and $(84)_1$ our equation $(88)_2$ becomes

$$\lambda_A M^{jA}{}_B \frac{\partial h'^k}{\partial \lambda_B} + h' \delta^{kj} = 0 \, . \tag{89}$$

If we want to write explicitly this condition, starting from its expression $(88)_1$ and using (61), we obtain

$$\begin{aligned}&\left(\lambda_A F^A - h\right)\delta^{kj} + \sum_{n=1}^{N} n\lambda_{i_1\cdots i_{n-1}j} G^{ki_1\cdots i_{n-1}} \\ &\quad + \sum_{R=0}^{M-1} \lambda^R_{i_1\cdots i_R} G^{kj_1j_2i_2\cdots i_R}_{R+1}\left[(N+M+1-2R)\delta^{i_1}_{j_1}\delta^{j}_{j_2} + R\delta^{i_1j}\delta_{j_1j_2}\right] \\ &+ \lambda^M_{i_1\cdots i_M} G^{kj_1j_2i_2\cdots i_M l_1 l_1\cdots l_{\frac{N-M-1}{2}} l_{\frac{N-M-1}{2}}}\left[(N-M+1)\delta^{i_1}_{j_1}\delta^{j}_{j_2} + M\delta^{i_1j}\delta_{j_1j_2}\right] \\ &\qquad = 0\,, \qquad (90)\end{aligned}$$

where it is intended that the term with $R = 0$ of the last summation is $\lambda^0 G_1^{kj}(N + M + 1)$.

We outline now some other important aspects.

- We note that from (83) and (49) it follows

$$\frac{\partial \lambda_C}{\partial u^j} = -M^{jB}{}_D X^D{}_C(-\vec{u})\lambda^I_B\,.$$

  Inserting here the expression (80) for λ^I_B, we obtain

$$\begin{aligned}\frac{\partial \lambda_C}{\partial u^j} &= -M^{jB}{}_D X^D{}_C(-\vec{u})\lambda_A X^A{}_B(\vec{u}) = -X^B{}_D(-\vec{u})M^{jD}{}_C\lambda_A X^A{}_B(\vec{u}) \\ &= -M^{jA}{}_C\lambda_A\,, \qquad (91)\end{aligned}$$

  where in the first passage we have used eq. (50). The result is the counterpart for the Lagrange multipliers of what eq. (56) is for the moments; the only difference is that there is a contraction of the upper index of $M^{jA}{}_C$, while in eq. (56) its down index is contracted.

- From eq. $(84)_1$ it follows

$$h' = -h + \lambda_A X^A{}_B(\vec{u})F^{IB} = -h + \lambda^I_B F^{IB}\,,$$

  where we have used eq. (80). The result allows to understand that h' is non convective because h is non convective. If we impose that $\frac{\partial h'}{\partial u^j} = 0$, that is $\frac{\partial h'}{\partial \lambda_C}\frac{\partial \lambda_C}{\partial u^j} = 0$ we find again (87) with the only use of (91).

- Similarly, from eq. (84) it follows

$$\begin{aligned}h'^k - h'u^k &= -(h^k - hu^k) + \lambda_A(G^{kA} - F^A u^k) \\ &= -(h^k - hu^k) + X^B{}_A(-\vec{u})\lambda^I_B(G^{kA} - F^A u^k)\end{aligned}$$

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$$= -(h^k - hu^k) + \lambda^I_B \left(G^{IkB} - \frac{F^{Ik}}{F^I} F^{IB} \right) ,$$

where in the second passage we have used (83), and in the last passage $(47)_2$. The result allows to understand that $h'^k - h'u^k$ is non convective because $h^k - hu^k$ is non convective. If we impose that $\frac{\partial(h'^k - h'u^k)}{\partial u^j} = 0$, that is $\frac{\partial h'^k}{\partial \lambda_C} \frac{\partial \lambda_C}{\partial u^j} - h'\delta^{kj} = 0$ we find again (89) with the only use of (91).

In other words, eqs. (87) and (89) are the counterparts, in the new variables, of eqs. (76); moreover, they are equivalent to $\frac{\partial h'}{\partial u^j} = 0$ and $\frac{\partial(h'^k - h'u^k)}{\partial u^j} = 0$, with use of (91).

- Obviously, also (87) and (89) do not depend on the reference frame. In fact, from (87), (89) with use of (80), it follows

$$0 = \lambda_A M^{jA}{}_B \frac{\partial h'}{\partial \lambda_B} = \lambda_A M^{jA}{}_B \frac{\partial h'}{\partial \lambda^I_C} X^B{}_C(\vec{u})$$
$$\stackrel{1}{=} \lambda_A \frac{\partial h'}{\partial \lambda^I_C} X^A{}_D(\vec{u}) M^{jD}{}_C \stackrel{2}{=} \lambda^I_D M^{jD}{}_C \frac{\partial h'}{\partial \lambda^I_C} ,$$

$$0 = \lambda_A M^{jA}{}_B \frac{\partial h'^k}{\partial \lambda_B} + h'\delta^{kj} = \lambda_A M^{jA}{}_B \frac{\partial h'^k}{\partial \lambda^I_C} X^B{}_C(\vec{u}) + h'\delta^{kj}$$
$$\stackrel{1}{=} \lambda_A \frac{\partial h'^k}{\partial \lambda^I_C} X^A{}_D(\vec{u}) M^{jD}{}_C + h'\delta^{kj} \stackrel{2}{=} \lambda^I_D M^{jD}{}_C \frac{\partial h'^k}{\partial \lambda^I_C} + h'\delta^{kj} ,$$

where, in the passage denoted by $\stackrel{1}{=}$ we have used (50) and in that denoted by $\stackrel{2}{=}$ eq. (80) has been used.

Let us conclude this section paying our attention to a particular partial differential equation which comes out from the above results but only in the particular case of ideal gases. To this end let us take the trace of eq. (90) and take into account the symmetry of G^{kA} (which holds for ideal gases), so obtaining

$$3\left(\lambda_A F^A - h\right) + \sum_{n=1}^{N} n\lambda_{i_1\cdots i_n} G^{i_1\cdots i_n} + \lambda^M_{i_1\cdots i_M} G^{i_1\cdots i_M l_0 l_0 l_1 l_1 \cdots l_{\frac{N-M-1}{2}} l_{\frac{N-M-1}{2}}}(N+1)$$
$$+ \sum_{R=1}^{M-1} \lambda^R_{i_1\cdots i_R} G^{i_1\cdots i_R ll}_{R+1}(N+M+1-R) + \lambda^0 G^{ll}_1(N+M+1) = 0 .$$

Writing explicitly the product $\lambda_A F^A$ and using (62), this equation becomes

$$-3h + \sum_{n=0}^{N}(n+3)\lambda_{i_1\cdots i_n} F^{i_1\cdots i_n} + (N+4)\lambda^M_{i_1\cdots i_M} F^{i_1\cdots i_M}_M + \qquad (92)$$

$$+\sum_{R=0}^{M-1}(N+M+4-R)\lambda^R_{i_1\cdots i_R}F_R^{i_1\cdots i_R}=0 \quad \text{that is},$$

$$-3h+\sum_{n=0}^{N}(n+3)\lambda_{i_1\cdots i_n}F^{i_1\cdots i_n}+\sum_{R=0}^{M}(N+M+4-R)\lambda^R_{i_1\cdots i_R}F_R^{i_1\cdots i_R}=0\,. \tag{93}$$

By using $(84)_1$ this equation becomes

$$3h'+\sum_{n=1}^{N}n\lambda_{i_1\cdots i_n}F^{i_1\cdots i_n}+\sum_{R=0}^{M}(N+M+1-R)\lambda^R_{i_1\cdots i_R}F_R^{i_1\cdots i_R}=0\,.$$

In these equations we use eqs. $(78)_1$ and $(86)_1$ so that they become

$$-3h+\sum_{n=0}^{N}(n+3)\frac{\partial h}{\partial F^{i_1\cdots i_n}}F^{i_1\cdots i_n}+\sum_{R=0}^{M}(N+M+4-R)\frac{\partial h}{\partial F_R^{i_1\cdots i_R}}F_R^{i_1\cdots i_R}=0\,, \tag{94}$$

$$3h'+\sum_{n=1}^{N}n\lambda_{i_1\cdots i_n}\frac{\partial h'}{\partial \lambda_{i_1\cdots i_n}}+\sum_{R=0}^{M}(N+M+1-R)\lambda^R_{i_1\cdots i_R}\frac{\partial h'}{\partial \lambda_R^{i_1\cdots i_R}}=0\,.$$

The eqs. (94) are partial differential equations for the functions h and h' respectively. To solve eq. $(94)_1$ it is not restrictive to suppose that h has the form

$$h=F\,H\left(F\,,\,F^{i_1\cdots i_n}(F)^{-\frac{n+3}{3}}\,,\,F_R^{i_1\cdots i_R}(F)^{-\frac{N+M+4-R}{3}}\right)$$
$$\text{for } n=1,\cdots,N \text{ and } R=0,\cdots,M. \tag{95}$$

By substituting this in eq. $(94)_1$, it becomes $\frac{\partial H}{\partial F}=0$, so that (95) becomes

$$h=F\,H\left(F^{i_1\cdots i_n}(F)^{-\frac{n+3}{3}}\,,\,F_R^{i_1\cdots i_R}(F)^{-\frac{N+M+4-R}{3}}\right) \tag{96}$$
$$\text{for } n=1,\cdots,N \text{ and } R=0,\cdots,M.$$

To solve eq. $(94)_2$ it is not restrictive to suppose that h' has the form

$$h'=(\lambda_{ll})^{-\frac{3}{2}}\,H'\left(\lambda_{ll}\,,\,\lambda_{<rs>}(\lambda_{ll})^{-1}\,,\,\lambda^{i_1\cdots i_n}(\lambda_{ll})^{-\frac{n}{2}}\,,\,\lambda_R^{i_1\cdots i_R}(\lambda_{ll})^{-\frac{N+M+1-R}{2}}\right)$$
$$\text{for } n=0,1,3,\cdots,N \text{ and } R=0,\cdots,M. \tag{97}$$

By substituting this in eq. $(94)_2$, it becomes $\frac{\partial H'}{\partial \lambda_{ll}}=0$, so that (97) becomes

$$h'=(\lambda_{ll})^{-\frac{3}{2}}\,H'\left(\lambda_{<rs>}(\lambda_{ll})^{-1}\,,\,\lambda^{i_1\cdots i_n}(\lambda_{ll})^{-\frac{n}{2}}\,,\,\lambda_R^{i_1\cdots i_R}(\lambda_{ll})^{-\frac{N+M+1-R}{2}}\right)$$
$$\text{for } n=0,1,3,\cdots,N \text{ and } R=0,\cdots,M. \tag{98}$$

5. A Comparison with a Particular Decomposition in Non Convective Quantities

In the previous part of this paper, the vector $\vec{u}$ was the velocity of any point of a frame moving with respect to the other, with a translational rectilinear uniformly motion. Consequently $\vec{u}$ was constant, in the sense that it did not depend on space and time; but the Lagrangian relativity principle has to be imposed for each of these frames, so that $\vec{u}$ is an arbitrary constant vector. For this reason it has been treated as another independent variable. Instead of this, usually it has been defined as $u^i = v^i$ with

$$v^i = \frac{F^i}{F} \, ,$$

so that it depends on space and time and from $(43)_1$ with $n = 0, 1$ and (3), it follows

$$F^{Ii} = 0 \, .$$

Almost all the previous considerations in the present paper continue to hold except for considering $u^i = \frac{F^i}{F}$,$F^{Ii} = 0$; only the passages where F^{Ii} was an independent variable are not acceptable. For example, we can still define λ^I_B with (80); but the question is if they are non convective quantities. With the other approach, this fact was a consequence of (81); now this equation continue to hold (because it is a consequence of (79)) but except for $F^{IB} = F^{Ii}$. Consequently, from (81) it follows that all the λ^I_B different from λ^I_i are non convective. Moreover, we can use (82) and note that there the coefficient of $\lambda^I_{i_1}$ is $M^{ji_1}{}_C F^{IC}$. But from the description of $M^{jA}{}_B$ reported after eq. (49) we see that $M^{ji_1}{}_C$ is zero for $C \neq 0$, while it is δ^{ji_1} if $C = 0$. Consequently, in (82) the term for $B = i_1$ is $\lambda^I_j F$. Therefore, we can use (82) to obtain λ^I_j as function of non convective quantities; as a consequence, it itself is non convective.

In other words, with the first approach the fact that λ^I_j is non convective can be deduced from (81), while with the second approach this fact can be deduced from (82). With the first approach, eq. (82) gave the counterpart of (76) in the new reference frame, while with the second approach it is the definition of λ^I_j.

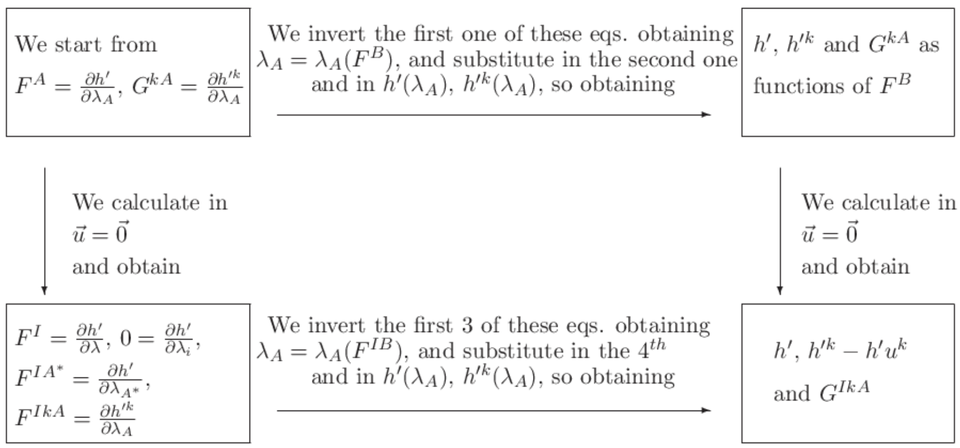
What to say about (86)? They give F^A and G^{kA} as derivatives of the 4-potentials h' and h'^k with respect to the Lagrange Multipliers; consequently they lead to a symmetric hyperbolic system if h' is a convex function of the Lagrange multipliers. To combine these equations with the Galilean relativity principle, the procedure used in literature was to transform the equations $(85)_{1,2}$, from which they were deduced, but taking as independent variables v^i and the λ^I_B different from λ^I_i; this procedure has the drawback to lose the symmetric form of the field equations and leads also to cumbersome calculations.

To avoid them, in the papers [23], [24] another approach has been described; it was already used in refs. [25], [26] and it is equivalent to the following passages: Let us continue to take eq. (86) in the variables λ_A and impose on h' and h'^k the partial differential equations (87) and (89). In the case of ideal gases, let us impose also eq. (62) and the further symmetries.

After that, we deduce F^A and G^{kA} from (86); finally, we deduce λ_A as function of F_A and substitute them in h', h'^k, G^{kA} so obtaining the constitutive functions in terms of the variable F^A. This passage is correct for the following

Proposition. *The decomposition of* h', h'^k, F^A *and* G^{kA}, *in non convective parts and in velocity dependent parts, is a consequence of eqs.* (87), (89), (86).

We will prove below this proposition in order not to lose the sight of what we are saying. So let us come back to our previous discussion. The second approach was followed in literature because it gives the non convective parts of the constitutive functions h', h'^k and G^{kA}. But this result can be realized also with the first approach by simply substituting $\vec{u} = \vec{0}$ in the expressions h', h'^k and G^{kA} previously found; this can be done also before deducing the Lagrange multipliers from $(86)_1$ and substituting them in $(86)_2$, in h' and in h'^k. In fact, it is obvious that the following diagram is commutative, where λ_{A^*} denotes the λ_A different from λ_i.



We note that the expression $0 = \frac{\partial h'}{\partial \lambda_i}$, which is present in the box on the left below, does not mean that h' does not depend on λ_i; it is only an implicit equation which, jointly with $F^I = \frac{\partial h'}{\partial \lambda}$, $F^{IA^*} = \frac{\partial h'}{\partial \lambda_{A^*}}$, gives a system from which to deduce $\lambda_A = \lambda_A(F^B)$. This is equivalent to what done in [23] and [24].

Let us now come back to prove the above Proposition.

Let us suppose to have deduced λ_A from $(86)_1$ in terms of F^A and to have substituted it in $(86)_2$; after that, let us take the derivative with respect to v^j of the function G^{kA} which comes out. We will have $\frac{\partial G^{kA}}{\partial v^j} = \frac{\partial G^{kA}}{\partial \lambda_B}\frac{\partial \lambda_B}{\partial v^j}$ with $\frac{\partial \lambda_B}{\partial v^j}$ obtained taking the derivative with respect to v^j of $(86)_1$, that is

$$\frac{\partial F^A}{\partial v^j} = \frac{\partial^2 h'}{\partial \lambda_C \partial \lambda_A} \frac{\partial \lambda_C}{\partial v^j} \,. \tag{99}$$

Now, taking the derivative of (87) with respect to λ_C and using eq. $(86)_1$, we obtain

$$M^{jC}{}_B F^B = -\lambda_D M^{jD}{}_B \frac{\partial^2 h'}{\partial \lambda_C \partial \lambda_B}$$

which we can substitute in (56) so obtaining

$$\frac{\partial F^A}{\partial v^j} = -\lambda_D M^{jD}{}_C \frac{\partial^2 h'}{\partial \lambda_A \partial \lambda_C} \,;$$

by substituting this in the left hand side of eq. (99) and taking into account the fact that $\frac{\partial^2 h'}{\partial \lambda_A \partial \lambda_C}$ is an invertible matrix (more than that, it is positive definite), we deduce that

$$\frac{\partial \lambda_C}{\partial v^j} = -\lambda_D M^{jD}{}_C \,. \tag{100}$$

Let us take now the derivative of (89) with respect to λ_C and using eq. (86) and obtain

$$\frac{\partial^2 h'^k}{\partial \lambda_C \partial \lambda_B} \lambda_D M^{jD}{}_B = -M^{jC}{}_B G^{kB} - F^C \delta^{kj} \,. \tag{101}$$

Finally, let us take the derivative of $(86)_2$ with respect to v^j calculated in $\lambda_A = \lambda_A(F^B)$, that is

$$\frac{\partial G^{kA}}{\partial v^j} = \frac{\partial^2 h'^k}{\partial \lambda_A \partial \lambda_B} \frac{\partial \lambda_B}{\partial v^j} = -\frac{\partial^2 h'^k}{\partial \lambda_A \partial \lambda_B} \lambda_D M^{jD}{}_B = M^{jA}{}_B G^{kB} + F^A \delta^{kj} \,,$$

where in the second passage we have used eq. (100) and in the third one we have used (101). From the equation which we have obtained and from (56), it follows

$$\frac{\partial}{\partial v^j}(G^{kA} - F^A \frac{F^k}{F}) = M^{jA}{}_B (G^{kB} - F^B \frac{F^k}{F}) \,, \tag{102}$$

where we have used the fact that $\frac{F^k}{F} = \frac{F^{Ik} + F v^k}{F} = \frac{F^{Ik}}{F} + v^k$ whose derivative with respect to v^j is δ^{kj}. The resulting equation looks like eq. (56) except that now $G^{kA} - F^A \frac{F^k}{F}$ replaces F^A and $G^{kB} - F^B \frac{F^k}{F}$ replaces F^B. We recall that from the description of $M^{jA}{}_B$, reported after eq. (49), it follows that it may be different from zero only when B contains a total number of indexes (included those contracted with some $\delta^{\cdot\cdot}$) less than that of A. Consequently, eq. (102) links $G^{kA} - F^A \frac{F^k}{F}$ on

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the left hand side with analogous functions $G^{kB} - F^B \frac{F^k}{F}$, on the right hand side, which have less indexes. This will be useful in the sequel.

Let us prove now that from these results we can deduce eq. $(47)_2$ and let use prove it with the iterative procedure on the total number of indexes compacted in the multindex A, enclosed those contracted with some $\delta^{\cdots}$.

If $A = 0$, eq. $(47)_2$ holds because $G^k = F^k$ and $G^{Ik} = F^{Ik}$ so that both sides of $(47)_2$ are zero.

Let us suppose, for the iterative procedure, that eq. $(47)_2$ holds for all the indexes $A < \bar{A}$.

Let us compare now, for $A = \bar{A}$ the relations (102) and (59) remembering that this last one was obtained from $(47)_2$ for $A \leq \bar{A}$. From the comparison we deduce that the function $G^{k\bar{A}} - F^{\bar{A}} \frac{F^k}{F}$ deduced from (86), (87), (89) and the function $G^{k\bar{A}} - F^{\bar{A}} \frac{F^k}{F}$ deduced from $(47)_2$ have the same derivative with respect to v^j; consequently, their difference is a constant with respect to v^j. But, by calculating this difference in $\vec{v} = \vec{0}$ we find zero because both the functions are the same function in the reference frame denoted with the apex I. We deduce that this constant is zero and both the functions coincide.

This completes the proof of our property for F^A and G^{kA}. For what concerns h', we have

$$\frac{\partial h'}{\partial v^j} = \frac{\partial h'}{\partial \lambda_A} \frac{\partial \lambda_A}{\partial v^j} = -\lambda_D M^{jD}{}_A \frac{\partial h'}{\partial \lambda_A} = 0\,,$$

where in the second passage we have used eq. (100) and in the third passage we have used (87). Consequently, h' does not depend on velocity, as expected.

Finally we have

$$\begin{aligned}\frac{\partial}{\partial v^j}\left(h'^k - h' \frac{F^k}{F}\right) &= \frac{\partial h'^k}{\partial \lambda_A} \frac{\partial \lambda_A}{\partial v^j} - \frac{\partial h'}{\partial \lambda_A} \frac{\partial \lambda_A}{\partial v^j} \frac{F^k}{F} - h' \delta^{kj} \\ &= -\left(\frac{\partial h'^k}{\partial \lambda_A} - \frac{\partial h'}{\partial \lambda_A} \frac{F^k}{F}\right) \lambda_D M^{jD}{}_A - h' \delta^{kj} = 0\end{aligned}$$

where in the second passage we have used eq. (100) and in the third passage we have used (87) and (89). This completes the proof of our Proposition.

Note. We have proved eq. (100) through the procedure of inverting eq. $(86)_1$ and substituting in eq. $(86)_2$, h' and h'^k. We could prove the same thing also taking the derivative of eq. (83) with respect to v^j and using eq. (49), then eq. (50) and then again eq. (83). This second procedure is surely correct when $\vec{v}$ is the velocity of one of the two reference frames with respect to the other; in effect it is correct also when $v^i = \frac{F^i}{F}$ because this value of v^i does not influence the passages described before this note, nor eq. (83) which is a consequence of (80). For the sake of major clarity, we have preferred to prove (100) in another way.

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6. Conclusions

We have introduced different tensorial functions of a single vector $\vec{u}$ and found very interesting identities between them. Moreover, they are very important for Extended Thermodynamics with an arbitrary number of moments when the neglected equations are chosen according to the suggestions of the non relativistic limit of the corresponding relativistic model. In some previous paper in literature they were already introduced ad hoc and used, without proving them. So this paper fills a gap of literature in this context. It is also interesting how some complicate expressions can be put in a elegant and compact form. This, in our opinion, bears witness to the goodness of the model.

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