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ON CERTAIN APPLICATION OF FIRST ORDER DIFFERENTIAL SUBORDINATION TO ANALYTIC FUNCTIONS

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Abstract: In the present investigation, we obtain a first order differential subordination and discuss a few applications to analytic functions. Also we show that our results generalize certain known results.

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1. Introduction

Let $\mathcal{H}$ be the class of analytic functions in the open unit disk $\mathbb{U} := \{z \in \mathbb{C} : |z| < 1\}$ and $\mathcal{H}[a, n]$ denotes the subclass of $\mathcal{H}$ consisting of functions of the form $f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \ldots$. Let $\mathcal{A}_p$ denote the class of all analytic functions of the form

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k \quad (z \in \mathbb{U}) \tag{1.1}$$

and let $\mathcal{A}_1 := \mathcal{A}$. The class P is the well known class of Caratheodory functions consists of functions $p(z) = 1 + p_1 z + p_2 z^2 + ..., z \in \mathbb{U}$ such that $\operatorname{Re} p(z) > 0$. Let f and g are analytic functions. Then the function f is said to be *subordinate* to g, if there is a Schwarz function w with $|w(z)| \leq |z|$ such that $f(z) = g(w(z))$. Further

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if g is univalent, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(\mathbb{U}) \subseteq g(\mathbb{U})$. Recently some authors[4] have discussed the subordinations of the type

- $\alpha p(z) + \delta z p'(z) \prec w(z)$,
- $\alpha p(z) + \delta z p(z) p'(z) \prec w(z)$,
- $\alpha p(z) + \delta \frac{zp'(z)}{p(z)} \prec w(z)$,

Motivated by the above differential subordinations we discuss the subordination of the type

$$\alpha p(z) + \delta \frac{zp'(z)}{p(z)^k} \prec w(z).$$

In this way by assigning the particular values to α and δ, and k our results reduces to some recent results in [5, 6, 3]. Apart from that we discuss some applications of our results. In the present investigation we need the following result of Miller and Mocanu to prove our main result.

2. Preliminaries

Definition 1. [2, Definition 2, p.817] Denote by $\mathcal{Q}$, the set of all functions $f(z)$ that are analytic and injective on $\overline{\mathbb{U}} - E(f)$, where

$$E(f) = \{\zeta \in \partial\mathbb{U} : \lim_{z \to \zeta} f(z) = \infty\},$$

and are such that $f'(\zeta) \neq 0$ for $\zeta \in \partial\mathbb{U} - E(f)$.

Lemma 1 (cf. Miller and Mocanu[1, Theorem 3.4h, p.132]). Let $\psi(z)$ be univalent in the unit disk $\mathbb{U}$ and let θ and ϕ be analytic in a domain $D \supset \psi(\mathbb{U})$ with $\phi(z) \neq 0$, when $w \in \psi(\mathbb{U})$. Set

$$Q(z) := z\psi'(z)\phi(\psi(z)), \quad h(z) := \theta(\psi(z)) + Q(z).$$

Suppose that

1. $Q(z)$ is starlike univalent in $\mathbb{U}$, and
2. $\Re \frac{zh'(z)}{Q(z)} > 0$, $z \in \mathbb{U}$.

If $q(z)$ is analytic in $\mathbb{U}$, with $q(0) = \psi(0)$, $q(\mathbb{U}) \subset D$ and

$$\theta(q(z)) + zq'(z)\phi(q(z)) \prec \theta(\psi(z)) + z\psi'(z)\phi(\psi(z)), \tag{2.1}$$

then $q(z) \prec \psi(z)$ and $\psi(z)$ is the best dominant.

3. Main Results

Theorem 3.1. Let α and δ be complex numbers with $\delta \neq 0$. Assume that $0 \neq q(z) \in \mathcal{A}$ univalent in $\mathbb{U}$ and satisfy the following conditions for any real number k:

1. $Q(z) = \delta \frac{zq'(z)}{q(z)^k}$ is starlike $\quad (z \in \mathbb{U})$,
2. $Re \frac{1}{\delta} \left\{ \alpha q(z)^k + \delta \frac{zQ'(z)}{Q(z)} \right\} > 0$.

If $p(z) \in \mathcal{A}$ satisfies

$$\alpha p(z) + \delta \frac{zp'(z)}{p(z)^k} \prec \alpha q(z) + \delta \frac{zq'(z)}{q(z)^k}, \tag{3.1}$$

then $p(z) \prec q(z)$ and $q(z)$ is the best dominant.

Proof. Let us define θ and ϕ respectively by

$$\theta(w) := \alpha w$$

and

$$\phi(w) := \frac{\delta}{w^k}.$$

Then, obviously, $\phi(w) \neq 0$ and $\theta(w)$, $\phi(w)$ are analytic in $\mathbb{C}^* = \mathbb{C} - \{0\}$. Now, define the functions $Q(z)$ and $h(z)$ respectively by

$$Q(z) := zq'(z)\phi(z) = \delta \frac{zq'(z)}{q(z)^k}$$

and

$$h(z) := \theta(q(z)) + Q(z) = \alpha q(z) + \delta \frac{zq'(z)}{q(z)^k}.$$

Clearly, $Q(z)$ is starlike in $\mathbb{U}$ and a simple computation shows that

$$\Re \frac{zh'(z)}{Q(z)} = \frac{1}{\delta} \left\{ \alpha q(z)^k + \delta \frac{zQ'(z)}{Q(z)} \right\} > 0.$$

Thus, conditions (i) and (ii) of the Lemma 1 are satisfied.

In the view of (3.1), we have

$$\theta(q(z)) + zq'(z)\phi(q(z)) \prec \theta(\psi(z)) + z\psi'(z)\phi(\psi(z)).$$

By an application of Lemma 1 the result follows at once. $\square$

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Remark 3.2. 1. Putting $\alpha = 0$ and $k = 2$ in Theorem 3.1, we get the result due to Sushma et al. [6, Theorem 3.2].

2. Putting $\alpha = \lambda > 0, \beta = \gamma = 0, \delta = 1$ and $k = 0$ in Theorem 3.1, we obtain the result [3, Lemma 2.1] obtained by Sharma and Misra.

3. Putting $\alpha = \lambda > 0, \beta = \gamma = 0, \delta = 1$ and $k = 1$ in Theorem 3.1, we obtain the result [3, Lemma 2.2] due to Sharma and Misra.

Note that, throughout this paper unless otherwise stated specially, μ and ν are real numbers and can not take the value zero simultaneously. Now we derive the following subordination result by taking $p(z) = (f'(z))^\nu$ in the Theorem 3.1.

Theorem 3.3. Let $q(z)$ be univalent function with $q(z) \neq 0$ in $\mathbb{U}$. Assume that $\delta \frac{zq'(z)}{q(z)^k}$ is starlike in $\mathbb{U}$ and

$$Re \frac{1}{\delta} \left\{ \alpha q(z)^k + \delta \frac{zQ'(z)}{Q(z)} \right\} > 0.$$

If $f(z) \in \mathcal{A}$ and satisfies the differential subordination

$$\alpha \left(f'(z)\right)^\nu + \frac{\delta \nu \frac{zf''(z)}{f'(z)}}{\left(f'(z)\right)^{(k-1)\nu}} \prec \alpha q(z) + \delta \frac{zq'(z)}{q(z)^k},$$

then $\left(\frac{z}{f(z)}\right)^\mu (f'(z))^\nu \prec q(z)$ and $q(z)$ is the best dominant.

Putting $\alpha = 0, \nu = \lambda > 0$ and $k = 1$ in Theorem 3.3, we obtain the next result due to Sushma et al [5, Theorem 3.2].

Theorem 3.4. Let q $(q(z) \neq 0)$ be univalent function in $\mathbb{U}$ such that $Q(z) := \frac{zq'(z)}{q(z)}$ is starlike in $\mathbb{U}$. If $f \in \mathcal{A}$ $(f'(z) \neq 0)$ satisfies the differential subordinaton

$$\frac{zf''(z)}{f'(z)} \prec \frac{1}{\lambda} \frac{zq'(z)}{q(z)}, \quad z \in \mathbb{U},$$

then $(f'(z))^\lambda \prec q(z)$ and $q(z)$ is the best dominant.

Further by setting $\alpha = 0, \nu = -\lambda$ with $\lambda > 0$ and $k = 1$ in Theorem 3.3, we obtain the next result due to Sushma et al. [5, Theorem 3.3].

Theorem 3.5. Let q $(q(z) \neq 0)$ be univalent function in $\mathbb{U}$ such that $\frac{zq'(z)}{q(z)}$ is starlike in $\mathbb{U}$. If $f \in \mathcal{A}$ $(f'(z) \neq 0)$ satisfies the differential subordinaton

$$\frac{zf''(z)}{f'(z)} \prec -\frac{1}{\lambda} \frac{zq'(z)}{q(z)}, \quad z \in \mathbb{U},$$

then $\left(\frac{1}{f'(z)}\right)^\lambda \prec q(z)$ and $q(z)$ is the best dominant.

4. Applications

Here in this section we will discuss an interesting application of our Theorem 3.1. Taking $\alpha = 0$ in Theorem 3.1 and $q(z) = ((1+z)/(1-z))^\eta, 0 < \eta \leq 1$, we have the following corollary.

Corollary 4.1. Let $\delta \neq 0$ a be complex number and k be any real number. Let $p(z) \in \mathcal{A}$ satisfies the following subordination

$$\frac{zp'(z)}{p(z)^k} \prec \frac{2\eta\delta z}{1-z^2}\left(\frac{1+z}{1-z}\right)^{(1-k)\eta} \qquad (z \in \mathbb{U}, 0 < \eta \leq 1).$$

Then

$$p(z) \prec \left(\frac{1+z}{1-z}\right)^\eta.$$

Proof. Let

$$q(z) = \left(\frac{1+z}{1-z}\right)^\eta \quad 0 < \eta \leq 1.$$

Then q is convex in the unit disk $\mathbb{U}$ with $q(0) = 1$. A simple computation gives

$$\begin{aligned} Q(z) &= \frac{zq'(z)}{q(z)} \\ &= \frac{2\delta\eta z}{1-z^2}\left(\frac{1+z}{1-z}\right)^{(1-k)\eta}. \end{aligned}$$

and

$$\frac{zQ'(z)}{Q(z)} = 1 + [1 + (1-k)\eta]\frac{2z}{1-z^2}.$$

Therefore

$$\operatorname{Re}\frac{zQ'(z)}{Q(z)} = 1 + [1 + (1-k)\eta]\operatorname{Re}\frac{2z}{1-z^2}$$

and taking $z = e^{i\theta}, -\pi < \theta < \pi$, we have

$$\operatorname{Re}\frac{2z}{1-z^2} = \operatorname{Re}\frac{i}{\sin\theta} = 0.$$

Hence Q is starlike in the unit disk. Let $h(z) = Q(z)$. Then a calculation shows that

$$\begin{aligned} \operatorname{Re}\frac{zh'(z)}{Q(z)} &= \operatorname{Re}\frac{zQ'(z)}{Q(z)} \\ &= 1 > 0. \end{aligned}$$

Thus h is close to convex in the unit disk $\mathbb{U}$ and therefore univalent therein. Hence the proof is complete in the view of our Theorem 3.1. □

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