International Electronic Journal of Pure and Applied Mathematics

Volume 6 No. 3 2013, 139-158
ISSN: 1314-0744
url: http://www.e.ijpam.eu
doi: http://dx.doi.org/10.12732/iejpam.v6i3.5

MAXIMIN DEGREE DOMINATION NUMBER IN GRAPHS AND ITS CRITICAL ASPECTS

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Abstract: A set $D \subseteq V(G)$ of a graph G is called a *dominating set* of G if every vertex $u \in V - D$ is adjacent to at least one vertex $v \in D$. In this paper, we define a dominating set D of a graph G to be a *maximin degree dominating set* if $\delta(\langle D \rangle)$ is maximum, where the maximum of δ is taken over all the dominating sets D of the graph G. The minimum cardinality of a maximin degree dominating set is the *maximin degree domination number* of the graph G and is denoted by $\gamma_{\delta_{max}}(G)$. We study the different properties of $\gamma_{\delta_{max}}(G)$, obtain the maximin degree domination number of some class of graphs and obtain the relation of $\gamma_{\delta_{max}}$ with some known domination parameters of a graph.

This paper also discusses the critical aspects of the maximin degree domination number with respect to the vertices and edges of a given graph.

AMS Subject Classification: 05C69
Key Words: maximin degree domination number, domination, total domination, connected domination, complement of the graph, critical and redundant elements, fixed, free and totally free elements

1. Introduction

We consider here only the finite simple undirected graphs with no loops or multiple

Received: February 7, 2013

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International Electronic Journal of Pure and Applied Mathematics – IEJPAM, Volume 6, No. 3 (2013)

edges. For the notations and terminologies used here, the reader is referred to [7, 23], unless specified otherwise.

We begin with a brief introduction to domination in graphs.

Definition 1.1 ([18]). Let $G = (V, E)$ be a graph. Two vertices u and v dominate each other if they are adjacent in G. A set $D \subseteq V$ is called a dominating set if for every vertex $u \in V - D$, there exists a vertex $v \in D$ such that u and v are adjacent.

The minimum cardinality of a dominating set is called the domination number of G and is denoted by $\gamma(G)$.

Plenty of research has been carried out on domination since its introduction around 1962 and several interesting parameters such as connected domination number ([21]), independent domination number, total domination number ([5, 1]) etc. have been introduced by imposing different conditions on the dominating sets. As a consequence, there were studies on the properties of these parameters. When it was realized that formation of these parameters for a general graph is NP-complete, the first step was to obtain the bounds on the domination parameters and to study their basic properties. Henning et.al. [12, 14] and many others tried to improve the bounds on the new parameters. B.D. Acharya and E. Sampathkumar were, perhaps, the first to introduce a study on the effect on the domination number upon removal of any element of the graph. This concept was studied as the critical aspects of the domination number ([3, 10, 13, 17, 20, 22]). The reader is referred to [9, 8, 2, 4, 6, 11, 15, 16, 19] for more details. We will, however, quote a few of them here in the relevant places.

In this paper, we introduce a new domination parameter called the maximin degree domination number and study the critical aspects of the new parameter. We begin with the motivation.

2. Motivation

Degree of a vertex in a graph is a significant concept as it reflects the number of adjacencies, particularly in the light of its applications in the areas such as Computer Networks, Social Networks, Management setups, Electrical Circuits, etc. Naturally, we come across many practical models wherein the degree concept becomes relevant and important. For instance, consider a graph representing a social network, where the degree may represent the number of persons on whom a person has influential contacts. In a situation, where we want to set up a committee of people that can effectively dominate the network system, we will be ideally looking for a set of people who have the maximum influence over the society. Similar instances of applications may be cited in Computer Networks, Electrical Circuits, critical safety measures in industries, etc.

International Electronic Journal of Pure and Applied Mathematics – IEJPAM, Volume 6, No. 3 (2013)

3. New Concept and Basic Observations

Let $G = (V, E)$ be a graph on n vertices. For any set $S \subseteq V$, the *induced subgraph* $\langle S \rangle$ is the maximal subgraph of G with vertex set S.

Definition 3.1. A dominating set D of a graph G is said to be a maximin degree dominating set if the minimum degree of the induced subgraph $\langle D \rangle$ of D is maximum, where this maximum is taken over all the dominating sets of the graph.

The minimum cardinality of a maximin degree dominating set is called as the maximin degree domination number of the graph and is denoted by $\gamma_{\delta_{max}}(G)$.

A $\gamma_{\delta_{max}}-$ set is a maximin degree dominating set of minimum cardinality.

Example:

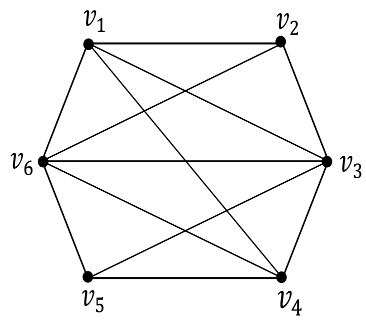

Figure 1: An illustration of the maximin degree domination concept.

In the Figure 1 there are five maximin degree dominating sets, viz., $D_1 = \{v_1, v_3, v_4, v_5, v_6\}$, $D_2 = \{v_3, v_4, v_5, v_6\}$, $D_3 = \{v_1, v_3, v_4, v_6\}$, $D_4 = \{v_1, v_3, v_4, v_2, v_6\}$ and the vertex set V itself. The minimum degree of the induced subgraphs of all the above five dominating sets is 3. But among the five D_2 and D_3 have the minimum cardinality and $\gamma_{\delta_{max}} = 4$ for the graph in Figure 1. There is, however, no change in the value of $\gamma_{\delta_{max}}$ of the new graph obtained by adding a pendant vertex to any of the vertex of the graph in Figure 1.

Let D be a $\gamma_{\delta_{max}}-$ set of a graph G.

1. If G is a disconnected graph, then $\gamma_{\delta_{max}}(G)$ is the sum of the maximin degree domination numbers of the components of the graph.

2. No vertex v in $V - D$ is adjacent to all the minimum degree vertices of $\langle D \rangle$.

The following results can be verified easily.

1. For any regular graph G of order n, $\gamma_{\delta_{max}}(G) = n$.

3. $\delta(\langle D \rangle) \geq \delta(G)$ where $\delta(G)$ is the minimum degree of the graph G. It is, however, possible that at times $\delta(\langle D \rangle) > \delta(G)$. For instance, consider the following example, in the Figure 2, wherein $D = \{v_3, v_4, v_5, v_6\}$ and $\delta(\langle D \rangle) = 3 > \delta(G) = 2$.

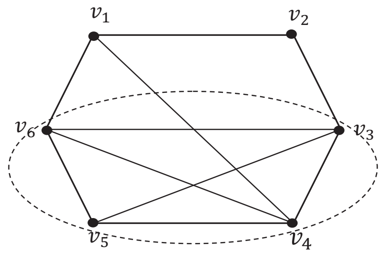

Figure 2: An illustration of a maximin degree dominating set having its minimum degree greater than $\delta(G)$.

2. $\gamma_{\delta_{max}}(K_n) = \gamma_{\delta_{max}}(C_n) = n$, where K_n is the complete graph on n vertices and C_n is the cycle on n vertices.

3. For a complete bipartite graph $K_{m,n}$,

$$\gamma_{\delta_{max}}(K_{m,n}) = 2min\{m, n\}.$$

Remark 3.2. Converse of the first observation, that is, the maximin degree domination number of a regular graph is the number of vertices of the graph, however, need not be true, as we see $\gamma_{\delta_{max}}(W_n) = n$, where W_n is the wheel graph, defined as in [7].

Proposition 3.3. *For any tree T, $\gamma_{\delta_{max}}(T) = \gamma_t(T)$, where $\gamma_t(G)$ is the total domination number of a graph G.*

Since any connected graph contains a spanning tree, the Proposition 3.3 implies the following corollary.

Corollary 3.4. *For any graph G with $\delta(G) \geq 1$, $\gamma_t(G) \leq \gamma_{\delta_{max}}(G)$.*

Proposition 3.5. *If G is a graph with $\delta(G) = 0$, then $\gamma_{\delta_{max}}(G) = \gamma(G)$.*

In a graph, a set A of vertices is said to be dominated by a vertex v if v is adjacent to each vertex of A.

Theorem 3.6. *Let $\gamma_{\delta_{max}}(G) = k$ and D be a $\gamma_{\delta_{max}}-$set of a graph G, with $\delta(G) \geq 1$. Then D is a $\gamma-$set and γ_t-set of G if and only if every vertex $u_i \in D$ dominates a set $V_i \subseteq V - D$ for $i = 1, 2, 3, ..., k$, such that, V_i does not contain a vertex v_i that dominates both V_i and V_j, where $i \neq j$.*

Proof. Let D be a $\gamma_{\delta_{max}}-$set of a graph G. Suppose that D is a $\gamma - set$ and γ_t-set of G and that there exists a vertex $u \in D$ which does not dominate any vertex of G uniquely. Then $D - \{u\}$ will still be a dominating set of G, which is a contradiction to our assumption. Therefore each vertex $u_i \in D$ dominates a set $V_i \subseteq V - D$ uniquely.

Let $u_i \in D$ dominate a set $V_i \subseteq V - D$ for $i = 1, 2, 3, ..., k$. Suppose $v_i \in V_i$ dominates each vertex of V_i and each vertex of V_j where $i \neq j$. Since $\delta(\langle D \rangle) \geq 1$, u_i and u_j are adjacent to some vertices of D, and hence u_i and u_j of D can be replaced by v_i. Thus $D' = (D - \{u_i, u_j\}) \cup \{v_i\}$ forms a dominating set of G with $|D'| < |D|$ and $\delta(\langle D' \rangle) = 0$ with $deg_{\langle D' \rangle}(v_i) = 0$ which is a contradiction. Thus no V_i contains a vertex v_i which dominates V_i and V_j, where $i \neq j$.

Conversely, since $\delta(G) \geq 1$, it is natural that, $\delta(\langle D \rangle) \geq 1$. Hence, $\delta(\langle D \rangle) \geq 1$ and hence D is a total dominating set of G. Also from the statement of the theorem, every vertex of D uniquely dominates some vertex of $V - D$. Therefore D is the smallest dominating set with $\delta(\langle D \rangle) \geq 1$. Hence D is a $\gamma-$set and a γ_t-set of the graph G. □

Theorem 3.7. *Let G be a graph with $\delta(G) = 1$, such that, G contains a support vertex v which is adjacent to at most one vertex of degree greater than or equal to 2. Then $\gamma_{\delta_{max}}(G) = \gamma_t(G)$.*

Proof. Let v be a support vertex of the graph G. Then v must lie in every dominating set of the graph. Suppose v is adjacent to at most one vertex of degree greater than or equal to 2 in G. Then removal of the pendant vertices adjacent to v in G, results in a graph in which v is a pendant vertex. Thus the minimum degree of any dominating set of G is 1. Thus $\gamma_{\delta_{max}}(G) = \gamma_t(G)$. □

Theorem 3.8. *In a graph G with $\delta(G) \geq 1$, $\gamma_{\delta_{max}}(G) \leq \gamma_i(G)$ if and only if $V - D$ contains an independent dominating set D' with $|D'| \geq |D| - \gamma_i(\langle D \rangle)$ such that D' is not dominated by any vertex of the γ_i-set of $\langle D \rangle$.*

Proof. Let $|D| = d$ and let D_i be a γ_i-set of $\langle D \rangle$ with $|D_i| = k$. Suppose $\gamma_{\delta_{max}}(G) \leq \gamma_i(G)$. Let D' be an independent dominating set of $\langle V - D \rangle$ and $|D'| < d - k$. Now, $D' \cup D_i$ forms an independent dominating set of G with $|D' \cup D_i| < d$, which is a contradiction. Hence $|D'| \geq d - k$. Suppose D' is dominated by some vertices of D_i. Since the remaining vertices of $V - D$ are dominated by the vertices of D, D_i together with few vertices of $D - D_i$, forms a γ_i- set of G, leading to $\gamma_i(G) < \gamma_{\delta_{max}}(G)$, a contradiction.

Conversely, let $D' \subseteq V - D$ be an independent dominating set of $V - D$ such that only D' of $V - D$ is not dominated by any vertex of D_i. Then $D' \cup D_i$ forms a γ_i-set of G with $|D' \cup D_i| \geq |D| - \gamma_i(\langle D \rangle) + \gamma_i(\langle D \rangle) = |D|$. Thus $\gamma_{\delta_{max}}(G) \leq \gamma_i(G)$. □

Remark 3.9. We note that in light of the maximin degree domination, independent domination number can also be looked upon as minimax degree domination number of the graph, where, a dominating set D of a graph is a minimax degree dominating set of the graph if $\Delta(\langle D \rangle)$ is minimum.

Note 3.10. 1. Since $\delta(G) \leq \gamma_{\delta_{max}}(G)$ for any graph G, we have the extended relation,

$$\kappa(G) \leq \lambda(G) \leq \delta(G) \leq \gamma_{\delta_{max}}(G).$$

where $\kappa(G)$ is the connectivity of the graph and $\lambda(G)$ is the edge connectivity and $\delta(G)$ is the minimum degree of the graph G ([7]).

2. We have $ir(G) \leq \gamma(G)$, where $ir(G)$ is the irredundance number [8] of the graph. Thus $ir(G) \leq \gamma_{\delta_{max}}(G)$

Theorem 3.11. *Let G be a graph with $\delta(G) \geq 1$. Let D be a $\gamma_{\delta_{max}}-$set of G. Then $V - D$ does not contain a maximin degree dominating set of G.*

Proof. We consider two cases.

Case 1: $V - D$ is not a dominating set of G. Then no subset of $V - D$ can be a dominating set of G.

Case 2: $V - D$ is a dominating set of G. Suppose $V - D$ contains a set T of vertices such that $|T| = |D|$ and T is a dominating set of G. Then obviously $\delta(\langle T\rangle) \leq \delta(\langle D\rangle)$. If $\delta(\langle T\rangle) = \delta(\langle D\rangle)$ then since T and D are dominating sets of G and $T \cap D = \phi$, the minimum degree vertices of D are adjacent to some vertices of T and vice-versa. Then the vertices of T together with the vertices of D forms a new dominating set D' of G such that $\delta(\langle D'\rangle) > \delta(\langle D\rangle)$. This is a contradiction to the fact that D is $\gamma_{\delta_{max}}$-set of G.

Thus T can not be a dominating set of G. Thus $V - D$ does not contain a maximin degree dominating set of G. □

4. Bounds on $\gamma_{\delta_{max}}(G)$

Since $\delta(G) \leq \delta(\langle D\rangle)$, every vertex v in a $\gamma_{\delta_{max}}$-set D of a graph G is adjacent to at least $\delta(G)$ number of vertices of D. Thus $\delta(G) + 1$ is an obvious lower bound for $\gamma_{\delta_{max}}(G)$. The following theorem gives a better lower bound for $\gamma_{\delta_{max}}(G)$.

Theorem 4.1. *For any (n, m) graph G with $\delta(G) \geq 1$, $\gamma_{\delta_{max}}(G) \geq \lceil \frac{n\delta}{\Delta+1} \rceil$.*

Proof. Note that in a graph G each vertex can dominate at most Δ number of vertices. In a $\gamma_{\delta_{max}}-$ set, every vertex is adjacent to at least δ number of vertices. Hence $\gamma_{\delta_{max}}(G) \geq \lceil \frac{n\delta}{\Delta+1} \rceil$. □

Theorem 4.2. *Let G be a graph such that no component of G is $K_{1,n}$. Then $\gamma_{\delta_{max}}(G) \leq |V| - \epsilon$, where ϵ is the number of pendant vertices in G.*

Proof. Let A be the set of all pendant vertices of the graph G. Note that the set $V - A$ forms a dominating set of the graph. Since the graph does not contain $K_{1,n}$ as its component, inclusion of any pendant vertex to this dominating set will not increase the minimum degree of the dominating set. Hence all pendant vertices lie in $V - D$. Thus $\gamma_{\delta_{max}}(G) \leq |V| - \epsilon$. □

The following result improves the upper bound of $\gamma_{\delta_{max}}$.

Theorem 4.3. *Let v be a vertex of a graph G with $\delta(G) \geq 1$ such that v satisfies the following conditions.*

1. *$deg(v) = \delta(G)$ and*
2. *$N(u)$ contains at least $\delta(G)$ number of vertices of degree greater than $\delta(G)$ for all $u \in N(v)$.*

Then $v \in V - D$, where D is a $\gamma_{\delta_{max}}$-set of G.

Proof. Let $u \in N(v)$, where $deg(v) = \delta(G)$. Suppose u satisfies the second condition in the statement of the theorem. Then the removal of the vertex v from $N(u)$ results in a graph G' with $\delta(G') \geq \delta(G)$. Hence $V - \{v\}$ forms a maximin degree dominating set of G. This proves the theorem. □

Remark 4.4. The vertices satisfying the conditions stated in the above theorem belong to $V - D$ for a $\gamma_{\delta_{max}}$-set D of G. Hence we define the following set and obtain an upper bound for $\gamma_{\delta_{max}}(G)$.

Definition 4.5. Let G be a graph with $\delta(G) \geq 1$. The set $M_\delta(G)$ is the set of all minimum degree vertices v of G such that $N(u)$ contains at least $\delta(G)$ number of vertices of degree greater than $\delta(G)$ for all $u \in N(v)$.

Theorem 4.6. *Suppose every minimum degree vertex of a graph G lies in $M_\delta(G)$ and the induced subgraph $\langle V(G) - M_\delta(G)\rangle$ of $V(G) - M_\delta(G)$ is a regular graph. Then $\gamma_{\delta_{max}}(G) = |V(G) - M_\delta(G)|$.*

Proof. Suppose every minimum degree vertices of G lies in $M_\delta(G)$. From Theorem 4.3, $M_\delta(G) \nsubseteq D$. Since the induced subgraph $\langle V(G) - M_\delta(G)\rangle$ of $V(G) - M_\delta(G)$ is a regular graph, every vertex of $V(G) - M_\delta(G)$ contributes to the minimum degree of $\langle D\rangle$. Thus $V(G) - M_\delta(G)$ forms a $\gamma_{\delta_{max}}-$set of G. □

Remark 4.7. The converse of the Theorem 4.6 need not be true. For example, the vertices v_{11} and v_{12} are the minimum degree vertices of the graph in the Figure 3, which lies in the set $M_\delta(G)$. $\gamma_{\delta_{max}}(G) = |V(G) - M_\delta(G)|$ for the graph G in the Figure 3. But the induced subgraph $\langle V(G) - M_\delta(G)\rangle$ of $V(G) - M_\delta(G)$ is not a regular graph.

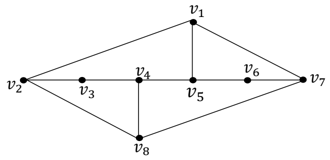

Figure 3: Illustration of the Remark 4.7

Theorem 4.8. *Let G be any graph with $\delta(G) \geq 1$. Then*

$$\gamma_{\delta_{max}}(G) \leq |V - M_\delta(G)|$$

Proof. The set $M_\delta(G)$ represents set of all minimum degree vertices such that all its neighbors are adjacent to at least $\delta(G)$ vertices of degree greater than $\delta(G)$. By Theorem 4.3, all such vertices lie outside the maximin degree dominating set of the graph. Hence the result. □

Note 4.9. 1. The upper bound in the Theorem 4.8 holds for the regular graphs.

2. In Figure 1, the vertices v_2 and v_5 are the minimum degree vertices satisfying the condition in the definition of $M_\delta(G)$. Thus $\gamma_{\delta_{max}} = 4 = |V - M_\delta(G)|$. However, in Figure 4 the vertices a, c, d, f are the minimum degree vertices. But none of them satisfy the condition in the definition of $M_\delta(G)$. Hence $\gamma_{\delta_{max}} = 4 < |V - M_\delta(G)| = 6$.

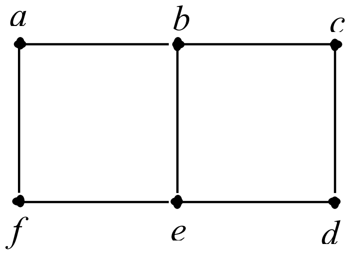

Figure 4: Illustration of the Note 4.9 (2)

The next result improves the bound further.

Theorem 4.10. *Let G be a (n, m) graph such that $\gamma_{\delta_{max}}(G) = d$. Then $m \geq \lceil \frac{d\ \delta(G)}{2} \rceil + n - d$.*

Proof. Let D be a $\gamma_{\delta_{max}}$−set of G. Then $|D| = d$ and since $\delta(\langle D \rangle) \geq \delta(G)$, each vertex of $\langle D \rangle$ must be adjacent to at least $\delta(G)$ vertices of D. Thus $\langle D \rangle$ should have

International Electronic Journal of Pure and Applied Mathematics – IEJPAM, Volume 6, No. 3 (2013)

at least $\lceil \frac{d\ \delta(G)}{2} \rceil$ edges. But $V - D$ has $n - d$ vertices and D is a dominating set of G. Hence each vertex of $V - D$ should be adjacent to at least one vertex of D. Hence there should be at least $n - d$ edges joining $V - D$ and D. Thus a (n, m) graph with $\gamma_{\delta_{max}}(G) = d$ should have at least $\lceil \frac{d\ \delta(G)}{2} \rceil + n - d$ edges. □

Note 4.11. In the above theorem equality holds for complete graphs, regular graphs, star graphs and complete bipartite graphs $K_{n,n}$.

As a consequence of the above theorem, we obtain an upper bound for $\gamma_{\delta_{max}}(G)$ for a graph with $\delta(G) \geq 3$.

Corollary 4.12. *For a graph G with $\delta(G) \geq 3$, $\gamma_{\delta_{max}}(G) \leq \frac{2(m-n)}{\delta(G)-2}$.*

Note 4.13. The bound in the corollary is attained for complete graphs, regular graphs and complete bipartite graphs $K_{n,n}$.

Star graphs are the only graphs with $\delta(G) = 1$, for which the upper bound in the corollary is attained.

5. Maximin Degree Domination Number of the Complement of a Graph

Lemma 5.1. *$\gamma_{\delta_{max}}(\overline{G}) = p$ if G is regular graph of order p.*

Definition 5.2 ([8]). Let G be a (p, q) graph. Then the corona $C(G)$ of the graph G is the graph obtained from G by adding a pendant edge to each vertex of the graph G.

Theorem 5.3. *$\gamma_{\delta_{max}}(C(K_n)) = \gamma_{\delta_{max}}(\overline{C(K_n)})$, where $C(K_n)$ represents the corona of K_n.*

Proof. Since $\gamma_{\delta_{max}}(K_n) = n$, we must have $\gamma_{\delta_{max}}(C(K_n)) = n$. Let $D_1 = V(K_n)$ and let D_2 be the set of all pendant vertices of $C(K_n)$. Note that $|D_1| = |D_2| = n$. In $\overline{C(K_n)}$ every vertex of D_1 is adjacent to $n - 1$ vertices of D_2. Every vertex of D_2 is adjacent to $n - 1$ vertices of D_1 and $n - 1$ vertices of D_2. Hence the degree of every vertex in D_2 is $2n - 2$ in $\overline{C(K_n)}$. Note that minimum degree of $\overline{C(K_n)}$ is $n - 1$. Removal of the vertices of D_1 from $\overline{C(K_n)}$ will reduce the degree of each vertex of D_2 by $n - 1$ resulting in a complete subgraph K_n with vertex set as D_2. Also this vertex set forms a maximin degree dominating set of $\overline{C(K_n)}$. Thus $\gamma_{\delta_{max}}(C(K_n)) = \gamma_{\delta_{max}}(\overline{C(K_n)})$. □

As a consequence, we have the following corollary.

Corollary 5.4. $\gamma_{\delta_{max}}(P_4) = \gamma_{\delta_{max}}(\overline{P}_4)$.

Theorem 5.5. *$\gamma_{\delta_{max}}(\overline{P}_n) = n$ for any path P_n if $n \geq 6$.*

Proof. Let P_n be a path with $n \geq 6$ vertices. P_n has 2 vertices of degree 1 and all the remaining $n-2$ vertices are of degree 2. Then $\overline{P}_n$ will have two vertices of degree $n-2$ and $n-2$ vertices of degree $n-3$. If any vertex, say v, is removed from $\overline{P}_n$ then the minimum degree of $\overline{P}_n$ will reduce by 1, since there exist only two vertices of degree $n-2$ and hence v must be adjacent to at least one vertex of degree $n-3$. Thus $\gamma_{\delta_{max}}(\overline{P}_n) = n$ if $n \geq 6$. □

Theorem 5.6. *For a complete bipartite graph $K_{m,n}$,*

$$\gamma_{\delta_{max}}(\overline{K}_{m,n}) = \gamma_{\delta_{max}}(K_{m,n}) = 2min\{m, n\}.$$

Proof. The partite sets V_1 and V_2 of $K_{m,n}$ are the independent sets of $K_{m,n}$ and they form complete graphs as components of $\overline{K}_{m,n}$. Since the given graph is complete bipartite, no vertex of V_1 is adjacent to any vertex of V_2 in $\overline{K}_{m,n}$. Hence the result is true. □

Corollary 5.7. $\gamma_{\delta_{max}}(P_n) = \gamma_{\delta_{max}}(\overline{P}_n)$ *for n = 2,3.*

Lemma 5.8. *If $\Delta(G) = n-1$ and $\delta(G) = 1$ for a graph with n vertices then,* $\gamma_{\delta_{max}}(\overline{G}) = 2$.

Proof. Since G has a vertex of degree $n-1$, $\delta(\overline{G}) = 0$. The vertex of degree 1 in G will dominate all the remaining vertices of $\overline{G}$. Thus $\gamma_{\delta_{max}}(\overline{G}) = 2$. □

6. Critical Aspects of Maximin Degree Domination in Graphs

B.D. Acharya and H.B. Walikar [22] observed that removal of an edge from a graph G may or may not result in a graph whose domination number is different from that of G. They introduced this concept as follows:

Definition 6.1. [22] Let G be a graph and x be any edge of G. An edge x is γ-critical, if $\gamma(G-x) \neq \gamma(G)$ and γ-redundant otherwise. If all the edges of the given graph is γ-critical then the graph is said to be γ-critical graph. Otherwise it is called a γ-durable graph.

E. Sampathkumar and Prabha S. Neeralagi [20] extended this concept to any element of the graph and also defined some other related parameters as follows.

Definition 6.2. [20] Let t be any parameter defined on the graph G and e be any element of graph G. Then the element e is said to be

1. t-critical if $t(G-e) \neq t(G)$.

2. t^+-critical if $t(G-e) > t(G)$.

3. t^--critical if $t(G-e) < t(G)$.

International Electronic Journal of Pure and Applied Mathematics – IEJPAM, Volume 6, No. 3 (2013)

4. t-fixed if e belongs to every t-set.

5. t-free if e belongs to some t-sets but not to all t-sets .

6. t-totally free if e belongs to no t-set.

We extend the above concepts to maximin degree domination number of the graph.

Definition 6.3. Let G be a graph and let e be any element of the graph G. Then the element e is said to be

1. $\gamma_{\delta_{max}}-$ critical if $\gamma_{\delta_{max}}(G-e) \neq \gamma_{\delta_{max}}(G)$

2. $\gamma^{+}_{\delta_{max}}-$ critical if $\gamma_{\delta_{max}}(G-e) > \gamma_{\delta_{max}}(G)$

3. $\gamma^{-}_{\delta_{max}}-$ critical if $\gamma_{\delta_{max}}(G-e) < \gamma_{\delta_{max}}(G)$

4. $\gamma_{\delta_{max}}-$ redundant if $\gamma_{\delta_{max}}(G-e) = \gamma_{\delta_{max}}(G)$

5. $\gamma_{\delta_{max}}-$ fixed if e belongs to every $\gamma_{\delta_{max}}-$set.

6. $\gamma_{\delta_{max}}-$ free if e belongs to some $\gamma_{\delta_{max}}-$sets but not all $\gamma_{\delta_{max}}-$sets.

7. $\gamma_{\delta_{max}}-$ totally free if e belongs to no $\gamma_{\delta_{max}}-$set.

7. Value of $\gamma_{\delta_{max}}$ of Some Classes of Graphs on Removal of an Element e from the Graph

In this section we observe the variation of $\gamma_{\delta_{max}}$ of some graphs upon the removal of an edge or a vertex from the graph.

Critical aspects of $\gamma_{\delta_{max}}$ with respect to a path: We begin with the critical aspects of maximin degree domination of paths. Let $P_n = v_1v_2v_3.......v_n$ be a given path. Then

$$\gamma_{\delta_{max}}(P_n - v_i) = \gamma_{\delta_{max}}(P_{i-1}) + \gamma_{\delta_{max}}(P_{n-i})$$
$$\gamma_{\delta_{max}}(P_n - v_iv_{i+1}) = \gamma_{\delta_{max}}(P_i) + \gamma_{\delta_{max}}(P_{n-i})$$

Depending upon the values of n and i, some elements are $\gamma_{\delta_{max}}-$redundant and some are $\gamma_{\delta_{max}}-$critical.

The Table 1 gives the summary of the critical aspects of $\gamma_{\delta_{max}}$ of a path with respect to vertex v_i, except for $i = 2$ and $i = n-1$.

Similarly, with respect to the edges, we observe the following. Let $P_n = v_1v_2v_3......v_n$ be a path on n vertices. Let $x = v_iv_{i+1}$ be any arbitrary edge of the path P_n. For different values of i and n some edges are critical and some edges are redundant as given in the Table 2.

International Electronic Journal of Pure and Applied Mathematics – IEJPAM, Volume 6, No. 3 (2013)

	$n \equiv 0(mod\ 4)$	$n \equiv 1(mod\ 4)$	$n \equiv 2(mod\ 4)$	$n \equiv 3(mod\ 4)$
$i \equiv 0(mod\ 4)$	$\gamma_{\delta_{max}}-$ redundant	$\gamma^{-}_{\delta_{max}}-$ redundant	$\gamma_{\delta_{max}}-$ redundant	$\gamma_{\delta_{max}}-$ redundant
$i \equiv 1(mod\ 4)$	$\gamma_{\delta_{max}}-$ redundant	$\gamma^{-}_{\delta_{max}}-$ critical	$\gamma^{-}_{\delta_{max}}-$ critical	$\gamma_{\delta_{max}}-$ redundant
$i \equiv 2(mod\ 4)$	$\gamma^{+}_{\delta_{max}}-$ critical	$\gamma_{\delta_{max}}-$ redundant	$\gamma^{-}_{\delta_{max}}-$ critical	$\gamma_{\delta_{max}}-$ redundant
$i \equiv 3(mod\ 4)$	$\gamma^{+}_{\delta_{max}}-$ critical	$\gamma^{+}_{\delta_{max}}-$ critical	$\gamma_{\delta_{max}}-$ redundant	$\gamma_{\delta_{max}}-$ redundant

Table 1: Critical aspects of $\gamma_{\delta_{max}}$ of a path with respect to vertices

	$n \equiv 0(mod\ 4)$	$n \equiv 1(mod\ 4)$	$n \equiv 2(mod\ 4)$	$n \equiv 3(mod\ 4)$
$i \equiv 0(mod\ 4)$	$\gamma_{\delta_{max}}-$ redundant	$\gamma_{\delta_{max}}-$ redundant	$\gamma_{\delta_{max}}-$ redundant	$\gamma_{\delta_{max}}-$ redundant
$i \equiv 1(mod\ 4)$	$\gamma^{+}_{\delta_{max}}-$ critical	$\gamma_{\delta_{max}}-$ redundant	$\gamma^{+}_{\delta_{max}}-$ critical	$\gamma^{+}_{\delta_{max}}-$ critical
$i \equiv 2(mod\ 4)$	$\gamma^{+}_{\delta_{max}}-$ critical	$\gamma^{+}_{\delta_{max}}-$ critical	$\gamma_{\delta_{max}}-$ redundant	$\gamma^{+}_{\delta_{max}}-$ critical
$i \equiv 3(mod\ 4)$	$\gamma^{+}_{\delta_{max}}-$ critical	$\gamma^{+}_{\delta_{max}}-$ critical	$\gamma_{\delta_{max}}-$ redundant	$\gamma_{\delta_{max}}-$ redundant

Table 2: Critical aspects of $\gamma_{\delta_{max}}$ of a path with respect to edges

Note 7.1. From the table 2 it is clear that, no edge of path P_n is $\gamma^{-}_{\delta_{max}}-$critical.

The following results are evident by the respective classes of graphs.

Theorem 7.2. *Every element e of a regular graph G is $\gamma^{-}_{\delta_{max}}-$critical.*

Theorem 7.3. *Let V_1 and V_2 be the partite sets of a complete bipartite graph $K_{m,n}$ with $|V_1| > |V_2|$. Let e be any element of $K_{m,n}$. Then*

$$\gamma_{\delta_{max}}(K_{m,n} - e) = \begin{cases} \gamma_{\delta_{max}}(K_{m,n}), & \text{if } e \in V_1 \text{ or } e \text{ is any edge} \\ \gamma_{\delta_{max}}(K_{m,n}) - 2, & \text{if } e \in V_2 \end{cases}$$

As a consequence of the above result we have,

Corollary 7.4. *Every vertex of $K_{n,n}$ is $\gamma^{-}_{\delta_{max}}-$ critical. Moreover*

$$\gamma_{\delta_{max}}(K_{n,n} - v) = \gamma_{\delta_{max}}(K_{n,n}) - 2$$

for any vertex v of $K_{n,n}$.

Theorem 7.5. *Let* $W_n = K_1 + C_{n-1}$ *be a wheel graph with* $n \geq 3$. *Every element e of wheel is* $\gamma^{-}_{\delta_{max}}$*−critical. Moreover,*

$$\gamma_{\delta_{max}}(W_n - e) = \begin{cases} n-1, & \text{if } e \in V(K_1) \\ 3, & \text{if } v \in V(C_{n-1}) \text{ or } e \text{ is an edge of } W_n \end{cases}$$

8. Properties of $\gamma_{\delta_{max}}$ Fixed, Free and Totally Free Elements of a Graph

The property of an element depends on many factors, like, the degree of the element, the adjacency and many others.

Theorem 8.1. *Let* $D_1, D_2, ..., D_n$ *be the* $\gamma_{\delta_{max}} - sets$ *of a graph G. A vertex v of graph G is* $\gamma_{\delta_{max}} - fixed$ *vertex if and only if one of the following holds.*

1. *v uniquely dominates some vertices of G.*
2. *v is adjacent to some minimum degree vertices of* $\langle D_i \rangle$ *for all i.*

Proof. Let $D_1, D_2, ..., D_n$ be the $\gamma_{\delta_{max}}$−sets of a graph G. Let v be a $\gamma_{\delta_{max}}$−fixed vertex of graph G. Then $v \in \cap_{i=1}^{n} D_i$. Suppose both the conditions in the statement are not true. Suppose the vertex v does not dominate any vertex of G uniquely. Then removal of v from any of the dominating set of G will not affect the domination property of D. But it may affect the minimum degree of the $\gamma_{\delta_{max}}$−set of the graph. Suppose v is not adjacent to any minimum degree vertex of $\langle D_i \rangle$ for some i. Then $D_i - v$ is a dominating set of G with $\delta(\langle D_i \rangle - v) = \delta(\langle D_i \rangle)$. Hence $D_i - v$ forms a $\gamma_{\delta_{max}}$−set of G. This is a contradiction to the fact that $\langle D_i \rangle$ is a $\gamma_{\delta_{max}}$−set of G.

Conversely, let us assume that the vertex v uniquely dominates some vertices of G. Then v must lie in all the dominating sets of G and hence in each $\gamma_{\delta_{max}}$−sets of G. Thus v is $\gamma_{\delta_{max}}$−fixed vertex of G. Suppose v is adjacent to some minimum degree vertices of $\langle D_i \rangle$ for all i. Then v contributes something to the minimum degree of each $\langle D_i \rangle$. Thus v lies in each $\langle D_i \rangle$ and hence, v is $\gamma_{\delta_{max}}$−fixed vertex of G. □

Remark 8.2. A vertex v of a graph G is $\gamma_{\delta_{max}}$− free if and only if G has a $\gamma_{\delta_{max}}$−set D such that $v \notin D$.

Theorem 8.3. *A vertex v of a graph G is* $\gamma_{\delta_{max}}$*−totally free vertex of G if and only if one of the following conditions holds.*

1. *Every neighbor of v is adjacent to at least deg(v) number of vertices of degree greater than deg(v).*

2. *There exists a vertex $u \notin N(v)$ with $deg(u) < deg(v)$ such that u dominates some vertices of G uniquely and v is dominated by some neighbor of u in G.*

Proof. Suppose v is a $\gamma_{\delta_{max}}-$totally free vertex of G. Then the vertex v does not belong to any $\gamma_{\delta_{max}}-$set of G. Suppose v is the minimum degree vertex of the graph. Since v is $\gamma_{\delta_{max}}-$totally free, $\delta(\langle V - \{v\}\rangle) \geq \delta(G) = deg(v)$. Thus every neighbor of v must be adjacent to at least $\delta(G)$ number of vertices of degree greater than $\delta(G)$. If there is at least one vertex of degree equal to the degree of v, to which the neighbor of v is adjacent, then that vertex can be replaced by v to obtain another $\gamma_{\delta_{max}}-$set of G which implies that v is $\gamma_{\delta_{max}}-$ free vertex of G. This is a contradiction.

Suppose $deg(v) > \delta(G)$. If $\delta(\langle D\rangle) < deg(v)$, then D must contain a vertex u such that, $deg(u) < deg(v)$ in G. Thus D must contain a vertex u such that, $deg(u) < deg(v)$ in G. Such a vertex u can lie in D only if u dominates some vertex of G uniquely. If $u \in N(v)$ then $v \in D$, which is a contradiction. Suppose v is not dominated by any neighbor of u, then v must lie in the dominating set of G, which is a contradiction.

If $\delta(\langle D\rangle) \geq deg(v)$ then v lies outside the $\gamma_{\delta_{max}}-$set if every neighbor of v is adjacent to at least $deg(v)$ number of vertices of degree greater than the $deg(v)$.

Conversely, let v be the minimum degree vertex and the condition 1 of the theorem holds. Then $\delta(\langle V - \{v\}\rangle) \geq \delta(G) = deg(v)$. Hence $V - \{v\}$ forms a maximin degree dominating set of G. Thus v is $\gamma_{\delta_{max}}-$totally free vertex of G.

Let $u, v \in V(G)$ with $u \notin N(v)$ and $deg(u) < deg(v)$ such that u dominates some vertex of G uniquely. Then $u \in D$ and $N(u) \subseteq D$. If v is dominated by some neighbor of u then $v \notin D$. Thus v is a $\gamma_{\delta_{max}}-$totally free vertex of G. □

Lemma 8.4. *If $x = uv$ is a $\gamma_{\delta_{max}}-$fixed edge of G, then there exists no vertices a and b in G such that $D' = (D - \{u, v\}) \cup \{a, b\}$ forms a $\gamma_{\delta_{max}}-$set of G.*

Proof. Suppose $D' = (D - \{u, v\}) \cup \{a, b\}$ forms a $\gamma_{\delta_{max}}-$set of G. Then D' is a $\gamma_{\delta_{max}}-$set of G which does not contain the edge uv. This is a contradiction to the fact that $x = uv$ is a $\gamma_{\delta_{max}}-$fixed edge of G. □

From the definition of the $\gamma_{\delta_{max}}$ fixed, free and totally free edges of a graph we can observe the following.

For a graph G, an edge $x = uv$ is

1. $\gamma_{\delta_{max}}-$ fixed if and only if both the end vertices u and v of x are $\gamma_{\delta_{max}}-$ fixed vertices of G.

2. $\gamma_{\delta_{max}}-$ free edge of G if and only if both the end vertices of x share at least one $\gamma_{\delta_{max}}-$set in common, but not all.

3. $\gamma_{\delta_{max}}-$totally free if both the end vertices of x does not share a $\gamma_{\delta_{max}}-$set in common.

9. Properties of $\gamma_{\delta_{max}}$ Critical and Redundant Elements

In this section let us see under what circumstances, a $\gamma_{\delta_{max}}$ fixed, free and totally free element becomes a critical or a redundant element of the graph.

First let us see the critical aspect with respect to the vertices.

Theorem 9.1. *A $\gamma_{\delta_{max}}-$totally free vertex v of a graph G is $\gamma_{\delta_{max}}-$ critical if and only if $V(G-v)-D$ contains a set A of vertices with $deg_{G-v}(u) \geq \delta(\langle D\rangle)$ for all $u \in A$, which are adjacent to a set $D' \subseteq D$, where D' is a dominating set of $G-v$ and $|D'| \neq |D|-|A|$.*

Proof. Suppose v is a $\gamma_{\delta_{max}}-$totally free vertex of G which is also $\gamma_{\delta_{max}}-$ critical. Suppose $deg_{G-v}(u) < \delta(D)$ for all $u \in V(G-v)-D$. Then induced subgraph of any dominating set of $G-v$ will not have minimum degree greater than $\delta(\langle D\rangle)$. Also since v is a $\gamma_{\delta_{max}}-$totally free vertex of G, removal of v from G will not affect the minimum degree of $\langle D\rangle$. Hence $\gamma_{\delta_{max}}(G-v) = |D| = \gamma_{\delta_{max}}(G)$. This is a contradiction to the fact that v is $\gamma_{\delta_{max}}-$critical. Hence $V(G-v)-D$ contains few vertices of degree greater than or equal to $\delta(\langle D\rangle)$. Let $B = \{u \in V(G-v)-D | deg_{G-v}(u) \geq \delta(\langle D\rangle)\}$. Since v is $\gamma_{\delta_{max}}-$totally free vertex of G, $v \notin D$ for any $\gamma_{\delta_{max}}-$set D of G and removal of v from G does not change the minimum degree of any $\gamma_{\delta_{max}}-$set of G. Let $D' \subseteq D$ be a dominating set of $G-v$. Suppose $A \subseteq B$ be the set of vertices which are adjacent to all the vertices of D'. Then $A \cup D'$ is a maximin degree dominating set of $G-v$. If $|D'| = |D|-|A|$ then $\gamma_{\delta_{max}}(G-v) = |A \cup D'| = |A|+|D'| = |D| = \gamma_{\delta_{max}}(G)$. This is a contradiction to the fact that v is $\gamma_{\delta_{max}}-$critical. Hence $|D'| \neq |D|-|A|$. Conversely suppose v is a $\gamma_{\delta_{max}}-$totally free vertex of G. Let $D' \subseteq D$ be a dominating set of $G-v$ and $V(G-v)-D$ contains a set A of vertices with $deg_{G-v}(u) \geq \delta(D)$ for all $u \in A$,. Suppose every vertex of A is adjacent to each vertex of D' and $|D'| \neq |D|-|A|$. Then $A \cup D'$ forms a maximin degree dominating set of $G-v$ and also $\gamma_{\delta_{max}}(G-v) = |A \cup D'| = |A|+|D'| \neq |D| = \gamma_{\delta_{max}}(G)$. Thus v is $\gamma_{\delta_{max}}-$critical. □

Theorem 9.2. *Let v be a $\gamma_{\delta_{max}}-$fixed vertex of a graph which is not a support vertex of G. Then v is $\gamma_{\delta_{max}}-$redundant vertex of G if and only if $V-D$ contains a vertex u such that*

1. *u is adjacent to all vertices of $N(v) \cap D$ if $D-v$ is a dominating set of $G-v$.*

2. *u is adjacent to at least those vertices w of $V-D$ such that $N(w) \cap D = \{v\}$ if $D-v$ is not a dominating set of $G-v$.*

Proof. Let v be a $\gamma_{\delta_{max}}-$fixed vertex of G which is not a support vertex of G. Then $\delta(G-v) \geq 1$ and $v \in D$ for all $\gamma_{\delta_{max}}-$sets D of G. Hence removal of v either affects the domination property of D or reduces the minimum degree of $\langle D\rangle$.

International Electronic Journal of Pure and Applied Mathematics – IEJPAM, Volume 6, No. 3 (2013)

Case 1: $D - v$ is a dominating set of $G - v$.

Then clearly $\delta(\langle D - v\rangle) < \delta(\langle D\rangle)$. Suppose $V - D$ does not contain a vertex u which satisfies either of the conditions in the statement. Then $(D - v) \cup \{u\}$ for any vertex $u \in V - D$ does not form a $\gamma_{\delta_{max}}-$set of $G - v$. Hence $D - v$ is the $\gamma_{\delta_{max}}-$set of $G - v$. This if a contradiction to the fact that v is $\gamma_{\delta_{max}}-$redundant.

Case 2: $D - v$ is not a dominating set of $G - v$.

Then since v is $\gamma_{\delta_{max}}-$redundant vertex of G, there must be some vertex u in $V - D$ which is adjacent to at least those vertices of $V - D$, which are uniquely dominated by v.

Conversely, Suppose $u \in V - D$ such that u is adjacent to all vertices of $N(v) \cap D$. Then $(D - v) \cup \{u\} = D_1$ forms a dominating set of $G - v$ with $\delta(\langle D_1\rangle) \geq \delta(\langle D\rangle)$. Suppose $u \in V - D$ such that u is adjacent to at least those vertices of $V - D$ which are uniquely dominated by v. Then $(D - v) \cup \{u\}$ forms a dominating set of $G - v$. This proves the result. □

Theorem 9.3. *A $\gamma_{\delta_{max}}-$ free vertex v of G is always $\gamma_{\delta_{max}}-$redundant vertex of G.*

Proof. The theorem follows from the fact that if v is a $\gamma_{\delta_{max}}-$free vertex of the graph then G always contains a $\gamma_{\delta_{max}}-$set D_1 such that $v \notin D_1$. □

Theorem 9.4. *Let G be a graph of order n such that $\gamma_{\delta_{max}}(G) < n$. If an edge $e = uv$ of G is $\gamma^+_{\delta_{max}}-$critical, then for every $\gamma_{\delta_{max}}-$set D, any one of the following two conditions hold.*

1. *$u \in D$ and $v \in V - D$ implies $N(v) \cap D = \{u\}$.*
2. *$u, v \in D$.*

Proof. Suppose $e = uv$ is a $\gamma^+_{\delta_{max}}-$critical edge of the graph G. Let us assume that none of the two conditions is true for e. Then there exists a particular $\gamma_{\delta_{max}}-$set D of G such that $u \in D$ and $v \in V - D$ but $N(v) \cap D \neq u$. Since D is dominating set of G, we must have $|N(v) \cap D| \geq 2$. Thus v is vertex of $V - D$ which has at least two neighbors in D and hence removal of the edge e does not change the minimum degree of the subgraph $\langle D\rangle$ and domination property of D. This is a contradiction. □

Corollary 9.5. *For any graph G, suppose there exists a $\gamma_{\delta_{max}}-$set D such that $e = uv$ is an edge of G with $u \in D$ and $v \in V - D$ with $|N(v) \cap D| \geq 2$ then the edge e is $\gamma_{\delta_{max}}-$redundant edge of G.*

Theorem 9.6. *Let $x = uv$ be a $\gamma_{\delta_{max}}-$ totally free edge of G and D be a $\gamma_{\delta_{max}}-$set of G. Then D is not a $\gamma_{\delta_{max}}-$set of $G - x$ if and only if $u \in D$ and $v \in V - D$ such that v is uniquely dominated by u in D.*

International Electronic Journal of Pure and Applied Mathematics – IEJPAM, Volume 6, No. 3 (2013)

Proof. Let $x = uv$ be a $\gamma_{\delta_{max}}-$ totally free edge of G and D be a $\gamma_{\delta_{max}}-$set of G. Then there are only two possibilities. Either both u and v are $\gamma_{\delta_{max}}-$ totally free vertices of G or one of them is $\gamma_{\delta_{max}}-$totally free and the other is either $\gamma_{\delta_{max}}-$ free or $\gamma_{\delta_{max}}-$ fixed vertex of G.

Let D be a $\gamma_{\delta_{max}}-$set of G which is not a $\gamma_{\delta_{max}}-$set of $G - x$. If both u and v are $\gamma_{\delta_{max}}-$totally free vertices, then removal of the edge x from G will not affect the minimum degree and the domination property of any $\gamma_{\delta_{max}}-$ set of G. Hence D is the $\gamma_{\delta_{max}}-$set of G. This is a contradiction. Thus $u \in D$ and $v \in V - D$. Suppose there exists at least one vertex w in D such that v is dominated by the vertex w in D. Then D remains the dominating set of $G - x$. Since the minimum degree of $\langle D\rangle$ is not affected on the removal of the edge x, D remains the $\gamma_{\delta_{max}}-$set of $G - x$. This is a contradiction.

Conversely, suppose $x = uv$ is a $\gamma_{\delta_{max}}-$totally free edge of G with $u \in D$ and $v \in V - D$ such that v is uniquely dominated by u in D. Then removal of x from G will not affect the minimum degree of $\langle D\rangle$ but $N(v) \cap D = \phi$. Then D is not a dominating set of $G - x$. Thus we need to pick some other vertices from $V - D$ to form a $\gamma_{\delta_{max}}-$set of $G - x$. □

Corollary 9.7. *Let $x = uv$ be a $\gamma_{\delta_{max}}-$totally free edge of G and D be a $\gamma_{\delta_{max}}-$set of G. Then D is the $\gamma_{\delta_{max}}-$set of $G - x$ if and only if one of the following is true.*

1. *both u and v are $\gamma_{\delta_{max}}-$totally free vertices of G.*

2. *$u \in D$ and $v \in V - D$ such that there exists at least one vertex w in D, where $w \neq u$, such that v is dominated by w.*

Theorem 9.8. *Let $x = uv$ be a $\gamma_{\delta_{max}}-$ free edge of a graph G where $G \ncong K_{1,n}$. The edge x is $\gamma_{\delta_{max}}-$critical edge of G if and only if u is $\gamma_{\delta_{max}}-$ fixed vertex and v is $\gamma_{\delta_{max}}-$ free vertex of G such that v is not dominated by any vertex, except u, of the $\gamma_{\delta_{max}}-$ sets D_i of G for which $v \notin D_i$.*

Proof. Let $x = uv$ be a $\gamma_{\delta_{max}}-$ free edge of a graph G. Then x belongs to some $\gamma_{\delta_{max}}-$ sets but not to all. Hence both u and v can not be $\gamma_{\delta_{max}}-$ fixed vertices of G and none of them can be $\gamma_{\delta_{max}}-$ totally free vertices of G. Thus either both u and v must be $\gamma_{\delta_{max}}-$ free vertices of G or one of them is $\gamma_{\delta_{max}}-$ fixed and the other is $\gamma_{\delta_{max}}-$ free vertex of G.

Suppose $x = uv$ is $\gamma_{\delta_{max}}-$ critical edge of G. Without loss of generality let us assume that v is $\gamma_{\delta_{max}}-$ free vertex of G. Suppose there exists at least one $\gamma_{\delta_{max}}-$ set D_i such that $u \notin D_i$. Since D_i is dominating set of G, $N(u) \cap D_i \neq \phi$. Since $u \notin D_i$, the removal of x from G does not affect the minimum degree of $\langle D_i\rangle$. Thus $\gamma_{\delta_{max}}(G - x) = |D_i| = \gamma_{\delta_{max}}(G)$. This is a contradiction. Hence the vertex u must be $\gamma_{\delta_{max}}-$ fixed vertex of G.

Since v is $\gamma_{\delta_{max}}-$ free vertex, there exists few $\gamma_{\delta_{max}}-$ sets D_i of G such that $v \notin D_i$. Suppose v is dominated by some vertex $w \in D_i$, where $w \neq u$ Then D_i is a dominating set of $G - x$. Since $v \notin D_i$, the removal of $x = uv$ does not affect the minimum degree of $\langle D_i \rangle$. Thus $\gamma_{\delta_{max}}(G - x) = |D_i| = \gamma_{\delta_{max}}(G)$. This is a contradiction.

Conversely, let $x = uv$ be a $\gamma_{\delta_{max}}-$ free edge of a graph G such that, u is $\gamma_{\delta_{max}}-$ fixed vertex, v is $\gamma_{\delta_{max}}-$ free vertex of G and v is not dominated by any vertex, except u, of the $\gamma_{\delta_{max}}-$ sets D_i of G for which $v \notin D_i$. Then the removal of x from G will affect the domination property of D_i if $v \notin D_i$ and the minimum degree of D_i is reduced if $v \in D_i$. Hence to obtain the $\gamma_{\delta_{max}}-$set if $G - x$ some vertices must be added or some vertices must be removed from the $\gamma_{\delta_{max}}-$set D_i of G. Thus x is $\gamma_{\delta_{max}}-$ critical edge of G. □

Theorem 9.9. *Let $x = uv$ be a $\gamma_{\delta_{max}} - fixed$ edge of a graph G, which is not a tree. Then x is $\gamma_{\delta_{max}}-$redundant if and only if one of the following holds.*

1. *Both u and v are of degree greater than the minimum degree of D.*

2. *One of u and v is a minimum degree vertex of $\langle D \rangle$ such that, both u and v dominates some vertex of G uniquely and if a vertex $w \in D$ does not dominate any vertex of G uniquely then w must be adjacent to some minimum degree vertex of $\langle D \rangle - x$.*

Proof. Let $x = uv$ be a $\gamma_{\delta_{max}}-$fixed edge of a graph G. Suppose one of u and v is a minimum degree vertex of $\langle D \rangle$ such that, one of u and v does not dominate any vertex of G uniquely. Let us assume that, u is the minimum degree vertex and v does not dominate any vertex of G uniquely. Since $\delta(G) \geq 1$, v is adjacent to some vertex other than u of D and hence, $D - \{v\}$ forms a maximin degree dominating set of $G - x$. Hence x is $\gamma_{\delta_{max}}-$critical, which is a contradiction. Thus both u and v must dominate some vertices of G uniquely and hence $u, v \in D'$ for any $\gamma_{\delta_{max}}-$set D' of $G - x$. Suppose $w \in D$ does not dominate any vertex of G uniquely and w is not adjacent to any minimum degree vertex of $\langle D \rangle - x$. Then $D - \{w\}$ is a dominating set of $G - x$ and $\delta(\langle D \rangle - w) \geq \delta(\langle D \rangle - x)$. Thus $\gamma_{\delta_{max}}(G - x) \leq \gamma_{\delta_{max}}(G)$, which is a contradiction.

Conversely, let $x = uv$ be a $\gamma_{\delta_{max}}-$fixed edge of a graph G. Suppose degree of both u and v is greater than the minimum degree of $\langle D \rangle$ in $\langle D \rangle$. Since x is $\gamma_{\delta_{max}}-$fixed edge, the end vertices u and v lie in all the $\gamma_{\delta_{max}}-$sets of the graph. Thus the removal of x from G does not affect the domination property and the minimum degree of $\langle D \rangle$. Thus D is the $\gamma_{\delta_{max}}-$set of $G - x$.

Let one of u and v is minimum degree vertex of $\langle D \rangle$ such that both u and v dominates some vertex of G uniquely. Then $u, v \in D'$ for any $\gamma_{\delta_{max}}-$set D' of $G - x$. If every vertex of D dominates some vertex of G uniquely, then $D' = D$. Suppose the vertices which does not dominate any vertex of G uniquely, are adjacent to some minimum degree vertices of $\langle D \rangle - x$. Hence every vertex of $\langle D \rangle$ is contributing something to the minimum degree of $\langle D \rangle - x$. Thus D is the $\gamma_{\delta_{max}}$−set of $G - x$ and hence x is $\gamma_{\delta_{max}}$−redundant edge of G. □

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