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WALLMAN COMPACTIFICATION AND SOBER SPACES

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Abstract: We give necessary and sufficient conditions on the space X in order to get it's Wallman compactification sober.

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1. Introduction

A nonempty topological space X is irreducible if any two nonempty open subsets of X intersect. Hence X is irreducible if X is not the union of two closed subsets distinct from X. A subset Y of X is irreducible if it is an irreducible topological space with the induced topology.

A generic point of a topological space X is a point x whose closure is all of X. In a T_0-space X every irreducible closed subset has at most one generic point.

A topological space X is said to be sober if every nonempty irreducible closed set C of X has a unique generic point x, [2].

On an other register, an embedding of a topological space X as a dense subset of a compact space is called a compactification of X. The most known and the simplest compactification is the (Alexandroff) one-point compactification.

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In 2010, Echi and Lazaar [1] have prove that the one point compactification of X is sober if and only if X is sober, and they conclude their paper with the following question: "Let X be a T_1-space. Is the Wallman compactification of X sober if and only if X is sober?".

The first part of this paper contains some remarks and properties of closed sets of the wallman compactification of a T_1-space. The second part deals with the characterization of spaces such that their Wallman compactification is a sober space.

2. Spaces such that their Wallman Compactification is Sober

Let us recall the Wallman compactification construction of a T_1-space, [4].

Let $\mathcal{P}$ be a class of subsets of a topological space X which is closed under finite intersections and finite unions.

A $\mathcal{P}$-filter on X is a collection $\mathcal{F}$ of nonempty elements of $\mathcal{P}$ with the properties:

(i) $\mathcal{F}$ is closed under finite intersections.

(ii) $P_1 \in F$, $P_1 \subseteq P_2$ implies $P_2 \in \mathcal{F}$.

A $\mathcal{P}$-ultrafilter is a maximal $\mathcal{P}$-filter. When $\mathcal{P}$ is the class of closed sets of X, then the $\mathcal{P}$ -filters are called closed filters.

The points of the Wallman compactification wX of a space X are the closed ultrafilters on X. For $x \in X$, let $w_X(x) = \{A \mid A \text{ is a closed set of } X \text{ and } x \in A\}$. Then w_X is an embedding of X into wX. Thus, if $x \in X$, then $w_X(x)$ will be identified to x.

For each closed set $D \subseteq X$, define D^* to be the set $D^* = \{\mathcal{F} \in wX \mid D \in \mathcal{F}\}$. Thus $C = \{D^* \mid D \text{ is a closed set of } X\}$ is a base for the closed sets of a topology on wX.

Let U be an open set of X, we define $U^* \subseteq wX$ by $U^* = \{\mathcal{F} \in wX \mid F \subseteq U$ for some F in $\mathcal{F}\}$, it is easily seen that the class $\{U^* \mid U \text{ is an open set of } X\}$ is a base for open sets of the topology of wX.

Remark that if $\mathcal{F} \in wX - X$, then each closed set $F \in \mathcal{F}$ is a non compact set of X.

The following proposition induces a necessary condition on a space X to have its compactification $K(X)$ a sober space.

Proposition 1. *Let X be a topological and $K(X)$ be a compactification of X. If $K(X)$ is a sober T_1-space, then X is sober.*

Proof. Let F be an irreducible closed set of X.

First, let us prove that the closure F of $\overline{F}^{K(X)}$ in $K(X)$ is an irreducible closed set of $K(X)$.

Let $\mathcal{U}$ an $\mathcal{V}$ be two open sets of $K(X)$ such that $\mathcal{U} \cap \overline{F}^{K(X)} \neq \emptyset$ and $\mathcal{V} \cap \overline{F}^{K(X)} \neq \emptyset$. Then $U = \mathcal{U} \cap X$ and $V = \mathcal{V} \cap X$ are two nonempty open sets of X such that $U \cap F \neq \emptyset$ and $V \cap F \neq \emptyset$. Since F is an irreducible set of X, $(U \cap V) \cap F \neq \emptyset$. Hence $(\mathcal{U} \cap \mathcal{V}) \cap \overline{F}^{K(X)} \neq \emptyset$. Thus $\overline{F}^{K(X)}$ is an irreducible closed set of $K(X)$.

Now, since $K(X)$ is a sober T_1-space, there exists $p \in K(X)$ such that $\overline{F}^{K(X)} = \{p\}$. That $F = \{p\}$ is immediate. Therefore X is sober. □

Corollary 2. *Let X be a T_1-space. If wX is sober then X is sober.*

Definition 3. Let X be a topological space and $\mathcal{W}$ be a collection of open sets of X.

(1) $\mathcal{W}$ is called the open collection w-associate to the closed set $H = wX - \cup(U^* : U \in \mathcal{W})$.

(2) $\mathcal{W}$ is called a bad covering of X (b-covering for short) if each finite subcollection $\mathcal{W}'$ of $\mathcal{W}$ is not a covering of X.

Proposition 4. *Let X be a T_1-space and $\mathcal{W}$ be a collection of open sets w-associate to the nonempty closed set H. Then $H \subseteq wX - X$ if and only if, $\mathcal{W}$ is a b-covering of X.*

Proof. Necessary condition. Set $H = wX - \cup(U^* : U \in \mathcal{W})$. Suppose that there exists a finite subset $\mathcal{W}'$ of $\mathcal{W}$ such that $X = \cup(U : U \in \mathcal{W}')$. So $wX = \cup(U^* : U \in \mathcal{W}')$ contradicting the fact that H is a nonempty set of wX.
Sufficient condition. Immediate. □

For a characterization of spaces such that their Wallman compactification is sober, we introduce the following definitions.

Definition 5. Let X be a T_1-space and $\mathcal{K}$ be collection of closed sets of X satisfying the finite intersection property (FIP for short).

(1) Let U be an open set of X. We say that U w-meet $\mathcal{K}$ if there exists a closed set F of X such that $F \subseteq U$ and $\mathcal{K} \cup \{F\}$ has the FIP.

(2) We say that $\mathcal{K}$ is w-irreducible if for each two open sets U and V of X such that U w-meet $\mathcal{K}$ and V w-meet $\mathcal{K}$, $U \cap V$ w-meet $\mathcal{K}$.

Lemma 6. *Let X be a T_1-space and $\mathcal{K}$ be a collection of closed sets of X satisfying the FIP. If U is an open set of X such that U w-meet $\mathcal{K}$, then $U^* \cap (wX - \underset{K \in \mathcal{K}}{\cup}(X - K)^*) \neq \emptyset$.*

Proof. Since U w-meet $\mathcal{K}$, there exists a closed set F of X such that $F \subseteq U$ and $\mathcal{K} \cup \{F\}$ has the FIP. Hence there exists $\mathcal{F} \in wX$ such that $\mathcal{K} \cup \{F\} \subseteq \mathcal{F}$, so $\mathcal{F} \in U^*$.

Suppose that $\mathcal{F} \notin wX - \underset{K\in\mathcal{K}}{\cup}(X-K)^*$, then there exists $K \in \mathcal{K}$ such that $\mathcal{F} \in (X-K)^*$. Thus $X - K \in \mathcal{F}$ contradicting the fact that $K \in \mathcal{F}$, since $\mathcal{K} \subseteq \mathcal{F}$. Therefore $\mathcal{F} \in wX - \underset{K\in\mathcal{K}}{\cup}(X-K)^*$. □

Lemma 7. *Let X be a T_1-space and $\mathcal{K}$ be collection $\mathcal{K}$ of closed sets of X satisfying the FIP. If $\mathcal{K}$ is w-irreducible, then $wX - \underset{K\in\mathcal{K}}{\cup}(X-K)^*$ is an irreducible closed set of wX.*

Proof. Set $\mathcal{H} = wX - \underset{K\in\mathcal{K}}{\cup}(X-K)^*$. Let U and V be two open sets of X such that $U^* \cap \mathcal{H} \neq \emptyset$ and $V^* \cap \mathcal{H} \neq \emptyset$.

Since $U^* \cap \mathcal{H} \neq \emptyset$, there exists $\mathcal{F} \in wX$ such that $\mathcal{F} \in U^* \cap \mathcal{H}$. Hence $\mathcal{F} \notin (X-K)^*$, for each $K \in \mathcal{K}$. So, for each $K \in \mathcal{K}$, $\mathcal{F} \notin wX - K^*$. Thus $\mathcal{F} \in K^*$. Consequently, $K \in \mathcal{F}$. Since $\mathcal{F} \in U^*$, there exists $F \in \mathcal{F}$ such that $F \subseteq U$. So that, $\mathcal{K} \cup \{F\}$ has the FIP.

Using the same argument we prove that there exists a closed set G of X such that $G \subseteq V$ and $\mathcal{K} \cup \{G\}$ has the FIP.

Since $\mathcal{K}$ is w-irreducible, U w-meet $\mathcal{K}$ and V w-meet $\mathcal{K}$, $U \cap V$ w-meet $\mathcal{K}$. So there exists $T \subseteq U \cap V$ such that $\mathcal{K} \cup \{T\}$ has the FIP. Then there exists $\mathcal{T} \in wX$ such that $\mathcal{K} \cup \{T\} \subseteq \mathcal{T}$ and $\mathcal{T} \in U^* \cap V^*$. Now, since $\mathcal{K} \cup \{T\} \subseteq \mathcal{T}$, $\mathcal{T} \notin (X-K)^*$, for each $K \in \mathcal{K}$. So, $\mathcal{T} \in wX - \underset{K\in\mathcal{K}}{\cup}(X-K)^* = \mathcal{H}$. Therefore $\mathcal{H}$ is an irreducible closed set of wX. □

Now, we are in position to give a characterization of spaces such that their Wallman compactification is a sober space.

Theorem 8. *Let X be a T_1-space. Then the following statements are equivalent:*

(1) *wX is sober.*

(2) *For each w-irreducible collection $\mathcal{K}$ of X and for each two closed sets F and G of X such that $\mathcal{K} \cup \{F\}$ and $\mathcal{K} \cup \{G\}$ have the FIP, $F \cap G \neq \emptyset$.*

Proof. (1) $\Longrightarrow$ (2). Let $\mathcal{K}$ be a w-irreducible collection of closed sets of X, F and G are two closed sets of X such that $\mathcal{K} \cup \{F\}$ and $\mathcal{K} \cup \{G\}$ have the FIP. Let us prove that $\mathcal{K} \cup \{F \cap G\}$ has the FIP.

Set $\mathcal{H} = wX - \underset{K\in\mathcal{K}}{\cup}(X-K)^*$. Using Lemma 7, $\mathcal{H}$ is an irreducible closed set of wX. Since wX is sober, $\mathcal{H}$ is a singleton set. Since $\mathcal{K} \cup \{F\}$ and $\mathcal{K} \cup \{G\}$ have the FIP, there exist $\mathcal{F}$ and $\mathcal{G} \in wX$ such that $\mathcal{K} \cup \{F\} \subseteq \mathcal{F}$ and $\mathcal{K} \cup \{G\} \subseteq \mathcal{G}$. Then

$\mathcal{F} \notin (X - K)^*$ and $\mathcal{G} \notin (X - K)^*$, for each $K \in \mathcal{K}$. So $\mathcal{F}$ and $\mathcal{G} \in \mathcal{H}$. Since $\mathcal{H}$ is a singleton set, $\mathcal{F} = \mathcal{G}$. Therefore $\mathcal{K} \cup \{F \cap G\}$ has the FIP.

(2) $\Longrightarrow$ (1). Let $\mathcal{H}$ be a nonempty closed irreducible set of wX. Since $\mathcal{H}$ is a closed set of wX, there exists a collection $\mathcal{W}$ of open sets of X such that $\mathcal{H} = wX - \bigcup_{W \in \mathcal{W}} W^*$. Set $\mathcal{K} = \{X - W \mid W \in \mathcal{W}\}$. Let $\mathcal{F} \in \mathcal{H}$. It is clear that $\mathcal{F} \notin W^*$, for each $W \in \mathcal{W}$. Hence $(X - W) \cap F \neq \emptyset$, for each $F \in \mathcal{F}$. By maximality $X - W \in \mathcal{F}$, for each $W \in \mathcal{W}$. Thus $\mathcal{K} \subseteq \mathcal{F}$ and so $\mathcal{K}$ has the PIF.

Let us prove that $\mathcal{K}$ is a w-irreducible collection of X.

Let U and V are two open sets of X such that U and V w-meet $\mathcal{K}$. Then, there exist two closed sets F and G of X such that $F \subseteq U$, $G \subseteq V$, $\mathcal{K} \cup \{F\}$ and $\mathcal{K} \cup \{G\}$ have the FIP. Let $\mathcal{F}, \mathcal{G} \in wX$ such that $\mathcal{K} \cup \{F\} \subseteq \mathcal{F}$ and $\mathcal{K} \cup \{G\} \subseteq \mathcal{G}$. Hence $\mathcal{F} \in U^* \cap \mathcal{H}$ and $\mathcal{G} \in V^* \cap \mathcal{H}$. Since $\mathcal{H}$ is an irreducible closed set of wX, there exists $\mathcal{T} \in wX$, such that $\mathcal{T} \in U^* \cap V^* \cap \mathcal{H}$. So $\mathcal{T} \in (U \cap V)^* \cap \mathcal{H}$. Thus there exists $T \in \mathcal{T}$ such that $T \subseteq U \cap V$. On the other hand, $\mathcal{K} \subseteq \mathcal{T}$, since $\mathcal{T} \in \mathcal{H}$. So $\mathcal{K} \cup \{T\}$ has the FIP. Then $\mathcal{K}$ is a w-irreducible collection of X.

Let us prove, now, that $\mathcal{H}$ is a singleton set of wX. Let $\mathcal{F}$ and $\mathcal{G}$ are two elements of $\mathcal{H}$. Then $\mathcal{K} \cup \{F\}$ and $\mathcal{K} \cup \{G\}$ have the FIP, for each $F \in \mathcal{F}$ and for each $G \in \mathcal{G}$. Since $\mathcal{K}$ is w-irreducible collection of wX, $F \cap G \neq \emptyset$, for each $F \in \mathcal{F}$ and each $G \in \mathcal{G}$. Since $\mathcal{F}$ and $\mathcal{G}$ are ultrafilters, $\mathcal{F} = \mathcal{G}$. □

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