International Electronic Journal of Pure and Applied Mathematics

Volume 6 No. 1 2013, 51-62
ISSN: 1314-0744
url: http://www.e.ijpam.eu
doi: http://dx.doi.org/10.12732/iejpam.v6i1.4

EXISTENCE OF POSITIVE UNBOUNDED SOLUTIONS FOR ϕ-LAPLACIAN BVPS ON THE HALF LINE

Abdelhamid Benmezaï[1], Salima Mechrouk[2], Sotiris K. Ntouyas[3 §]

[1]Faculty of Mathematics
USTHB, Po. Box 32, El-Alia Bab-ezouar, Algiers, ALGERIA
[2]Faculty of Sciences
University M'Hmed Bouguerra
Boumerdes, ALGERIA
[3]Department of Mathematics
University of Ioannina
451 10, Ioannina, GREECE

Abstract: We provide in this work sufficient conditions for existence of positive unbounded solutions to the boundary value problem

$$\begin{cases} -\left(\phi\left(u'\right)\right)' = a(t)f(t,u), \quad t \in (0,+\infty), \\ u\left(0\right) = 0, \ \lim\limits_{t\to+\infty} u'(t) = 0 \end{cases}$$

where $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ is an increasing homeomorphism with $\phi\left(0\right) = 0$, $a : (0,+\infty) \to \mathbb{R}^+$ is a measurable function which does not vanish identically on $(0,+\infty)$ and the function $f : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is continuous.

AMS Subject Classification: 34B15, 34B40
Key Words: positive unbounded solutions, boundary value problem

1. Introduction

In this paper, we are concerned with existence of positive unbounded solutions to

Received: January 14, 2013

[§]Correspondence author

the $\phi-$Laplacian boundary value problem

$$\begin{cases} -(\phi(u'))' = a(t)f(t,u), \quad t \in (0,+\infty), \\ u(0) = 0, \lim\limits_{t\to+\infty} u'(t) = 0 \end{cases} \tag{1.1}$$

where $\phi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is an increasing homeomorphism with $\phi(0) = 0$, $\mathbb{R}^+ = [0,+\infty)$, $a : (0,+\infty) \longrightarrow \mathbb{R}^+$ is a measurable function and $f : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous function.

We mean, by a positive solution to Problem (1.1), a function $u \in C^1(\mathbb{R}^+, \mathbb{R}^+)$ such that $\phi(u') \in C((0,+\infty),\mathbb{R})$ and $u(t_0) > 0$ for some $t_0 > 0$, satisfying all equations in (1.1).

Our main result will be obtained under the following assumptions:

$$\begin{cases} \text{There exist } m \in C((0,+\infty),\mathbb{R}^+) \text{ and } g \in C(\mathbb{R}^+,\mathbb{R}^+) \\ \text{such that } f(t,(1+t)w) \le m(t)g(w) \text{ for all } t,w \in \mathbb{R}^+ \\ \text{and } \displaystyle\int_0^{+\infty} a(t)m(t)dt < \infty, \end{cases} \tag{1.2}$$

$$\left\{ \lim_{t\to+\infty} \frac{1}{1+t} \int_0^t \psi\left(\int_s^{+\infty} a(\tau)m(\tau)d\tau\right) ds = 0, \right. \tag{1.3}$$

$$\begin{cases} \lim\limits_{t\to+\infty} t\psi\left(\displaystyle\int_t^{+\infty} a(t)f(t,\lambda)\,dt\right) = +\infty \text{ uniformly for } \lambda \\ \text{in compact intervals of } (0,+\infty), \end{cases} \tag{1.4}$$

and

$$\begin{cases} \text{there exists } \alpha > 0 \text{ such that for all } t \in [0,1] \text{ and } u \in \mathbb{R}^+, \\ \phi(tu) \ge t^\alpha \phi(u), \end{cases} \tag{1.5}$$

where ψ denotes the inverse function of ϕ.

Because of their mathematical and physical interest, the study of second order differential equations posed on the half line and subject to various boundary conditions have received a great deal of attention during the latter two decades; see [9]-[14] and references therein.

This work is motivated by that in [12], where D. O'Regan *et al.* consider the Problem (1.1) with $\phi(u) = u$ and f may be singular at $u = 0$. They obtain existence and multiplicity results for positive solutions in the functional space constituted by the functions $u \in C(\mathbb{R}^+, \mathbb{R}^+)$ satisfying $\lim_{t\to+\infty} u(t)/(1+t) = 0$ endowed with the norm $\|u\| = \sup_{t\in\mathbb{R}^+} \dfrac{|u(t)|}{1+t}$. Clearly, the choice of this space is motivated by the boundary condition in Problem (1.1), $u'(+\infty) = 0$ and fortunately, this space provide a good framework where the fixed point index theory or theorems of cone expansion and compression in a Banach space can be used.

The unique disadvantage of this framework is that we know nothing about the boundeness of the obtained positive solutions.

The main goal of this paper is to provide existence results for positive unbounded solutions under additionnal assumptions on the behavior of the ratio $f(t,u)/\phi(u)$ at 0 and $+\infty$ as those obtained in [1]-[6]. We will make use of a theorem of cone expansion and compression in a Banach space in the same framework as that in [12]. In contrast to [12], here hypothesis (1.4) ensure that the obtained positive solution is unbounded.

2. Preliminaries

The main tool of this paper is the following theorem of cone expansion and compression in a Banach space.

Theorem 2.1. *([8]) Let X be a Banach space and P be a cone of X. Assume Ω_1, Ω_2 are open bounded subsets of X with $0 \in \Omega_1 \subset \bar{\Omega_1} \subset \Omega_2$ and let $A : P \cap \left(\bar{\Omega_2} \backslash \Omega_1\right) \longrightarrow P$ be a completely continuous operator such that either*

i) $\|Au\| \leq \|u\|$ *for all* $u \in P \cap \partial\Omega_1$ *and* $\|Au\| \geq \|u\|$ *for all* $u \in P \cap \partial\Omega_2$,

ii) $\|Au\| \leq \|u\|$ *for all* $u \in P \cap \partial\Omega_2$ *and* $\|Au\| \geq \|u\|$ *for all* $u \in P \cap \partial\Omega_1$.

Then A has at least one fixed point in $P \cap \left(\bar{\Omega_2} \backslash \Omega_1\right)$.

In all this paper E denotes the Banach space defined

$$E = \left\{ u \in C\left(\mathbb{R}^+, \mathbb{R}^+\right) : \lim_{t \to +\infty} \frac{u(t)}{1+t} = 0 \right\}$$

equiped with the norm $\|\cdot\|_E$, defined for $u \in E$ by $\|u\|_E = \sup_{t \geq 0} \left| \frac{u(t)}{1+t} \right|$. In order to prove the compacteness of some operator we will make use of the following lemma.

Lemma 2.2. *([7], [12], [13]) A nonempty subset M of E is relatively compact if the following conditions hold:*

(a) M is bounded in E,

(b) the functions belonging to $\left\{ u : u(t) = \frac{x(t)}{1+t},\ x \in M \right\}$ are locally equicontinuous on $[0, +\infty)$, that is, equicontinuous on every compact interval of $\mathbb{R}^+$ and

International Electronic Journal of Pure and Applied Mathematics – IEJPAM, Volume 6, No. 1 (2013)

(c) the functions belonging to $\left\{u : u(t) = \dfrac{x(t)}{1+t},\ x \in M\right\}$ *are equiconvergent at* $+\infty$, *that is, given* $\epsilon > 0$, *there corresponds* $T(\epsilon) > 0$ *such that* $|x(t) - x(+\infty)| < \epsilon$ *for any* $t \geq T(\epsilon)$ *and* $x \in M$.

Throughout, K is the cone of E given by

$$K = \{u \in E,\ u \geq 0,\ u(0) = 0 \text{ and u is concave in } (0, +\infty)\}. \tag{2.1}$$

Lemma 2.3. *Let* $u \in K$ *and* $\theta \in (1, +\infty)$. *Then*

$$u(t) \geq \frac{1}{\theta} \|u\|_E\,,\ for\ all\ t \in \left(\frac{1}{\theta}, \theta\right).$$

Proof. First, note that the function $h(t) = \dfrac{u(t)}{1+t}$ is continuous and satisfies $h(0) = h(+\infty) = 0$, so, it achieves its maximum at some $t_0 > 0$.

Then, since u is concave and nondecreasing on $(0, +\infty)$, we have for all $t \in \left[\dfrac{1}{\theta}, \theta\right]$

$$\begin{aligned} u(t) &\geq \min_{t \in [\frac{1}{\theta}, \theta]} u(t) = u\left(\frac{1}{\theta}\right) = u\left(\frac{\theta - 1 + \theta t_0}{\theta + \theta t_0}\frac{1}{\theta - 1 + \theta t_0} + \frac{1}{\theta + \theta t_0} t_0\right) \\ &\geq \frac{\theta - 1 + \theta t_0}{\theta + \theta t_0} u\left(\frac{1}{\theta - 1 + \theta t_0}\right) + \frac{1}{\theta + \theta t_0} u(t_0) \\ &\geq \frac{1}{\theta + \theta t_0} u(t_0) = \frac{u(t_0)}{\theta(1 + t_0)} = \frac{1}{\theta}\|u\|_E\,. \end{aligned}$$

The lemma is proved. □

Remark 2.4. We have from the above lemma that for all $u \in K$ and $\theta > 1$

$$\frac{u(t)}{1+t} \geq \frac{1}{\theta(1+\theta)} \|u\|_E\ \ for\ all\ t \in \left(\frac{1}{\theta}, \theta\right).$$

Let ψ be the inverse function of ϕ and consider the operator $T : K \to K$ defined for $u \in K$ by

$$Tu(t) = \int_0^t \psi\left(\int_s^{+\infty} a(r) f(r, u(r))\, dr\right) ds.$$

Lemma 2.5. *Assume that* (1.2) *and* (1.3) *hold. Then the operator* T *is well defined.*

Proof. Let $u \in K$ and $v = Tu$. We have for arbitrary $\delta > 0$,

$$v(t) = \int_0^t \psi\left(\int_s^{+\infty} a(r) f(r, u(r))\, dr\right) ds$$

$$
\begin{aligned}
&\le \int_0^t \psi\left(\int_s^{+\infty} a(r)m(r)g\left(\frac{u(r)}{1+r}\right)dr\right)ds \\
&\le \overline{g}\int_0^\delta \psi\left(\int_s^{+\infty} a(r)m(r)dr\right)ds < \infty \ for\ all\ t\in[0,\delta]
\end{aligned}
$$

and

$$v(t_2)-v(t_1)\le \overline{g}\int_{t_1}^{t_2}\psi\left(\int_s^{+\infty} a(r)m(r)dr\right)ds$$

for all $t_1, t_2 \in [0,\delta]$ with $t_1 < t_2$, where $\overline{g} = \max\{g(t),\ t\in[0,\|u\|_E]\}$.

Clearly, the above inequalities mean that v is defined and continuous on $[0,\delta]$ and since δ is arbitrary v is continuous on $\mathbb{R}^+$.

We have obviously that $v(0) = 0$ and $(\phi(v'))'(t) = -a(t) f(t, u(t)) \le 0$. Thus, the facts that ϕ is increasing and $\lim_{t\to+\infty} v'(t) = 0$ imply that v' is nonincreasing and nonnegative on $(0,+\infty)$. That is, v is concave.

Finally we have

$$
\begin{aligned}
\frac{v(t)}{1+t} &= \frac{1}{1+t}\int_0^t \psi\left(\int_s^{+\infty} a(r)f(r,u(r))\,dr\right)ds \\
&\le \overline{g}\frac{1}{1+t}\int_0^t \psi\left(\int_s^{+\infty} a(r)m(r)dr\right)ds
\end{aligned}
$$

which combined with hypothesis (1.3), imply that $\lim\limits_{t\to+\infty}\dfrac{Tu(t)}{1+t} = 0$, and $Tu \in K$. □

Lemma 2.6. *Assume that* (1.2) *and* (1.3) *hold. Then the operator T is completely continuous.*

Proof. In order to prove that T is continuous, let $(u_n)_n \subset K$ be such that $\lim u_n = u$ in E and $R > 0$ be such that $\|u_n\|_E \le R$ for all positive integer n.

We have $\lim_{n\to\infty} f(t, u_n(t)) = f(t, u(t))$ for all $t \ge 0$ and for all $s \ge 0$

$$
\begin{aligned}
\Big|\int_s^{+\infty} a(t)f(t,u_n(t))\,dt &- \int_s^{+\infty} a(t)f(t,u(t))\,dt\Big| \\
&\le \int_0^{+\infty} a(t)m(t)\left|\frac{f(t,u_n(t))}{m(t)} - \frac{f(t,u(t))}{m(t)}\right|dt.
\end{aligned}
$$

Since $\lim_{n\to\infty} f(t, u_n(t)) = f(t, u(t))$ for all $t \ge 0$ and

$$\left|\frac{f(t,u_n(t))}{m(t)} - \frac{f(t,u(t))}{m(t)}\right| \le 2g_R = 2\max_{\eta\in[0,R]} g(\eta),$$

the sequence

$$\left(\int_s^{+\infty} a(t)f(t,u_n(t))\,dt\right)$$

converge uniformly to

$$\left(\int_s^{+\infty} a(t)f\left(t,u(t)\right)dt\right)$$

by Lebesgue dominated convergence theorem.

Thus, the uniform continuity of ψ on compact intervals of $\mathbb{R}^+$ implies that for all $\epsilon>0$ there exists a positive integer n_ϵ such that for all $n\geq n_\epsilon$ and all $s\geq 0$

$$\left|\psi\left(\int_s^{+\infty} a(t)f\left(t,u_n(t)\right)dt\right)-\psi\left(\int_s^{+\infty} a(t)f\left(t,u(t)\right)dt\right)\right|\leq\epsilon.$$

Then

$$\begin{aligned}
&\left\|Tu_n-Tu\right\|_E\\
=\ &\sup_{t\in\mathbb{R}^+}\left|\frac{\int_0^t\left[\psi\left(\int_s^{+\infty} a(r)f\left(r,u_n(r)\right)dr\right)-\psi\left(\int_s^{+\infty} a(r)f\left(r,u(r)\right)dr\right)\right]ds}{1+t}\right|\\
\leq\ &\sup_{t\in\mathbb{R}^+}\left[\frac{\int_0^t\epsilon dr}{1+t}\right]=\epsilon.
\end{aligned}$$

Let $\delta>0$ and $B_\delta=\{u\in K:\ \left\|u\right\|_E\leq\delta\}$. We have for all $u\in B_\delta$

$$\begin{aligned}
\left\|Tu\right\|_E &= \sup_{t\in\mathbb{R}^+}\left[\frac{1}{1+t}\int_0^t\psi\left(\int_s^{+\infty} a(r)f\left(r,u(r)\right)dr\right)ds\right]\\
&\leq \sup_{t\in\mathbb{R}^+}\left[\frac{1}{1+t}\int_0^t\psi\left(\int_s^{+\infty} a(r)m(r)g\left(\frac{u(r)}{1+r}\right)dr\right)ds\right]\\
&\leq g_\delta\Delta,
\end{aligned}$$

where $g_\delta=\sup_{\eta\in[0,\delta]}g\left(\eta\right)$ and $\Delta=\sup_{t\in\mathbb{R}^+}\left[\frac{1}{1+t}\int_0^t\psi\left(\int_s^{+\infty} a(r)m(r)dr\right)ds\right]$. This means that $T\left(B_\delta\right)$ is uniformly bounded.

Let $A>0$ and $t_1,t_2\in[0,A]$. We have:

$$\begin{aligned}
\left|\frac{Tu(t_1)}{1+t_1}-\frac{Tu(t_2)}{1+t_2}\right| &= \left|\frac{1}{1+t_1}\int_0^{t_1}\psi\left(\int_s^{+\infty} a(r)f\left(r,u(r)\right)dr\right)ds\right.\\
&\quad\left.-\frac{1}{1+t_2}\int_0^{t_2}\psi\left(\int_s^{+\infty} a(r)f\left(r,u(r)\right)dr\right)ds\right|\\
&\leq \left|\frac{1}{1+t_1}\int_{t_2}^{t_1}\psi\left(\int_s^{+\infty} a(r)f\left(r,u(r)\right)dr\right)ds\right|\\
&\quad+\left|\left(\frac{1}{1+t_1}-\frac{1}{1+t_2}\right)\int_0^{t_2}\psi\left(\int_s^{+\infty} a(r)f\left(r,u(r)\right)dr\right)ds\right|
\end{aligned}$$

$$\leq \quad g_\delta \left| \int_{t_2}^{t_1} \psi \left(\int_s^{+\infty} a(r)m\,(r)\,dr \right) ds \right| + |t_1 - t_2|\, g_\delta \Delta.$$

This together the uniform continuity of the function

$$t \to \int_0^t \psi \left(\int_s^{+\infty} a(r)m\,(r)\,dr \right) ds$$

on the interval $[0, A]$ imply that $T(B_\delta)$ is equicontinuous on $[0, A]$.

At this stage we have for all $u \in B_\delta$

$$\left| \frac{Tu(t)}{1+t} - \lim_{t\to+\infty} \frac{Tu(t)}{1+t} \right| = \frac{Tu(t)}{1+t} \leq g_\delta \frac{1}{1+t} \int_0^t \psi \left(\int_s^{+\infty} a(r)m(r)dr \right) ds.$$

This together with hypothesis (1.3) implies that $T\,(B_\delta)$ is equiconvergent.

Therefore we deduce from Lemma 2.2 that $T\,(B_\delta)$ is relatively compact in E and the operator T is completly continuous. □

It is easy to prove the following lemma.

Lemma 2.7. *Assume that* (1.2) *and* (1.3) *hold. Then* u *is a positive solution to Problem (1.1) if and only if* u *is a positive fixed point of* T.

Lemma 2.8. *Assume that* (1.2), (1.3) *and* (1.4) *hold. Then any positive solution* u *of Problem (1.1) is unbounded (ie:* $\lim_{t\to+\infty} u(t) = +\infty$*).*

Proof. Let u be a positive solution of Problem (1.1). We have from Lemma 2.7 that

$$\begin{aligned} u(t) &= \int_0^t \psi \left(\int_s^{+\infty} a(\tau) f\,(\tau, u(\tau))\,d\tau \right) ds \\ &\geq t\psi \left(\int_t^{+\infty} a(\tau) f\,(\tau, u(\tau))\,d\tau \right). \end{aligned}$$

Suppose that u is bounded and let $u_\infty = \lim_{t\to+\infty} u(t) > 0$. Let $\epsilon_0 > 0$ be such that $u_\infty - \epsilon_0 > 0$. There exists $t_\infty > 0$ such that $u(t) \geq u_\infty - \epsilon_0$ for all $t \geq t_\infty$.

Thus we obtain from hypothesis (1.4) and the above inequality the contradiction

$$+\infty > u_\infty = \lim_{t\to+\infty} u(t) \geq \lim_{t\to+\infty} t\psi \left(\int_t^{+\infty} a(\tau) f\,(\tau, u(\tau))\,d\tau \right) = +\infty.$$

This completes the proof. □

3. The Main Result

The statement of the main result of this paper needs to introduce the following notations. We have from hypothesis (1.5) that

$$t\psi(x) \geq \psi(t^{\alpha}x) \text{ for all } t \in [0,1] \text{ and } x \in \mathbb{R}^{+} \tag{3.1}$$

or in another manner

$$\psi(tx) \leq t^{1/\alpha}\psi(x) \text{ for all } t \in [0,1] \text{ and } x \in \mathbb{R}^{+}. \tag{3.2}$$

In fact (1.5) is a technical condition and we often met in $\phi-$Laplacian BVPs literature conditions looking like it, see [1]-[6] and references therein.

Let $\theta > 1$ be fixed and set $I_{\theta} = [1/\theta, \theta]$,

$$g^{0} = \limsup_{w \longrightarrow 0} \frac{g(w)}{\phi(w)}, \qquad g^{\infty} = \limsup_{w \to +\infty} \frac{g(w)}{\phi(w)},$$

$$f_{0} = \liminf_{w \longrightarrow 0} \left(\min_{t \in I_{\theta}} \frac{f(t,(1+t)w)}{\phi(w)} \right), \quad f_{\infty} = \liminf_{w \longrightarrow +\infty} \left(\min_{t \in I_{\theta}} \frac{f(t,(1+t)w)}{\phi(w)} \right)$$

$$\Gamma = \left(\int_{0}^{+\infty} a(r)m(r)dr \right)^{-1}, \qquad \Theta(\theta) = (1+\theta)^{2\alpha}\theta^{\alpha} \left(\int_{\frac{1}{\theta}}^{\theta} a(r)dr \right)^{-1}.$$

Theorem 3.1. *Assume that hypotheses* (1.2) – (1.5) *and one of the following conditions*

$$g^{0} < \Gamma, \quad \Theta(\theta) < f_{\infty} \tag{3.3}$$

and

$$g^{\infty} < \Gamma, \quad \Theta(\theta) < f_{0} \tag{3.4}$$

hold true. Then Problem (1.1) has at least one unbounded positive solution.

Proof. **a).** We consider first the case where (3.3) is satisfied. Let $\epsilon > 0$ be such that $(g^{0} + \epsilon) < \Gamma$. For a such ϵ, there exists $R_1 > 0$ such that $g(u) \leq (g^{0}+\epsilon)\phi(u)$ for all $u \in [0, R_1]$. Thus we have for all $u \in K \cap \partial\Omega_1$, where $\Omega_1 = \{u \in E : \|u\|_E < R_1\}$, that

$$\begin{aligned}
\|Tu\|_E &= \sup_{t\in\mathbb{R}^+} \left[\frac{1}{1+t} \int_0^t \psi\left(\int_s^{+\infty} a(r) f(r, u(r))\, dr \right) ds \right] \\
&\leq \sup_{t\in\mathbb{R}^+} \left[\frac{1}{1+t} \int_0^t \psi\left(\int_s^{+\infty} a(r)m(r) g\left(\frac{u(r)}{1+r} \right) dr \right) ds \right] \\
&\leq \sup_{t\in\mathbb{R}^+} \left[\frac{t}{1+t} \psi\left(\int_0^{+\infty} a(r)m(r)(g^0+\epsilon)\phi\left(\frac{u(r)}{1+r} \right) dr \right) \right] \\
&\leq \psi\left(\phi\left(\|u\|_E \right) (g^0+\epsilon)\Gamma^{-1} \right) < \|u\|_E .
\end{aligned}$$

Now, let $\nu > 0$ be such that $(f_\infty - \nu) > \Theta(\theta)$. There exists $R_2 > R_1$ such that $f(r,u) > (f_\infty - \nu)\phi\left(\frac{u}{1+r}\right)$ for all $u \geq R_2$ and all $r \in I_\theta$. Thus we have for all $u \in K \cap \partial\Omega_2$, where $\Omega_2 = \{u \in E : \|u\|_E < \theta(1+\theta)R_2\}$, that

$$\begin{aligned} \|Tu\|_E &\geq \frac{\theta Tu(1/\theta)}{1+\theta} \geq \frac{\theta}{1+\theta}\int_0^{\frac{1}{\theta}} \psi\left(\int_{\frac{1}{\theta}}^{\theta} a(r) f(r,u(r))\,dr\right) ds \\ &> \frac{1}{1+\theta}\psi\left(\int_{\frac{1}{\theta}}^{\theta} a(r)(f_\infty - \nu)\phi\left(\frac{u(r)}{1+r}\right) dr\right) \\ &\geq \frac{1}{1+\theta}\psi\left(\int_{\frac{1}{\theta}}^{\theta} a(r)(f_\infty - \nu)\phi\left(\frac{1}{\theta(1+\theta)}\|u\|_E\right) dr\right) \\ &\geq \psi\left((f_\infty - \nu)\frac{1}{(1+\theta)^\alpha}\frac{1}{\theta^\alpha(1+\theta)^\alpha}\left(\int_{\frac{1}{\theta}}^{\theta} a(r)dr\right)\phi(\|u\|_E)\right) \\ &= \psi\left((f_\infty - \nu)(\Theta(\theta))^{-1}\phi(\|u\|_E)\right) \geq \|u\|_E. \end{aligned}$$

Therfore, we deduce from i) of Theorem 2.1 that T admits a fixed point $u \in K$ with $R_1 < \|u\|_E < R_2$ which is, by Lemmas 2.7 and 2.8, a positive unbounded solution to Problem (1.1).

b). Now we consider the case where condition (3.4) is satisfied. Let $\epsilon > 0$ be such that $(f_\infty - \epsilon) > \Theta(\theta)$. There exists $R_1 > 0$ such that $f(r,u) > (f_\infty - \nu)\phi\left(\frac{u}{1+r}\right)$ for all $u \leq R_1$ and all $r \in I_\theta$. Thus we have for all $u \in K \cap \partial\Omega_1$, where $\Omega_1 = \{u \in E : \|u\|_E < \theta(1+\theta)R_1\}$, that

$$\begin{aligned} \|Tu\|_E &\geq \frac{\theta Tu(1/\theta)}{1+\theta} \geq \frac{\theta}{1+\theta}\int_0^{\frac{1}{\theta}} \psi\left(\int_{\frac{1}{\theta}}^{\theta} a(t) f(r,u(r))\,dr\right) ds \\ &> \frac{1}{1+\theta}\psi\left(\int_{\frac{1}{\theta}}^{\theta} a(r)(f_\infty - \epsilon)\phi\left(\frac{u(r)}{1+r}\right) dr\right) \\ &\geq \frac{1}{1+\theta}\psi\left(\int_{\frac{1}{\theta}}^{\theta} a(r)(f_\infty - \epsilon)\phi\left(\frac{1}{\theta(1+\theta)}\|u\|_E\right) dr\right) \\ &\geq \psi\left((f_\infty - \epsilon)\frac{1}{(1+\theta)^\alpha}\frac{1}{\theta^\alpha(1+\theta)^\alpha}\left(\int_{\frac{1}{\theta}}^{\theta} a(r)dr\right)\phi(\|u\|_E)\right) \\ &= \psi\left((f_\infty - \epsilon)(\Theta(\theta))^{-1}\phi(\|u\|_E)\right) > \|u\|_E. \end{aligned}$$

Let $\nu > 0$ be such that $(g^\infty + \nu) < \Gamma$. Then there exists $c > 0$ and $R_2 > R_1$ such that

$$g(u) \leq (g^\infty + \nu)\phi(u) + c \text{ for all } t, u \geq 0$$

International Electronic Journal of Pure and Applied Mathematics – IEJPAM, Volume 6, No. 1 (2013)

and

$$\left(g^{\infty}+\nu\right)\Gamma^{-1}\phi\left(R_2\right)+c\Gamma^{-1}<\phi\left(R_2\right).$$

Thus we have for all $u\in K\cap\partial\Omega_2$, where $\Omega_2=\left\{u\in E:\ \|u\|_E<R_2\right\}$, that

$$\begin{aligned}\|Tu\|_E &\leq \psi\left(\int_0^{+\infty}a(r)m(r)g\left(\frac{u(r)}{1+t}\right)dr\right)\\ &\leq \psi\left(\int_0^{+\infty}a(r)m(r)\left((g^{\infty}+\nu)\phi\left(\frac{u(r)}{1+r}\right)+c\right)dr\right)\\ &\leq \psi\left(\Gamma^{-1}(g^{\infty}+\nu)\phi\left(\|u\|\right)_E+c\Gamma^{-1}\right)\leq\|u\|_E.\end{aligned}$$

We deduce from ii) of Theorem 2.1 that T admits a fixed point $u\in K$ with $R_1<\|u\|_E<R_2$ which is, by Lemmas 2.7 and 2.8 a positive unbounded solution to Problem (1.1). This ends the proof. □

Example 3.2. Consider the boundary value problem (1.1) with $\phi(u)=u^{p-1}+u^{q-1}$, $2<p<q$, $a(t)=\dfrac{1}{(1+t)^p}$ and $f(t,u)=\dfrac{t}{1+t}\dfrac{Au^{p-1}}{(1+t)^{p-1}}$.

Clearly (1.5) is satisfied with $\alpha=q-1$ and (1.2) is satisfied with $m(t)=1$ and $g(w)=Aw^{p-1}$.

Straighforward computations lead to

$$\int_s^{+\infty}a(t)m(t)dt=\int_s^{+\infty}\frac{dt}{(1+t)^p}=\frac{1}{(p-1)(1+s)^{p-1}}.$$

Then we have from (3.2)

$$\psi\left(\int_s^{+\infty}a(t)m(t)dt\right)\leq(1+s)^{-\frac{p-1}{q-1}}\psi\left(\frac{1}{p-1}\right)\ for\ all\ s>0.$$

Integrating we get

$$\int_0^t\psi\left(\int_s^{+\infty}a(t)m(t)dt\right)ds\leq\frac{q-1}{q-p}\psi\left(\frac{1}{p-1}\right)\left(1-(1+t)^{\frac{p-q}{q-1}}\right)\leq\frac{q-1}{q-p}\psi\left(\frac{1}{p-1}\right)$$

leading to

$$\lim_{t\to+\infty}\frac{1}{1+t}\int_0^t\psi\left(\int_s^{+\infty}a(t)m(t)dt\right)ds=0$$

and (1.3) is satisfied.

Let $\lambda>0$. We have

$$\int_t^{+\infty}a(s)f(s,\lambda)ds = A\lambda^{p-1}\left(\frac{1}{2(p-1)}\frac{1}{(1+t)^{2(p-1)}}-\frac{1}{2p-1}\frac{1}{(1+t)^{2p-1}}\right)$$

$$\geq \quad A\lambda^{p-1}\frac{1}{2\,(p-1)}\frac{t}{(1+t)^{2(p-1)}}.$$

Then from (3.2) we get

$$\psi\left(\int_t^{+\infty} a(s)f(s,\lambda)ds\right) \geq \psi\left(\frac{A\lambda^{p-1}}{2\,(p-1)}\right)\frac{t^{\frac{1}{p-1}}}{(1+t)^2}$$

and

$$\lim_{t\to+\infty} t\psi\left(\int_t^{+\infty} a(s)f(s,\lambda)ds\right) \geq \psi\left(\frac{A\lambda^{p-1}}{2\,(p-1)}\right)\lim_{t\to+\infty}\frac{t^{\frac{p}{p-1}}}{(1+t)^2} = +\infty,$$

and thus (1.4) is satisfied.

Let $\theta > 1$ be fixed. By simple computations we get

$$g^{\infty} = 0 \quad \text{and} \quad f_0 = \frac{A}{1+\theta}.$$

Thus, we conclude from Theorem 3.1 that Problem (1.1) has at least one positive unbounded solution if $A > (1+\theta)\,\Theta\,(\theta)$.

Acknowledgments

A. Benmezai and S. Mechrouk would like to thank their laboratory Fixed point theory and apllications for supporting this work.

References

[1] N. Benkaci-Ali, A. Benmezai and J. Henderson, Existence of positive solutions to three point $\phi-$ Laplacian BVPs via homotopic deformations, *Electron. J. Differential Equations, Vol. 2012 (2012), No. 126, 1-8.*

[2] A. Benmezai, S. Djebali and T. Moussaoui, Positive solution for $\phi-$Laplacian Dirichlet BVPs, *Fixed Point Theory* **8**, No. 2 (2007), 167-186.

[3] A. Benmezai, S. Djebali and T. Moussaoui, Multiple positive solution for $\phi-$ Laplacian Dirichlet BVPs, *Panamer. Math. J.* **17**, No. 3 (2007), 53-73.

[4] A. Benmezai, S. Djebali and T. Moussaoui, Existence results for one dimensional Dirichlet $\phi-$Laplacian BVPs: Fixed point approach, *Dynam. Systems Appl.*, **17** (2008), 149-166.

[5] A. Benmezai and S. Mechrouk, Positive solutions to the abstract nonlinear Hammerstein equation and application to $\phi-$Laplacian BVPs, *Nonlinear Differ. Equ. Appl., DOI 10.1007/s00030-012-0162-5*.

[6] A. Benmezai, S. Mechrouk and S. K. Ntouyas, Global curve of positive solutions for $\phi-$ Laplacian Dirichlet BVP with at most one turning point, *Differ. Equ. Appl.*, *to appear.*

[7] C. Corduneanu, *Integral Equations and Stability of Feedback Systems*, Academic Press, New York, 1973.

[8] D. Guo and V. Lakshmikantaham, *Nonlinear problems in abstract cones*, Academic Press, San Diego, 1988.

[9] P. Kang and Z. Wei, Multiple solutions of second-order three-point singular boundary value problems on the half-line, *Appl. Math. Comput.* **203**, (2008) 523-535.

[10] S. Liang and J. Zhang, The existence of countably many positive solutions for nonlinear singular m¡point boundary value problems on the half line, *J. Comput. Appl. Math.*, **222** (2008) 229-243.

[11] S. Liang and J. Zhang, The existence of countably many positive solutions for one-dimensional $p-$Laplacian with infinitely many singularities on the half-line, *Appl. Math. Comput.*, **201** (2008), 210-220.

[12] D. O'Regan, B. Yan and R.P. Agarwal, Solutions in weighted spaces of singular boundary value problems on the half-line, *J. Comput. Appl. Math.*, **205** (2007), 751-763.

[13] B. Yan, D. O'Regan and R.P. Agarwal, Unbounded solutions for singular boundary value problems on the semi-infinite interval: Upper and lower solutions and multiplicity, *J. Comput. Appl. Math.*, **197** (2006), 365-386.

[14] Y. Wei, P.J.Y. Wong and W. Ge, The existence of multiple positive solutions to boundary value problems of nonlinear delay differential equations with countably many singularities on infinite interval, *J. Comput. Appl. Math.*, **233** (2010), 2189-2199.