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ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF A NONLINEAR VOLTERRA DIFFERENCE EQUATION

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Abstract: This article gives an answer to a question posed by I. Györi and E. Awwad in the paper "On the boundedness of the solutions in nonlinear discrete Volterra difference equations" published in *Advances in Difference Equations*, **2** (2012).

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Key Words: Volterra difference equations, boundedness, asymptotic behavior

1. Introduction and Main Results

Let us consider the Volterra difference equation

$$x_{n+1} = \sum_{j=0}^{n} q^{n-j} x_j^p, \quad n = 0, 1 \dots \tag{1}$$

where $p > 1$, $q \in (0, 1)$. The following conjecture has been formulated in [1].

Conjecture 1. *(See [1], Conjecture 6.8). Let $p > 1$ and $0 < q < 1$. Then there exists a constant $\kappa < 1$ such that the solution of (1) with initial condition $x_0 > 0$ is bounded whenever $x_0 \in [0; \kappa)$ and it is unbounded whenever $x_0 > \kappa$.*

In the following theorem we will confirm the conjecture. In addition, we determine the exact value for κ and qualitative characteristics of the solutions of (1).

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Theorem 1. *Let $p > 1$ and $0 < q < 1$, and let $(x_n)_{n=0}^{\infty}$ be the solution of* (1), *corresponding to the initial value x_0. Then the following statements are true:*

1. *If $x_0 < (1-q)^{\frac{1}{p(p-1)}}$ then $(x_n)_{n=0}^{\infty}$ monotonically tends to zero as $n \to \infty$;*
2. *If $x_0 = (1-q)^{\frac{1}{p(p-1)}}$ then $x_n = (1-q)^{\frac{1}{p-1}}$ for all $n \geqslant 1$;*
3. *If $x_0 > (1-q)^{\frac{1}{p(p-1)}}$ then $(x_n)_{n=0}^{\infty}$ is strictly increasing for $n \geqslant 1$ and $(x_n)_{n=0}^{\infty}$ is unbounded.*

Proof. Equation (1) with the initial value x_0 is equivalent to the first order nonlinear difference equation

$$x_{n+1} = qx_n + x_n^p, \quad n \geq 1, \tag{2}$$

with the initial condition $x_1 = x_0^p$. Now let us introduce the function

$$f(x) = qx + x^p, \quad x \geqslant 0. \tag{3}$$

Then (2) can be rewritten as

$$x_{n+1} = f(x_n), \quad n = 1, 2, \ldots \tag{4}$$

By letting $f(x) = x$, we find that there are two fixed points $x_1^* = 0$ and $x_2^* = (1-q)^{\frac{1}{p-1}}$. One can easily verify that equilibrium point $x_1^* = 0$ is an attractor (see [2]) and equilibrium point $x_2^* = (1-q)^{\frac{1}{p-1}}$ is a repeller. Indeed, the function (3) for $x \geqslant 0$ is monotonically increasing, and $f(x) < x$ for $0 < x < (1-q)^{\frac{1}{p-1}}$, $f(x) > x$ for $x > (1-q)^{\frac{1}{p-1}}$. In addition, $f'(x_1^*) = q < 1$ and $f'(x_2^*) = q + p(1-q) > q + (1-q) = 1$. Therefore, the solutions of (2) monotonically tend to zero if $x_1 < (1-q)^{\frac{1}{p-1}}$, and increase unboundedly if $x_1 > (1-q)^{\frac{1}{p-1}}$ (see Fig. 1). As $x_1 = x_0^p$, this completes the proof. □

The cobweb diagram in Figure 1 illustrates the proof of Theorem 1.

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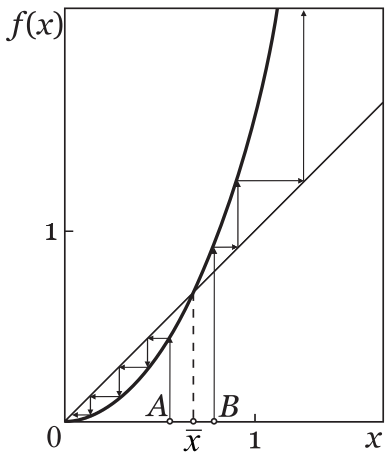

Figure 1: Iterations (4).

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