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EXISTENCE OF NONOSCILLATORY SOLUTIONS OF FIRST ORDER NONLINEAR NEUTRAL DELAY DIFFERENCE EQUATIONS WITH FORCING TERM

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Abstract: In this paper, the authors consider the first order nonlinear neutral delay difference equation with forcing term

$$\Delta\left(x(n)+p(n)x(n-\tau)\right)+q_1(n)x(n-\sigma_1)+q_2(n)x(n)x(n-\sigma_2)=e(n)$$

where τ, σ_1, σ_2 are positive integers $p(n), q_1(n), q_2(n)$ and $e(n)$ are real sequences. By using Krasnoselskii's fixed point theorem, we obtain sufficient conditions for the existence of nonoscillatory solutions. The Examples are illustrated with MATLAB Programming.

AMS Subject Classification: 39A10, 39A11
Key Words: difference equations, nonoscillation, Krasnoselskii's fixed point theorem

1. Introduction

Theory of oscillation and nonoscillation of solutions of difference equations have developed very rapidly in recent years, refer [1-4]. This is a field with interesting

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applications in real world life problems. There has been growing interest in the study of discrete time models described by difference equations which arise quite naturally in population dynamics, epidemic models and electronic circuit analysis [5, 6, 7].

In this paper, we consider the first order nonlinear neutral delay difference equation with forcing term

$$\Delta\left(x(n)+p(n)x(n-\tau)\right)+q_1(n)x(n-\sigma_1)+q_2(n)x(n)x(n-\sigma_2)=e(n), \qquad (1)$$

where Δ is the forward difference operator defined by $\Delta x(n)=x(n+1)-x(n)$, τ,σ_1,σ_2 are positive integers, $p(n),q_1(n),q_2(n)$ and $e(n)$ are real sequences defined for all $n\in\mathbb{N}(n_0)=\{n_0,n_0+1,n_0+2,\ldots\}$, n_0 a positive integer. Let $\rho=\max\{\tau,\sigma_1,\sigma_2\}$. By a solution of equation (1), we mean a real sequence $x(n)$ defined for all $n\geq\mathbb{N}(n_0-\rho)$ and satisfies equation (1).

A solution of the difference equation (1) is called eventually positive if there exists a positive integer n_0 such that $x(n)>0$ for $n\in N(n_0)$. If there exists a positive integer n_0 such that $x(n)<0$ for $n\in N(n_0)$, then (1) is called eventually negative.

The solution of the difference equation (1) is said to be oscillatory if it is neither eventually positive nor eventually negative. Otherwise, it is called nonoscillatory.

2. Preliminary Results

In this paper, using Krasnoselskii's fixed point theorem, we obtain some sufficient conditions for the existence of a bounded nonoscillatory solution of equation (1).

The space ℓ^∞ is the set of real sequences defined on the set of positive integers where any individual sequence is bounded with respect to the usual supremum norm. It is well known that under the supremum norm ℓ^∞ is a Banach space.

Definition 2.1. A subset Ω of a Banach Space X is relatively compact if every sequence in Ω converges to an element of X.

Definition 2.2. A set S of sequences in ℓ^∞ is uniformly Cauchy (or equi-Cauchy) if for every $\epsilon>0$, there exists an integer N such that

$$|x_i-x_j|<\epsilon$$

whenever $i,j>N$ for any $x=\{x_k\}$ in S.

Lemma 2.3. *(Discrete Arzela-Ascoli's Theorem) A bounded, uniformly Cauchy subset Ω of ℓ^∞ is relatively compact.*

Theorem 2.4. *(Krasnoselskii's Fixed Point Theorem). Let X be a Banach space. Let Ω be a bounded convex subset of X and let S_1,S_2 be maps of Ω into X*

such that $S_1x + S_2y \in \Omega$ for every pair $x, y \in \Omega$. If S_1 is a contractive and S_2 is completely continuous then the equation

$$S_1x + S_2x = x$$

has a solution in Ω.

3. Existence of Nonoscillatory Solutions

In this section we establish sufficient condition for the existence of bounded nonoscillatory solution of equation (1).

Theorem 3.1. *Suppose that there exist nonnegative constants c_1 and c_2 such that $c_1 + c_2 < 1$, $-c_2 \le p(n) \le c_1$. Further, assume that $\sum_{s=n_0}^{\infty} q_i(s) < \infty, i = 1, 2$. Then equation (1) has a bounded nonoscillatory solution.*

Proof. We choose a $n_1 > n_0$ sufficiently large such that

$$\sum_{s=n_1}^{\infty} |q_1(s)| + M\,|q_2(s)| + |e(s)| < \frac{1 - c_1 - c_2}{4}, \tag{2}$$

$$\sum_{s=n_1}^{\infty} |q_1(s)| < \frac{1 - c_1}{2}, \tag{3}$$

where $M = \max_{\frac{1-c_1-c_2}{2} \le x,y \le 1} \{xy\}$. Let $\ell_{n_0}^{\infty}$ be the set of all real sequences $x = \{x_n\}_{n=n_0}^{\infty}$with the norm $\|x\| = sup_{n \ge n_0} |x(n)| < \infty$. Then $\ell_{n_0}^{\infty}$ is a Banach space. We define a closed, bounded and convex subset of Ω of $\ell_{n_0}^{\infty}$ as follows.

$$\Omega = \left\{ x = \{x(n)\} \in \ell_{n_0}^{\infty} : \frac{1 - c_1 - c_2}{2} \le x(n) \le 1, n \ge n_0 \right\}.$$

Define two maps S_1 and $S_2 : \Omega \to \ell_{n_0}^{\infty}$ as follows.

$$(S_1x)(n) = \begin{cases} \frac{3+c_1-3c_2}{4} - p(n)x(n-\tau) + \sum_{s=n}^{\infty} q_1(s)x(s-\sigma_1), n \ge n_1, \\ (S_1x)(n_1), n_0 \le n \le n_1. \end{cases}$$

$$(S_2x)(n) = \begin{cases} \sum_{s=n}^{\infty} q_2(s)x(s)x(s-\sigma_2) - e(s), n \ge n_1, \\ (S_2x)(n_1), n_0 \le n \le n_1. \end{cases}$$

(i) We shall show that for any $x, y \in \Omega$, $S_1x + S_2y \in \Omega$.

For every $x, y \in \Omega$ and $n \geq n_1$, we obtain

$$\begin{aligned}(S_1x)(n) + (S_2y)(n) \leq & \frac{3+c_1-3c_2}{4} - p(n)x(n-\tau) + \sum_{s=n}^{\infty} q_1(s)x(s-\sigma_1) \\ & + \sum_{s=n}^{\infty} q_2(s)y(s)y(s-\sigma_2) - e(s) \\ \leq & \frac{3+c_1-3c_2}{4} + c_2 + \sum_{s=n_1}^{\infty} |q_1(s)| \\ & + M\,|q_2(s)| + |e(s)| \\ \leq & \frac{3+c_1-3c_2}{4} + c_2 + \frac{1-c_1-c_2}{4} = 1\end{aligned}$$

Furthermore, we have

$$\begin{aligned}(S_1x)(n) + (S_2y)(n) \geq & \frac{3+c_1-3c_2}{4} - p(n)x(n-\tau) + \sum_{s=n}^{\infty} q_1(s)x(s-\sigma_1) \\ & + \sum_{s=n}^{\infty} q_2(s)y(s)y(s-\sigma_2) - e(s) \\ \geq & \frac{3+c_1-3c_2}{4} - c_1 - \sum_{s=n_1}^{\infty} |q_1(s)| + M\,|q_2(s)| + |e(s)| \\ \geq & \frac{3+c_1-3c_2}{4} - c_1 - \frac{1-c_1-c_2}{4} = \frac{1-c_1-c_2}{2}\end{aligned}$$

Hence

$$\frac{1-c_1-c_2}{2} \leq (S_1x)(n) + (S_2y)(n) \leq 1 \quad \textit{for} \quad n \geq n_0.$$

Thus we have proved that $S_1x + S_2y \in \Omega$ for any $x, y \in \Omega$.

(ii) We shall show that S_1 is a contraction mapping on Ω. In fact for $x, y \in \Omega$ and $n \geq n_1$ we have

$$\begin{aligned}|(S_1x)(n) - (S_1y)(n)| \leq & |p(n)|\,|x(n-\tau) - y(n-\tau)| \\ & + \sum_{s=n_1}^{\infty} q_1(n)\,|x(n-\sigma_1) - y(n-\sigma_1)| \\ \leq & \left(|p(n)| + \sum_{s=n_1}^{\infty} |q_1(s)|\right) \|x-y\| \\ = & \left(c_1 + \frac{1-c_1}{2}\right) \|x-y\| \\ \|S_1x - S_1y\| \leq & \frac{1+c_1}{2}\,\|x-y\|\,.\end{aligned}$$

We conclude that S_1 is a contraction mapping on Ω.

(iii) We now show that S_2 is completely continuous.

First, we shall show that S_2 is continuous. Let $x_k(n)$ be a sequence in Ω such that $x_k(n) \to x(n)$ as $k \to \infty$. Since Ω is closed, $x = \{x(n)\} \in \Omega$.

For $n \geq n_1$, we have,

$$|(S_2x_k)(n) - (S_2x)(n)| \leq \sum_{s=n_1}^{\infty} |q_2(n)| \, |x_k(s-\sigma_2)x_k(s) - x(s-\sigma_2)x(s)| \, .$$

Since

$$|x_k(s-\sigma_2)x_k(s) - x(s-\sigma_2)x(s)| \to 0 \quad \text{as } k \ \to \infty,$$

by applying the Lebesgue dominated convergence theorem we obtain that

$$\lim_{k\to\infty} \|(S_2x_k)(n) - (S_2x)(n)\| = 0.$$

This means that S_2 is continuous.

Next, we shall we prove that $(S_2x)(n)$ is uniformly Cauchy. For any given $\epsilon > 0$, there exists $n_1 \geq n_0$ such that

$$\sum_{s=n_1}^{\infty} M\,|q_2(s)| + |e(s)| \leq \frac{\epsilon}{2}.$$

Then for $x = \{x(n)\} \in \Omega$. $n_2 > n_1 \geq n_0$, we get

$$\begin{aligned} |(S_2x)(n_2) - (S_2x)(n_1)| &\leq |(S_2x)(n_2)| + |(S_2x)(n_1)| \\ &= \sum_{s=n_2}^{\infty} M\,|q_2(s)| + |e(s)| + \sum_{s=n_1}^{\infty} M\,|q_2(s)| + |e(s)| \\ &\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \end{aligned}$$

Therefore $(S_2x)(n)$ is uniformly Cauchy. By Lemma 2.3, $(S_2x)(n)$ is relatively compact. By Theorem 2.4, there is $x = \{x_n\} \in \Omega$ such that $S_1x + S_2x = x$. Clearly $x = \{x_n\}$, is a bounded positive solution for equation (1). This completes the proof of the theorem. $\square$

Example 3.2. Consider the following nonlinear delay difference equation

$$\begin{aligned} \Delta\left(x(n) + \frac{1}{2}x(n-1)\right) + \frac{1}{(n)(n-2)}x(n-1) + \frac{1}{(n-1)(n-2)}x(n)x(n-1) \\ = \frac{2n+3}{2n(n-1)} - \frac{1}{n+1}, n \geq 3. \end{aligned} \tag{4}$$

$p(n) = \frac{1}{2}$, $\sum_{s=n_0}^{\infty} q_i(s) < \infty, i = 1,2$. Since conditions of Theorem 3.1 are satisfied, the equation (4) has a nonoscillatory bounded positive solution.

Example 3.3. Consider the following nonlinear delay difference equation

$$\begin{aligned} \Delta\left(x(n) - \frac{1}{2}x(n-1)\right) + \frac{1}{n^2}x(n-1) + \frac{1}{n(n-1)}x(n)x(n-1) \\ = \frac{1}{n+1} + \frac{3-2n}{2n(n-1)} + \frac{1}{n(n-1)^2}, n \geq 2. \end{aligned} \tag{5}$$

$p(n) = -\frac{1}{2}$, $\sum_{s=n_0}^{\infty} q_i(s) < \infty, i = 1,2$. Since conditions of Theorem 3.1 are satisfied, the equation (5) has a nonoscillatory bounded positive solution.

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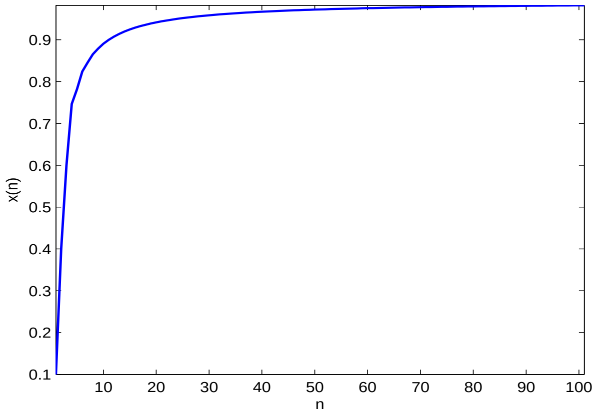

Figure 1: Bounded nonoscillatory positive solution of equation (4).

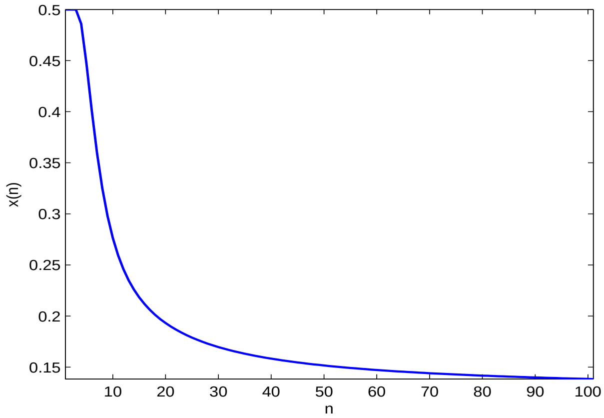

Figure 2: Bounded nonoscillatory positive solution of equation (5).

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