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A TWO-SIDED BOUNDARY PROBLEM FOR TWO-DIMENSIONAL RANDOM WALKS

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Abstract: A particle begins in the first quadrant at coordinates (h, k). On each independent step, it moves either up, down, left, or right one unit at a time with probabilities p_u, p_d, p_l, and p_r, respectively. We derive the probability that the particle hits the x-axis before reaching the y-axis. We then derive the expected value of the number of steps needed to hit the x-axis, and the conditional average for those paths that hit before ever having reached the y-axis. Finally, we give the average number of steps needed to hit an axis and the average number of steps needed to hit both axes.

AMS Subject Classification: 60G50, 60C05
Key Words: two-dimensional random walk, boundary problem, average number of steps

1. Introduction

A particle begins in the first quadrant at integer-valued coordinates (h, k). On each incremental step, the particle randomly moves one unit either up, down, left, or right with non-zero probabilities p_u, p_d, p_l, and p_r, respectively, where $p_u+p_d+p_l+p_r = 1$. The possible paths create a two-dimensional random walk on the integer lattice. From properties of one-dimensional random walks, the x-axis is hit almost surely if and only if $p_u \le p_d$, and the y-axis is hit almost surely if and only if $p_r \le p_l$. Regardless of these conditions, we shall derive the probability that a particle hits

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the x-axis before reaching the y-axis. Then for $p_u < p_d$, we derive the expected value of the number of steps needed to hit the x-axis, and the conditional average for those paths that have hit before ever having reached the y-axis. When $p_r < p_l$ also, we give the average number of steps needed to hit an axis and the average number of steps needed to hit both axes.

2. One-Dimensional Random Walks

As a particle moves randomly within the lattice, its position after t steps can be denoted by the coordinates (X_t, Y_t), where $(X_0, Y_0) = (h, k)$. But by counting just the steps that are either left or right, we let $\tilde{X}_m$ denote the distance to the y-axis after m horizontal movements, where $\tilde{X}_0 = h > 0$. Then $\tilde{X}$ is a one-dimensional random walk that increases or decreases one unit at a time with non-zero probabilities $\tilde{p}_r = p_r/(p_r + p_l)$ and $\tilde{p}_l = p_l/(p_r + p_l)$, respectively. Likewise, by counting just the steps that are either up or down, we let $\tilde{Y}_n$ denote the distance to the x-axis after n vertical movements, where $\tilde{Y}_0 = k > 0$. Then $\tilde{Y}$ is a separate one-dimensional random walk that increases or decreases one unit at a time with non-zero probabilities $\tilde{p}_u = p_u/(p_u + p_d)$ and $\tilde{p}_d = p_d/(p_u + p_d)$, respectively.

The classic boundary problem argument, detailed in [1], can be used to show that particles of the process $\tilde{Y}$ almost surely reach a boundary height of 0 or N, for any integer $N > k$, and that the probability of reaching height 0 before height N is

$$ {}_kP_0^N = \begin{cases} \dfrac{N-k}{N} & \text{if } \tilde{p}_u = \tilde{p}_d = 1/2 \\[2ex] \dfrac{(\tilde{p}_d/\tilde{p}_u)^k - (\tilde{p}_d/\tilde{p}_u)^N}{1 - (\tilde{p}_d/\tilde{p}_u)^N} & \text{if } \tilde{p}_u \neq \tilde{p}_d\,. \end{cases} \tag{1} $$

Also, the expected value of the number of vertical steps ${}_kV_0^N$ needed for $\tilde{Y}$ to reach a boundary height of 0 or N from initial height k is

$$ E[{}_kV_0^N] = \begin{cases} k(N-k) & \text{if } \tilde{p}_u = \tilde{p}_d = 1/2 \\[2ex] \dfrac{N(1 - {}_kP_0^N)}{\tilde{p}_u - \tilde{p}_d} - \dfrac{k}{\tilde{p}_u - \tilde{p}_d} & \text{if } \tilde{p}_u \neq \tilde{p}_d\,. \end{cases} \tag{2} $$

By taking the limit in (1) as N tends to infinity, we obtain the probability that $\tilde{Y}$ actually drops to height 0, which is given by

$$ {}_kP_0 = \lim_{N \to \infty} {}_kP_0^N = \begin{cases} 1 & \text{if } \tilde{p}_u \leq \tilde{p}_d \\ (\tilde{p}_d/\tilde{p}_u)^k & \text{if } \tilde{p}_u > \tilde{p}_d \end{cases} \tag{3} $$

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$$= \begin{cases} 1 & \text{if } p_u \leq p_d \\ (p_d/p_u)^k & \text{if } p_u > p_d \,. \end{cases}$$

The original two-dimensional random walk hits the x axis if and only if $\tilde{Y}$ drops to height 0, which happens almost surely if and only if $\tilde{p}_u \leq \tilde{p}_d$ if and only if $p_u \leq p_d$. And by the Monotone Convergence Theorem, the expected value of the number of vertical steps ${}_kV_0$ needed for $\tilde{Y}$ to drop to height 0, when $\tilde{p}_u \leq \tilde{p}_d$, is given by

$$E[{}_kV_0] = \lim_{N \to \infty} E[{}_kV_0^N] = \begin{cases} +\infty & \text{if } \tilde{p}_u = \tilde{p}_d = 1/2 \\ \dfrac{k}{\tilde{p}_d - \tilde{p}_u} & \text{if } \tilde{p}_u < \tilde{p}_d \,. \end{cases} \tag{4}$$

Equations (1) and (2) are typically derived using difference equations with the results stated in terms of probabilities $p = \tilde{p}_u$ and $q = \tilde{p}_d$, where $p + q = 1$. And the actual distribution of the number of steps needed for $\tilde{Y}$ to drop to height 0 is not needed to derive the expected value given in (4). However we shall need the distribution of ${}_kV_0$ in order to derive our desired probability of the two-dimensional random walk hitting the x-axis before hitting the y-axis.

3. The Distribution of Vertical Steps

We now shall derive $P({}_kV_0 = n)$, which is the probability that $\tilde{Y}$ drops to height 0 from height k in exactly n vertical steps. To end at height 0 after n vertical steps, where $n \geq k$, there must be a specific combination of the number of upward steps u and the number of the downward steps d satisfying $k + u - d = 0$ and $u + d = n$. Solving for u and d, we obtain $u = (n - k)/2$ and $d = (n + k)/2$.

But some sequences of upward and downward steps may yield a height of 0 before n steps are taken. So we must count the number of ways, denoted by $B(n, k)$, for a sequence of ups and downs to yield a height of 0 for the first time in exactly n steps when starting from height k. We note that there must be at least k downward steps. Of the remaining $n - k$ steps, there must be an equal number of upward and downward steps in order to end at height 0. Thus, $n - k$ must be even, which is equivalent to $n + k$ being even. So if $n + k$ is odd, then $B(n, k) = 0$.

We now assume that $n + k$ is even. Because the last step must be downward, we instead may count the number of ways for a path to lead from $(0, k)$ to $(n - 1, 1)$ while never dropping below height 1.

By re-labeling $(n - 1, 1)$ as $(0, 0)$ and traversing the paths backward, we instead can count the number of ways to go from $(0, 0)$ to $(n - 1, k - 1)$ while never dropping below height 0. And now we can use the Reflection Principle, also described in [1], to count the number of these paths.

In general, the number of paths that begin at (0, 0), drop to or *go under* height $s \le 0$, and end at coordinates (m, j) after m steps is given by

$$N_s(m, j) = \binom{m}{\frac{m+j-2s}{2}}.$$

By subtracting the number of paths that drop to or go under height $s-1$ and end at coordinates (m, j), we obtain the number of paths that have a *minimum* value of s, which is given by

$$N_s(m, j) - N_{s-1}(m, j) = \binom{m}{\frac{m+j-2s}{2}} - \binom{m}{\frac{m+j-2s}{2}+1}.$$

Letting $m = n-1$, $j = k-1$, and $s = 0$, we obtain the number of paths that go from $(0, 0)$ to $(n-1, k-1)$ and have a minimum value of 0, which is given by

$$N_0(n-1, k-1) - N_{-1}(n-1, k-1) = \binom{n-1}{\frac{n+k}{2}-1} - \binom{n-1}{\frac{n+k}{2}}.$$

This value gives our originally desired number of ways $B(n, k)$ for $\tilde{Y}$ to drop to height 0 from height k in exactly n vertical steps. So for $n \ge k$, we have

$$B(n, k) = \begin{cases} \binom{n-1}{\frac{n+k}{2}-1} - \binom{n-1}{\frac{n+k}{2}} & \text{if } n+k \text{ is even} \\ \\ 0 & \text{if } n+k \text{ is odd}. \end{cases} \tag{5}$$

For each of these paths, there is a total of $u = (n-k)/2$ upward steps and $d = (n+k)/2$ downward steps. Thus, for $n \ge k$, the pdf of ${}_kV_0$, which is the probability that $\tilde{Y}$ drops to height 0 from height k in exactly n vertical steps, is given by

$$f_V(n) = P({}_kV_0 = n) = B(n, k)\, \tilde{p}_u^{(n-k)/2}\, \tilde{p}_d^{(n+k)/2}. \tag{6}$$

In order to eliminate the cases of $n+k$ being even or odd in (5), we can write $n = k+2i$ for $i \ge 0$. Then $B(n, k)$ becomes

$$B(k+2i, k) = \binom{k+2i-1}{k+i-1} - \binom{k+2i-1}{k+i} = \frac{k}{k+2i}\binom{k+2i}{i},$$

which can be substituted into (6).

Using an analogous argument, we can analyze the number of horizontal steps ${}_hH_0$ needed for $\tilde{X}$ to reach the y-axis (i.e., drop to length 0 from length h). Its pdf is given by

$$f_H(m) = P({}_hH_0 = m) = B(m, h)\, \tilde{p}_r^{(m-h)/2}\, \tilde{p}_l^{(m+h)/2},$$

for $m \ge h$. Writing $m = h+2j$ in order to eliminate the cases of $m+h$ being even or odd, we obtain the following result:

Theorem 1. *(a) Let ${}_kV_0$ be the number of vertical steps needed for a two-dimensional random walk to reach the x-axis from initial integer-valued coordinates (h, k), where $k > 0$. Its pdf is given by*

$$f_V(k+2i) = P({}_kV_0 = k+2i) = \frac{k}{k+2i}\binom{k+2i}{i}(\tilde{p}_u\,\tilde{p}_d)^i\,\tilde{p}_d^{\,k},$$

for $i \geq 0$, where $\tilde{p}_u = p_u/(p_u + p_d)$ and $\tilde{p}_d = p_d/(p_u + p_d)$.

(b) Let ${}_hH_0$ be the number of horizontal steps needed for a two-dimensional random walk to reach the y-axis from initial integer-valued coordinates (h, k), where $h > 0$. Its pdf is given by

$$f_H(h+2j) = P({}_hH_0 = h+2j) = \frac{h}{h+2j}\binom{h+2j}{j}(\tilde{p}_r\,\tilde{p}_l)^j\,\tilde{p}_l^{\,h},$$

for $j \geq 0$, where $\tilde{p}_r = p_r/(p_r + p_l)$ and $\tilde{p}_l = p_l/(p_r + p_l)$.

We note that the cumulative distribution function of ${}_hH_0$, for $t \geq h$, gives the probability of $\tilde{X}$ dropping to length 0 within t horizontal steps. This cdf is given by

$$F_H(t) = P({}_hH_0 \leq t) = \sum_{m=h}^{t} f_H(m) = \sum_{j=0}^{\lfloor \frac{t-h}{2} \rfloor} f_H(h+2j)\,. \tag{7}$$

4. Some Infinite Series

Using Theorem 1(a) along with (3) and (4), we obtain some interesting infinite series. For simplicity, we shall let $p = \tilde{p}_u$ and $q = \tilde{p}_d$, so that $p + q = 1$. We also let $x = pq = p(1-p)$. Because $0 < p \leq 1$, we have $0 < x \leq 1/4$. We also have $p^2 - p + x = 0$. Solving for p, we obtain $p = (1 \pm \sqrt{1-4x})/2$, which implies that $\max(p, q) = (1 + \sqrt{1-4x})/2$.

Summing the pdf values $f_V(k+2i)$ over $i \geq 0$ gives the probability of $\tilde{Y}$ dropping to height 0 from height k, which also is given by (3). Thus we obtain

$$kq^k \sum_{i=0}^{\infty} \frac{\binom{k+2i}{i}}{k+2i}(pq)^i = \begin{cases} 1 & \text{if } p \leq q \\ (q/p)^k & \text{if } p > q, \end{cases}$$

which gives

$$\sum_{i=0}^{\infty} \frac{\binom{k+2i}{i}}{k+2i}(pq)^i = \frac{1}{k\,(\max(p,q))^k}\,.$$

Thus,

$$\sum_{i=0}^{\infty} \frac{\binom{k+2i}{i}}{k+2i}\, x^i = \frac{2^k}{k\left(1+\sqrt{1-4x}\right)^k}\ ,$$

for $k \geq 1$ and $0 < x \leq 1/4$.

The expected value of ${}_kV_0$, for $\tilde{p}_u \leq \tilde{p}_d$, is given by the sum $\sum_i (k+2i)\, f_V(k+2i)$, but also is given by (4). Thus we obtain

$$kq^k \sum_{i=0}^{\infty} (k+2i) \frac{\binom{k+2i}{i}}{k+2i} (p\,q)^i = \begin{cases} +\infty & \text{if } p = q \\ \dfrac{k}{q-p} & \text{if } p < q \end{cases}$$

which gives

$$\sum_{i=0}^{\infty} \binom{k+2i}{i} (p\,q)^i = \begin{cases} +\infty & \text{if } p = q \\ \dfrac{1}{(q-p)\,q^k} & \text{if } p < q\ . \end{cases}$$

Thus,

$$\sum_{i=0}^{\infty} \binom{k+2i}{i} x^i = \begin{cases} +\infty & \text{if } x = 1/4 \\ \dfrac{2^k}{\sqrt{1-4x}\left(1+\sqrt{1-4x}\right)^k} & \text{if } 0 < x < 1/4\ . \end{cases}$$

5. The Two-Sided Boundary Problem

We now derive the probability that a two-dimensional random walk starting from (h, k) hits the x-axis before reaching the y-axis. First, we let B_x be the set of all paths of independent left, right, upward, and downward steps that cause a particle to hit the x-axis before ever possibly reaching the y-axis. Such paths may or may not actually hit the y-axis if allowed to continue.

Next, we let A_x be the set of all paths that cause a particle to hit the x-axis at some point and A_y be the set of paths that cause a particle to hit the y-axis at some point. Then $A_x \cap A_y$ are those paths that eventually hit both axes. We partition the intersection as $A_x \cap A_y = A_{x,y} \cup A_{y,x}$, where $A_{x,y}$ are those paths that hit the x axis before later hitting the y-axis, and $A_{y,x}$ are those paths that hit the y axis before later hitting the x-axis. Then $B_x = A_x - A_{y,x}$. Hence our desired probability is $P(B_x) = P(A_x) - P(A_{y,x})$. But $P(A_x)$ is given in (3). So we just need to derive $P(A_{y,x})$. To continue, we now let $p_H = p_r + p_l$ be the probability of a horizontal movement on any step and let $p_V = p_u + p_d$ be the probability of a vertical movement.

To have eventually hit the x-axis from (h, k), it takes at least k steps. For $N \geq k$, we let C_N be the set of all sequences of paths that hit the x-axis for the first time in N steps. Then $A_{y,x} = \bigcup_{N \geq k} (A_{y,x} \cap C_N)$. But we can partition $A_{y,x}$ even finer. For each $N \geq k$, there must be at least k vertical movements in order to hit the x-axis. So for $k \leq n \leq N$, we let $C_{N,n}$ be the set of paths that hit the x-axis for the first time in N steps and which have exactly n vertical movements. Finally, these n vertical movements can occur in $\binom{N}{n}$ places within the N steps. So for $k \leq n \leq N$, and $1 \leq z \leq \binom{N}{n}$, we let $C_{N,n,z}$ be one of the distinct set paths that hit the x-axis for the first time in N steps and which have exactly n vertical movements, and thus have $N - n$ horizontal movements. Then

$$A_{y,x} = \bigcup_{N=k}^{\infty} \bigcup_{n=k}^{N} \bigcup_{z=1}^{\binom{N}{n}} (A_{y,x} \cap C_{N,n,z}).$$

By independence, the number of vertical movements in N steps is a binomial distribution with $P(C_{N,n,z}) = p_V^n \, p_H^{N-n}$ for all z and $P(C_{N,n}) = \binom{N}{n} p_V^n \, p_H^{N-n}$ for all n. By the Law of Total Probability, we then have

$$\begin{aligned} P(A_{y,x}) &= \sum_{N=k}^{\infty} \sum_{n=k}^{N} \sum_{z=1}^{\binom{N}{n}} P(C_{N,n,z}) \; P(A_{y,x} \mid C_{N,n,z}) \\ &= \sum_{N=k}^{\infty} \sum_{n=k}^{N} \sum_{z=1}^{\binom{N}{n}} p_V^n \, p_H^{N-n} \, P(A_{y,x} \mid C_{N,n,z}). \end{aligned}$$

However $P(A_{y,x} \mid C_{N,n,z}) = 0$ whenever the Nth step is horizontal because the Nth step must be vertical to hit the x-axis. So there are at most $\binom{N-1}{n-1}$ terms with $P(A_{y,x} \mid C_{N,n,z}) \neq 0$. In these cases, we must have $\tilde{Y}$ hitting the x-axis for the first time in exactly n vertical steps and $\tilde{X}$ hitting the y-axis *within* $N-n$ horizontal steps. By the independence of steps, we then have $P(A_{y,x} \mid C_{N,n,z}) = f_V(n) \times F_H(N-n)$. Thus we obtain

$$P(A_{y,x}) = \sum_{N=k}^{\infty} \sum_{n=k}^{N} \binom{N-1}{n-1} p_V^n \, p_H^{N-n} \, f_V(n) \, F_H(N-n). \tag{8}$$

Additionally, we only need to sum over terms n for which $n - k$ is even. Letting $n = k + 2i$, for $0 \leq i \leq \lfloor (N-k)/2 \rfloor$ and using Theorem 1(a) gives

$$\begin{aligned} p_V^n \, f_V(n) &= p_V^{k+2i} \, f_V(k+2i) \\ &= (p_u + p_d)^{k+2i} \, \frac{k}{k+2i} \binom{k+2i}{i} \frac{(p_u \, p_d)^i \, p_d^k}{(p_u + p_d)^{k+2i}} \\ &= \frac{k}{k+2i} \binom{k+2i}{i} (p_u \, p_d)^i \, p_d^k \,. \end{aligned} \tag{9}$$

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Equation (8) now becomes

$$
\begin{aligned}
P(A_{y,x}) &= \\
&= \frac{k\,p_d^k}{p_H^k} \sum_{N=k}^{\infty} \sum_{i=0}^{\lfloor \frac{N-k}{2} \rfloor} \frac{\binom{N-1}{k+2i-1}\binom{k+2i}{i}\, p_H^{N-2i}\,(p_u\,p_d)^i}{k+2i} F_H(N-k-2i) \\
&= \frac{k\,p_d^k}{p_H^k} \sum_{N=k}^{\infty} \sum_{i=0}^{\lfloor \frac{N-k}{2} \rfloor} \frac{(N-1)!\,p_H^{N-2i}\,(p_u\,p_d)^i}{(N-k-2i)!\,(k+i)!\,i!} F_H(N-k-2i). \qquad (10)
\end{aligned}
$$

Finally, from (7) and Theorem 1(b), we have

$$
\begin{aligned}
F_H(N-k-2i) &= \sum_{m=h}^{N-k-2i} f_H(m) = \sum_{j=0}^{\lfloor \frac{N-k-2i-h}{2} \rfloor} f_H(h+2j) \qquad (11) \\
&= \sum_{j=0}^{\lfloor \frac{N-k-2i-h}{2} \rfloor} \frac{h}{h+2j} \binom{h+2j}{j} \frac{(p_r\,p_l)^j\,p_l^h}{(p_r+p_l)^{h+2j}}.
\end{aligned}
$$

Substituting into (10) gives us our final form of $P(A_{y,x})$, and an analogous argument gives us $P(A_{x,y})$. We state both next:

Theorem 2. *Let (X,Y) be a two-dimensional random walk that begins at integer-valued coordinates (h,k), with $h>0$ and $k>0$.*

(a) The probability that a path hits the y-axis before later hitting the x-axis for the first time is

$$
P(A_{y,x}) = \frac{h\,k\,p_l^h\,p_d^k}{(p_r+p_l)^{h+k}} \times
$$

$$
\sum_{N=k}^{\infty} \sum_{i=0}^{\lfloor \frac{N-k}{2} \rfloor} \frac{(N-1)!(p_r+p_l)^{N-2i}(p_u p_d)^i}{(N-k-2i)!\,(k+i)!\,i!} \sum_{j=0}^{\lfloor \frac{N-k-2i-h}{2} \rfloor} \frac{\binom{h+2j}{j}(p_r p_l)^j}{(h+2j)(p_r+p_l)^{2j}}.
$$

(b) The probability that a path hits the x-axis before later hitting the y-axis for the first time is

$$
P(A_{x,y}) = \frac{h\,k\,p_l^h\,p_d^k}{(p_u+p_d)^{h+k}} \times
$$

$$
\sum_{M=h}^{\infty} \sum_{j=0}^{\lfloor \frac{M-h}{2} \rfloor} \frac{(M-1)!(p_u+p_d)^{M-2j}\,(p_r p_l)^j}{(M-h-2j)!\,(h+j)!\,j!} \sum_{i=0}^{\lfloor \frac{M-h-2j-k}{2} \rfloor} \frac{\binom{k+2i}{i}(p_u p_d)^i}{(k+2i)(p_u+p_d)^{2i}}.
$$

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Combining our results, we obtain our originally desired probabilities:

Theorem 3. *Let (X, Y) be a two-dimensional random walk that begins at integer-valued coordinates (h, k), with $h > 0$ and $k > 0$.*

(a) The probability that a path hits the x-axis before possibly reaching the y-axis is $P(B_x) = P(A_x) - P(A_{y,x})$, where $P(A_x) = 1$ if $p_u \le p_d$ and $P(A_x) = (p_d/p_u)^k$ if $p_u > p_d$.

(b) The probability that a path hits the y-axis before possibly reaching the x-axis is $P(B_y) = P(A_y) - P(A_{x,y})$, where $P(A_y) = 1$ if $p_r \le p_l$ and $P(A_y) = (p_l/p_r)^h$ if $p_r > p_l$.

Below are some immediate facts:

(i) The event A_x of a two-dimensional random walk ever hitting the x-axis depends only on the parameters p_u, p_d, and k; thus, A_x is independent of A_y. Thus we have, $P(A_{y,x}) + P(A_{x,y}) = P(A_x \cap A_y) = P(A_x)P(A_y)$, $P(A_x \cap A_y^c) = P(A_x)P(A_y^c)$, $P(A_y \cap A_x^c) = P(A_y)P(A_x^c)$, and $P(A_x \cup A_y) = P(A_x) + P(A_y) - P(A_x)P(A_y)$.

(ii) We have used $B_x = A_x - A_{y,x}$, but we alternately have $B_x = (A_x \cap A_y^c) \cup A_{x,y}$. Thus, $P(B_x) = P(A_x)P(A_y^c) + P(A_{x,y})$. Likewise, $P(B_y) = P(A_y)P(A_x^c) + P(A_{y,x})$. Clearly, B_x and B_y are disjoint with $B_x \cup B_y = A_x \cup A_y$ giving $P(A_x \cup A_y) = P(B_x) + P(B_y)$.

(iii) In the case of $h = k$, $p_r = p_u$, and $p_l = p_d$, the formulas for $P(A_{y,x})$ and $P(A_{x,y})$ are identical, as are the formulas for $P(A_x)$ and $P(A_y)$. Thus, $P(A_{y,x}) = P(A_{x,y})$ which gives $2P(A_{y,x}) = P(A_x \cap A_y) = P(A_x)^2$, and $P(A_{y,x}) = (1/2)P(A_x)^2$. It follows that $P(B_x) = 1/2$ if $p_u \le p_d$ and $P(B_x) = (p_d/p_u)^k\,(1 - (1/2)(p_d/p_u)^k)$ if $p_u > p_d$.

(iv) Among just paths that actually hit both axes, the probability that the x-axis is hit first is given by $P(B_x \mid A_x \cap A_y) = P(A_{x,y})/(P(A_x)\,P(A_y))$. Likewise, the probability that the y-axis is hit first is $P(B_y \mid A_x \cup A_y) = P(A_{y,x})/(P(A_x)\,P(A_y))$.

Example 1. Let (X, Y) start at $(2, 3)$ with $p_d = 0.24$, $p_u = 0.30$, $p_l = 0.21$, and $p_r = 0.25$. Then $P(A_x) = (0.24/0.30)^3 = 0.512$, $P(A_y) = (0.21/0.25)^2 = 0.7056$, $P(A_x \cap A_y) = P(A_x)P(A_y) = 0.3612672$, and $P(A_x \cup A_y) = 0.8563328$. The series given in Theorem 2 are slow to converge with no apparent reduced form, but by summing over $3 \le N \le 2000$ and $2 \le M \le 2000$, we obtain $P(A_{y,x}) \approx 0.21058621$ and $P(A_{x,y}) \approx 0.15060556$, giving $P(A_{y,x}) + P(A_{x,y}) \approx 0.36119 \approx P(A_x \cap A_y)$.

Thus we have $P(B_x) = P(A_x) - P(A_{y,x}) \approx 0.3014$ and $P(B_y) = P(A_y) - P(A_{x,y}) \approx 0.555$. So there is about a 30% chance of a particle hitting the x-axis

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before possibly reaching the y-axis. If we consider just paths that eventually hit both axes, then there is $P(A_{x,\,y})/(P(A_x)\,P(A_y)) \approx 0.41688$ probability of the x-axis being hit first.

6. Average Number of Steps

Suppose now that $p_u < p_d$ so that the two-dimensional random walk almost surely hits the x-axis. We now derive the expected value of the number of steps S_x needed to hit the x-axis, and the conditional average $E[S_x \mid B_x]$ for those that have hit before ever having reached the y-axis.

Theorem 4. *Let (X, Y) be a two-dimensional random walk that begins at integer-valued coordinates (h, k), with $h > 0$ and $k > 0$. Assume $p_u < p_d$, and let S_x be the number of steps needed to reach the x-axis.*

(a) The average number of steps needed to hit the x-axis is

$$E[S_x] = \frac{k}{p_d - p_u}\,.$$

(b) The weighted portion of $E[S_x]$ among paths that have first hit the y-axis is

$$E[S_x \mid A_{y,\,x}] \times P(A_{y,\,x}) = \frac{h\,k\,p_l^h p_d^k}{(p_r + p_l)^{h+k}} \times$$

$$\sum_{N=k}^{\infty} \sum_{i=0}^{\lfloor \frac{N-k}{2} \rfloor} \left(\frac{N!\,(p_r + p_l)^{N-2i}\,(p_u\,p_d)^i}{(N-k-2i)!\,(k+i)!\,i!} \sum_{j=0}^{\lfloor \frac{N-k-2i-h}{2} \rfloor} \frac{\binom{h+2j}{j}\,(p_r\,p_l)^j}{(h+2j)\,(p_r + p_l)^{2j}} \right).$$

(c) Among paths that hit the x-axis before ever possibly reaching the y-axis, the conditional average number of steps needed to hit the x-axis is

$$E[S_x \mid B_x] = \frac{E[S_x] - E[S_x \mid A_{y,\,x}]\,P(A_{y,\,x})}{1 - P(A_{y,\,x})}\,.$$

Proof. (a) Let ${}_kV_0$ be the number of vertical steps taken upon reaching the x-axis. Then $E[{}_kV_0]$ is given by (4), where $\tilde{p}_u = p_u/(p_u + p_d)$ and $\tilde{p}_d = p_d/(p_u + p_d)$. Given any N steps taken, the average number of vertical steps taken is $N \times (p_u + p_d)$. Thus,

$$\begin{aligned} \frac{k}{\tilde{p}_d - \tilde{p}_u} &= E[{}_kV_0] = \sum_{N=k}^{\infty} E[{}_kV_0 \mid S_x = N]\,P(S_x = N) \\ &= \sum_{N=k}^{\infty} N\,(p_u + p_d)\,P(S_x = N) = (p_u + p_d)E[S_x]\,, \end{aligned}$$

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and $E[S_x] = \dfrac{k}{(\tilde{p}_d - \tilde{p}_u)(p_u + p_d)} = \dfrac{k}{p_d - p_u}$.

(b) Let $C_{N,n,z}$ be the partition of paths used in the boundary problem. Then $P(S_x = N \cap A_{y,x} \mid C_{N,n,z}) = 0$ whenever the Nth step is horizontal, and $P(S_x = N \cap A_{y,x} \mid C_{N,n,z}) = f_V(n)F_H(N-n)$ otherwise. Thus,

$$E[S_x \mid A_{y,x}] \times P(A_{y,x}) = \sum_{N=k}^{\infty} N\, P(S_x = N \mid A_{y,x}) \times P(A_{y,x})$$

$$= \sum_{N=k}^{\infty} N\, P(S_x = N \cap A_{y,x})$$

$$= \sum_{N=k}^{\infty} N \sum_{n=k}^{N} \sum_{z=1}^{\binom{N}{n}} P(C_{N,n,z}) P(S_x = N \cap A_{y,x} \mid C_{N,n,z})$$

$$= \sum_{N=k}^{\infty} \sum_{n=k}^{N} N \binom{N-1}{n-1} p_V^n\, p_H^{N-n}\, f_V(n) F_H(N-n).$$

Expanding as in Equations (8) through (11) gives the result in (b).

(c) With $p_u < p_d$, we have the disjoint union $A_x = B_x \cup A_{y,x}$ with $P(A_x) = 1$. So $P(B_x) = 1 - P(A_{y,x})$. Thus, (c) follows from

$$E[S_x] = E[S_x \mid A_x] = E[S_x \mid B_x]P(B_x) + E[S_x \mid A_{y,x}]P(A_{y,x}). \tag{12}$$

□

We have analogous results, when $p_r < p_l$, for averages of the number of steps S_y needed to hit the y-axis. In particular, we have

$$E[S_y] = E[S_y \mid A_y] = E[S_y \mid B_y]P(B_y) + E[S_y \mid A_{x,y}]P(A_{x,y}). \tag{13}$$

When we have both $p_u < p_d$ and $p_r < p_l$, then almost surely we have $B_x = A_{x,y}$, $B_y = A_{y,x}$, and $B_x \cup B_y$ equaling the set of all paths. We now let $S_1 = \min(S_x, S_y)$ be the number of steps needed to hit either the x-axis or the y-axis, and $S_2 = \max(S_x, S_y)$ be the number of steps needed to hit both axes. Then

$$E[S_1] = E[S_x \mid B_x]\, P(B_x) + E[S_y \mid B_y]\, P(B_y) \tag{14}$$

and

$$E[S_2] = E[S_x \mid B_y]\, P(B_y) + E[S_y \mid B_x]\, P(B_x). \tag{15}$$

Substituting from (12) and (13), we obtain the following result:

Theorem 5. *Let (X, Y) be a two-dimensional random walk that begins at integer-valued coordinates (h, k), with $h > 0$ and $k > 0$. Assume $p_u < p_d$ and $p_r < p_l$.*

(a) The average number of steps needed to hit either the x-axis or the y-axis is

$$E[S_1] = E[S_x] + E[S_y] - E[S_x \,|A_{y,x}]P(A_{y,x}) - E[S_y \,|A_{x,y}]P(A_{x,y}).$$

(b) The average number of steps needed to hit both axes is

$$E[S_2] = E[S_x \mid A_{y,x}]P(A_{y,x}) + E[S_y \mid A_{x,y}]P(A_{x,y}).$$

Example 2. Let (X, Y) start at $(4, 3)$ with $p_d = 0.36$, $p_u = 0.24$, $p_l = 0.25$, and $p_r = 0.15$. Now $P(A_x) = P(A_y) = P(A_x \cap A_y) = P(A_x \cup A_y) = 1$. The series in Theorem 2 converge more quickly now, with sums over $3 \le N \le 800$ and $4 \le M \le 800$ giving $P(A_{y,x}) \approx 0.319252$ and $P(A_{x,y}) \approx 0.680747$. Also, $B_x = A_{x,y}$ a.s. and $B_y = A_{y,x}$ a.s. and $P(B_x) = P(A_{x,y}) = 1 - P(A_{y,x})$.

From Theorem 4(a), $E[S_x] = 25$. By summing the series in Theorem 4(b) for $3 \le N \le 1000$, we obtain $E[S_x \mid B_y]P(B_y) = E[S_x \,|\, A_{y,x}]P(A_{y,x}) \approx 15.856$. Then from Theorem 4(c) we have $E[S_x \mid A_{x,y}] = E[S_x \mid B_x] \approx 13.4323$.

By reversing the roles of x and y in Theorem 4, we obtain $E[S_y] = 40$, $E[S_y \mid B_x]P(B_x) = E[S_y \mid A_{x,y}]P(A_{x,y}) \approx 33.8255$, and then $E[S_y \mid A_{y,x}] = E[S_y \mid B_y] \approx 19.34$. Then from (14) or Theorem 5, the average number of steps needed to hit either axis is $E[S_1] \approx 15.3185$. Finally, the average number of steps needed to hit both axes is $E[S_2] \approx 49.6815$.

Depending on the behavior of the path after first hitting an axis, a path could be approaching either the second, third or fourth quadrants upon hitting the second axis. Fukai [2] gives an analysis of two-dimensional paths that actually enter the third quadrant. And Shimura [4] analyzes asymptotic behaviour of paths exitng a quadrant. Earlier work on two-dimensional boundary problems has been done by Mensing and David [3].

Finally, we also can determine the average of the x-coordinate when the two-dimensional random walk hits the x-axis from initial coordinates (h, k), for $k > 0$ and $p_u < p_d$. If T denotes the number of horizontal steps taken upon hitting the x-axis, then Theorem 4(a) and (4) gives

$$E[T] = \frac{k}{p_d - p_u} - \frac{k}{\tilde{p}_d - \tilde{p}_u} = \frac{k(p_r + p_l)}{p_d - p_u}.$$

After m horizontal steps, the average position of the one-dimensional random walk $\tilde{X}$ is $E[\tilde{X}_m] = h + m(\tilde{p}_r - \tilde{p}_l)$, where $\tilde{p}_r - \tilde{p}_l$ is the average change on the ith horizontal step. Then *Wald's Equation* gives

$$E[\tilde{X}_T] = h + E[T]\,(\tilde{p}_r - \tilde{p}_l) = h + \frac{k(p_r - p_l)}{p_d - p_u}.$$

For the values in Example 2, there are on average $E[T] = 10$ horizontal steps upon hitting the x-axis, with the the final x-coordinate being $E[\tilde{X}_T] = 1.5$.

The numerical results in Examples 1 and 2 can be verified with simulations of the scenario. As a project, one can write a program that performs multiple trials of a two-dimensional random walk, and then determines the proportion of those hitting one boundary before the other and the sample means of the number of steps needed to hit the boundaries in order to compare with the formulas of Theorems 2, 3, and 5. The results and simulations can be extended further to analyze two independent M/M/1 queues that each have independent exponential arrival and service times. Theorem 3 can be used to determine the probability of one queue clearing before the other. And with more work, the results in Section 6 can be used to determine the expected values of the *times* needed for one or both queues to clear.

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