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REPRESENTATION THEOREMS IN A 4-DIMENSIONAL EUCLIDEAN SPACE. A NEW CASE

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Abstract: In a 4-dimensional Euclidean space, representation theorems have been recently obtained for isotropic functions depending on an arbitrary number of scalars, skew-symmetric second order tensors and symmetric second order tensors; at least one of these last ones is assumed to have an eigenvalue with multiplicity 1. The case with at least a non null vector, among the independent variables, has already been treated in literature. Here the new case is considered where no symmetric tensor has eigenvalues with multiplicity 1, but there is at least one symmetric tensor with two distinct eigenvalues. The result is a finite, but long, set of scalar valued isotropic functions such that every other scalar function of the same variables can be expressed as a function of the elements of this set. Similarly, a set of symmetric tensor valued isotropic functions is found such that every other symmetric tensor valued function of the same variables can be expressed as a linear combination, trough scalar coefficients, of the elements of this set. Finally, we obtain also a set of skew-symmetric tensor valued isotropic functions such that every other skew-symmetric tensor valued function of the same variables can be expressed as a linear combination, through scalar coefficients, of the elements of this set.

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$$
\begin{tikzcd}
X_{\beta_1\cdots\beta_m} \arrow[r, "F^{\alpha_1\cdots\alpha_n}"] \arrow[d, "P^{\beta}_{\beta'}"] & F^{\alpha_1\cdots\alpha_n}(X_{\beta_1\cdots\beta_m}) \arrow[d, "P^{\alpha'}_{\alpha}"] \\
X_{\beta'_1\cdots\beta'_m} = P^{\beta_1}_{\beta'_1}\cdots P^{\beta_m}_{\beta'_m} X_{\beta_1\cdots\beta_m} \arrow[r, "F^{\alpha'_1\cdots\alpha'_n}"] & \begin{array}{l} P^{\alpha'_1}_{\alpha_1}\cdots P^{\alpha'_n}_{\alpha_n} F^{\alpha_1\cdots\alpha_n}(X_{\beta_1\cdots\beta_m}) = \\ F^{\alpha'_1\cdots\alpha'_n}(P^{\beta_1}_{\beta'_1}\cdots P^{\beta_m}_{\beta'_m} X_{\beta_1\cdots\beta_m}) \end{array}
\end{tikzcd}
$$

Figure 1: The condition defining isotropic functions.

1. Introduction

Representation theorems are the mathematical tool by which to impose the physical principle requiring that the laws of physics do not depend on the observer. Now these laws are expressed in terms of tensors $F^{\alpha_1\cdots\alpha_n}$ which, in turns, are functions of other tensors $X_{\beta_1\cdots\beta_m}$; moreover, we know the transformation laws of the components of a tensor, when the basis of the vectorial space is changed. Therefore, the above mentioned requirement amounts in imposing that the diagram in Figure 1 is commutative, where $P^{\alpha'}_{\alpha}$ is the orthonormal matrix of the change of basis. In other words, proceeding from the upper left corner of the figure and moving to the right hand side, we start from the independent variables $X_{\beta_1\cdots\beta_m}$ in the reference frame Σ, then we apply to them the function $F^{\alpha_1\cdots\alpha_n}$, in Σ; after that, proceeding to the lower side, we transform the result in the reference Σ'. Following the other side of the diagram, we transform the independent variables in Σ'; on the transformed variables, we apply the function $F^{\alpha'_1\cdots\alpha'_n}$, in Σ'. We require that the result is the same.

What has already been done, about this, in literature? Regarding the framework of a 3-dimensional Euclidean vectorial space, in [5], [6] the case has been considered with $n = 0$ (an arbitrary number of scalars), $n = 1$ (an arbitrary number of vectors), $n = 2$ (an arbitrary number of second order tensors, some of which are symmetric and the remaining ones are skew-symmetric); similar values have been considered for m, that is to distinguish the different types of tensorial functions. The result is a set S^0 of particular scalar functions such that every other scalar function of the same variables can be expressed as a function of the elements of S^0; similarly, for the other values of m we obtain a set S^m of particular tensorial functions of order m such that every other tensorial function of the same order, and depending on the same variables, can be expressed as a linear combination of the elements of

S^m through scalar coefficients. The sets S^m are called "representations"; they are called "irreducible" if no proper subset satisfies the same property. In [1] Boheler proved that the representations exposed in [5] are not irreducible, showing also some their redundant elements. In [3] Pennisi and Trovato proved that once eliminated the elements indicated by Boheler, the remaining elements furnish a complete and irreducible representation. In [2] complete representations were found also for the case $m = 3$, that is third order tensorial functions.

Regarding the case of a 4-dimensional vectorial space, Pennisi and Trovato furnished in [4] complete representations, but only for the case of a pseudo-Euclidean vectorial space and with the hypothesis that, among the independent variables, there is a time-like 4-vector. Obviously, their result holds also in the case of a 4-dimensional Euclidean vectorial space when, among the independent variables, there is a 4-vector different from zero. In references [7], [8], [9] the case has been considered where there are no vectors among the independent variables, but among the independent variables, there is at least a symmetric second order tensor $\mathcal{A}$ endowed with an eigenvalue a with multiplicity 1. We exploit here the new case where no symmetric tensor has eigenvalues with multiplicity 1, but there is at least one symmetric tensor with two distinct eigenvalues.

The treatment of this whole subject will be splitted in different parts and we will devote a section for each of them; those described in Section 2 express the results in terms of those obtained in refs. [7], [8], [9] and we will omit to report the corresponding tables, for the sake of brevity. The other parts are described in Sections 3-5 and their contribute to the sets S^0, S^2 is reported in partial tables. We report now in table S the union of the elements of the partial tables for scalars, i.e., their contribute to the set S^0; obviously, we have omitted the elements which can be expressed as functions of the remainder. Similarly, the contribute of the partial tables to the set S^2 is reported in table Sy for symmetric functions, and in table Sk for skew-symmetric functions. As before, we have omitted the elements which can be expressed as linear combinations of the remainder. Even if in the sequel we will indicate with $\mathcal{A}$ the symmetric tensor with two distinct double eigenvalues, in the following sets S, Sy and Sk the tensor $\mathcal{A}$ denotes a generic second order tensor; the reason is that we don' t know what tensor, among the symmetric ones, has two distinct double eigenvalues.

The set S

λ, $tr\mathcal{A}$, $tr\mathcal{A}^2$, $tr\mathcal{A}^3$, $tr\mathcal{A}^4$, $tr\mathcal{W}^2$, $tr\mathcal{W}^4$, $tr\mathcal{A}\mathcal{B}$, $tr\mathcal{A}^2\mathcal{B}$, $tr\mathcal{A}^2\mathcal{B}^2$, $tr\mathcal{B}\mathcal{A}^2\mathcal{B}$, $tr\mathcal{B}\mathcal{A}\mathcal{B}\mathcal{A}$, $tr(\mathcal{W}^2\mathcal{A})$,

$tr(\mathcal{W}^2\mathcal{A}^2)$, $tr\mathcal{W}\mathcal{A}\mathcal{W}\mathcal{A}$, $tr(\mathcal{W}^2\mathcal{A}\mathcal{W}^2\mathcal{A})$, $tr(\mathcal{V}\mathcal{W})$, $tr\mathcal{V}\mathcal{W}^3$, $tr\mathcal{V}\mathcal{W}\mathcal{V}\mathcal{W}$, $tr(\mathcal{V}^2\mathcal{W}^2)$, $tr\mathcal{A}\mathcal{B}\mathcal{C}$,

$tr[(\mathcal{B}\mathcal{C}+\mathcal{C}\mathcal{B})\mathcal{A}(\mathcal{B}\mathcal{C}+\mathcal{C}\mathcal{B})\mathcal{A}]$, $tr[(\mathcal{B}\mathcal{C}+\mathcal{C}\mathcal{B})^2\mathcal{A}]$, $tr\mathcal{A}\mathcal{B}\mathcal{W}$, $tr[(\mathcal{B}\mathcal{W}-\mathcal{W}\mathcal{B})\mathcal{A}(\mathcal{B}\mathcal{W}-\mathcal{W}\mathcal{B})\mathcal{A}]$,

$tr[(\mathcal{BW}-\mathcal{WB})^2\mathcal{A}]$, $tr(\mathcal{BWA}-\mathcal{WBA})$, $tr\mathcal{BWAW}$, $tr[(\mathcal{VW}+\mathcal{WV})\mathcal{A}(\mathcal{VW}+\mathcal{WV})\mathcal{A}]$,

$tr[(\mathcal{VW}+\mathcal{WV})^2\mathcal{A}]$, $tr(\mathcal{VWA}+\mathcal{WVA})$, $tr\mathcal{WAVA}$, $tr\mathcal{VWAWA}$, $tr(\mathcal{WVWA}+\mathcal{VW}^2\mathcal{A})$, $tr(\mathcal{AVAW}^3)$,

$tr[(\mathcal{AV}+\mathcal{VA})\mathcal{W}^3]$, $tr(\mathcal{AVAWVW})$, $tr[(\mathcal{AV}+\mathcal{VA})\mathcal{WVW}]$, $tr\mathcal{UWV}$, $tr\mathcal{VWUW}$, $tr\mathcal{ABCW}$,

$tr[\mathcal{BW}(\mathcal{AV}+\mathcal{VA})]$, $tr(\mathcal{BWAVA})$, $tr\mathcal{AUWV}$, $tr(\mathcal{AVAWUW})$, $tr[(\mathcal{AV}+\mathcal{VA})\mathcal{WUW}]$,

for all the scalars λ, all the symmetric tensors $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and skew-symmetric tensors $\mathcal{W}$, $\mathcal{V}$, $\mathcal{U}$.

The set Sy

$\mathcal{I}$, $\mathcal{A}$, $\mathcal{W}^2$, $\mathcal{WAW}$, $\mathcal{AW}-\mathcal{WA}$, $\mathcal{WV}+\mathcal{VW}$, $\mathcal{ABW}-\mathcal{WBA}$, $(\mathcal{AVW}+\mathcal{WVA})+(\mathcal{VAW}+\mathcal{WAV})$,

$\mathcal{AVAW}+\mathcal{WAVA}$, for all the symmetric tensors $\mathcal{A}$, $\mathcal{B}$, and skew-symmetric tensors $\mathcal{W}$, $\mathcal{V}$.

The set Sk

$\mathcal{W}$, $\mathcal{W}^3$, $\mathcal{AB}-\mathcal{BA}$, $\mathcal{AWA}$, $\mathcal{AW}+\mathcal{WA}$, $\mathcal{AWAW}-\mathcal{WAWA}$, $\mathcal{AW}^2-\mathcal{W}^2\mathcal{A}$, $\mathcal{W}^2\mathcal{AW}+\mathcal{WAW}^2$,

$\mathcal{WVW}$, $\mathcal{WV}-\mathcal{VW}$, $\mathcal{ABC}-\mathcal{CBA}$, $\mathcal{ABW}+\mathcal{WBA}$, $\mathcal{WAV}-\mathcal{VAW}$, $\mathcal{WVA}-\mathcal{AVW}$, $\mathcal{WAVA}-\mathcal{AVAW}$,

$\mathcal{WAVW}+\mathcal{WVAW}$, $\mathcal{WAVAW}$,

for all the symmetric tensors $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and skew-symmetric tensors $\mathcal{W}$, $\mathcal{V}$.

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2. Cases whose Results Will be Expressed in Terms of those in Refs. [7], [8], [9]

Let us begin now describing the first situation which we face. It can be expressed in terms of the scalar

$$8tr(\mathcal{B}\mathcal{A}\mathcal{B}\mathcal{A}) - 2(tr\mathcal{A})tr(\mathcal{B}\mathcal{A}\mathcal{B} + \mathcal{B}^2\mathcal{A}) + 2(tr\mathcal{A}^2)(tr\mathcal{B}^2) - 4(tr\mathcal{B}\mathcal{A})^2 \\ - (tr\mathcal{A}^2)(tr\mathcal{B})^2 + 2tr(\mathcal{B}\mathcal{A})(tr\mathcal{A})(tr\mathcal{B}). \quad (1)$$

2.1. Case 1: All the symmetric tensors have no eigenvalue with multiplicity one, but among them there are two tensors $\mathcal{A}$ and $\mathcal{B}$ such that the scalar (1) is different from zero

A first consequence of eq. (1) is that the tensor $\mathcal{A}$ has not an eigenvalue λ with multiplicity four, otherwise we would have $\mathcal{A} = \lambda\mathcal{I}$, where $\mathcal{I}$ is the identity tensor; in this case the scalar (1) would be zero.

Consequently the tensor $\mathcal{A}$ has two distinct eigenvalues with multiplicity 2. Let us choose a reference frame where $\mathcal{A} = diag(a, a, b, b)$ with $a > b$. This form remains unchanged with a rotation of the axis x_1, x_2; it remains unchanged also with a rotation of the axis x_3, x_4. Let us use these rotations so that $\mathcal{B}^{12} = 0$, $\mathcal{B}^{34} = 0$.

Let us define now

$$\mathcal{I}_1 = \frac{1}{a-b}(\mathcal{A} - b\mathcal{I}) \quad \text{and} \quad \mathcal{I}_2 = \frac{1}{b-a}(\mathcal{A} - a\mathcal{I})\,, \quad (2)$$

from which it follows

$$\mathcal{I}_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} ; \; \mathcal{I}_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} . \quad (3)$$

After that we can rewrite the hypothesis that (1) is not zero, as

$$4tr(\mathcal{B}\mathcal{I}_1\mathcal{B}\mathcal{I}_1) - 2(tr\mathcal{B}\mathcal{I}_1)^2 + 4tr(\mathcal{B}\mathcal{I}_2\mathcal{B}\mathcal{I}_2) - 2(tr\mathcal{B}\mathcal{I}_2)^2 \neq 0\,. \quad (4)$$

But

$$\mathcal{B}\mathcal{I}_1 = \begin{pmatrix} B^{11} & 0 & 0 & 0 \\ 0 & B^{22} & 0 & 0 \\ B^{31} & B^{32} & 0 & 0 \\ B^{41} & B^{42} & 0 & 0 \end{pmatrix} ; \; \mathcal{B}\mathcal{I}_2 = \begin{pmatrix} 0 & 0 & B^{13} & B^{14} \\ 0 & 0 & B^{23} & B^{24} \\ 0 & 0 & B^{33} & 0 \\ 0 & 0 & 0 & B^{44} \end{pmatrix} , \quad (5)$$

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so that eq.(4) becomes

$$2(B^{11} - B^{22})^2 + 2(B^{33} - B^{44})^2 \neq 0$$

from which it follows

$$B^{11} - B^{22} \neq 0 \quad \text{and/or} \quad B^{33} - B^{44} \neq 0\,.$$

In the first of these cases we have that

$$\mathcal{I}_1\mathcal{B}\mathcal{I}_1 = \begin{pmatrix} B^{11} & 0 & 0 & 0 \\ 0 & B^{22} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

has at least an eigenvalue with multiplicity 1 (B^{11} and/or B^{22}); in the second of the above cases we have that

$$\mathcal{I}_2\mathcal{B}\mathcal{I}_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & B^{33} & 0 \\ 0 & 0 & 0 & B^{44} \end{pmatrix}$$

has at least an eigenvalue with multiplicity 1.

Consequently we can apply the results of refs. [7], [8], [9] with $\mathcal{I}_1\mathcal{B}\mathcal{I}_1$ instead of $\mathcal{A}$ in the first case, and with $\mathcal{I}_2\mathcal{B}\mathcal{I}_2$ instead of $\mathcal{A}$ in the second case.

Obviously the tensor $\mathcal{B}$ must intervene when substituting $\mathcal{A}$ with $\mathcal{I}_1\mathcal{B}\mathcal{I}_1$, but it has also to be considered as every other tensor; in fact its components $\mathcal{B}^{13}$, $\mathcal{B}^{14}$, $\mathcal{B}^{23}$, $\mathcal{B}^{24}$ are not present in $\mathcal{I}_1\mathcal{B}\mathcal{I}_1$ and $\mathcal{I}_2\mathcal{B}\mathcal{I}_2$.

In the first of the above mentioned cases, the tensor $\mathcal{I}_1$ will appear in the results; it will be necessary to substitute it from eq.$(2)_1$. Every term of the representations, both for scalars and for second order tensors, will become an homogeneous polynomial in the coefficients $\frac{1}{a-b}$ and $\frac{-b}{a-b}$, so that it will be necessary to take separately the terms of the various degrees in such coefficients.

The same thing must be done in the second of the above mentioned cases but, obviously, the results will be the same of the first case.

In conclusion of this section, we can say that it is easy to write the representations for this case but they will be very long; also those of refs. [7], [8], [9] are very long and this fact gives an idea on how much more longer will be the present ones obtained starting from them. For this reason we omit to write them, knowing that if we want to write them, then we can. The second situation we want to face is expressed in terms of the scalar

$$\begin{aligned} &8tr[(\mathcal{BC}+\mathcal{CB})\mathcal{A}(\mathcal{BC}+\mathcal{CB})\mathcal{A}] - 2(tr\mathcal{A})tr[(\mathcal{BC}+\mathcal{CB})\mathcal{A}(\mathcal{BC}+\mathcal{CB}) + (\mathcal{BC}+\mathcal{CB})^2\mathcal{A}] \\ &+2(tr\mathcal{A}^2)[tr(\mathcal{BC}+\mathcal{CB})^2] - 4[tr(\mathcal{BCA}+\mathcal{CBA})]^2 - (tr\mathcal{A}^2)[tr(\mathcal{BC}+\mathcal{CB})]^2 \\ &+2[tr(\mathcal{BCA}+\mathcal{CBA})](tr\mathcal{A})[tr(\mathcal{BC}+\mathcal{CB})]. \end{aligned} \tag{6}$$

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2.2. Case 2: All the symmetric tensors have no eigenvalue with multiplicity one, and for every couple of them $(\mathcal{A}, \mathcal{B})$ we have that the scalar (1) is zero, but three of them $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ exist such that the scalar (6) is different from zero

We note that the expression (6) is the same of (1), but with $\mathcal{BC}+\mathcal{CB}$ instead of $\mathcal{B}$. By repeating the considerations of the previous subsection, we can apply the results of refs. [7], [8], [9] with $\mathcal{I}_1(\mathcal{BC}+\mathcal{CB})\mathcal{I}_1$ instead of $\mathcal{A}$. Obviously the tensors $\mathcal{B}$ and $\mathcal{C}$ intervene when substituting $\mathcal{A}$ with $\mathcal{I}_1(\mathcal{BC}+\mathcal{CB})\mathcal{I}_1$, but they have also to be considered as every other tensor. Some term of the representations could be omitted by using the fact that (1) is zero and by using the expression of the scalar (6); but, with this exploitation, the treatment will become heavier; so we avoid to do it.

In the results, the tensor $\mathcal{I}_1$ will intervene; it will be necessary to substitute it from eq.$(2)_1$. Every term of the representations, both for scalars and for second order tensors, will become an homogeneous polynomial in the coefficients $\frac{1}{a-b}$ and $\frac{-b}{a-b}$, so that it will be necessary to take separately the terms of the various degrees in such coefficients.

Tshe third situation we want to face is expressed in terms of the scalar

$$\begin{aligned}&8tr[(\mathcal{BW}-\mathcal{WB})\mathcal{A}(\mathcal{BW}-\mathcal{WB})\mathcal{A}]\\&\quad-2(tr\mathcal{A})tr[(\mathcal{BW}-\mathcal{WB})\mathcal{A}(\mathcal{BW}-\mathcal{WB})+(\mathcal{BW}-\mathcal{WB})^2\mathcal{A}]\\&\quad+2(tr\mathcal{A}^2)[tr(\mathcal{BW}-\mathcal{WB})^2]-4[tr(\mathcal{BWA}-\mathcal{WBA})]^2\\&\quad-(tr\mathcal{A}^2)[tr(\mathcal{BW}-\mathcal{WB})]^2+2tr(\mathcal{BWA}-\mathcal{WBA})(tr\mathcal{A})[tr(\mathcal{BW}-\mathcal{WB})].\end{aligned}\tag{7}$$

2.3. Case 3: All the symmetric tensors have no eigenvalue with multiplicity one, for every couple of them $(\mathcal{A}, \mathcal{B})$ we have that the scalar (1) is zero, however we take three of them $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ we have that the scalar (6) is zero, but two symmetric tensors $(\mathcal{A}, \mathcal{B})$ and a skew-symmetric one $\mathcal{W}$ exist such that the scalar (7) is different from zero

We note that the expression (7) is the same of (1), but with $\mathcal{BW}-\mathcal{WB}$ instead of $\mathcal{B}$. By repeating the considerations of subsection 2.1, we can apply the results of refs.[7], [8], [9] with $\mathcal{I}_1(\mathcal{BW}-\mathcal{WB})\mathcal{I}_1$ instead of $\mathcal{A}$. Obviously the tensors $\mathcal{B}$ and $\mathcal{W}$ intervene when substituting $\mathcal{I}_1(\mathcal{BW}-\mathcal{WB})\mathcal{I}_1$, but they have also to be considered as every other tensor. In the results, the tensor $\mathcal{I}_1$ will intervene; it will be necessary to substitute it from eq.$(2)_1$. Every term of the representations, both for scalars and for second order tensors, will become an homogeneous polynomial in the coefficients $\frac{1}{a-b}$ and $\frac{-b}{a-b}$, so that it will be necessary to take separately the terms of the various degrees in such coefficients.

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The fourth situation we want to face is expressed in terms of the scalar

$$\begin{aligned}&8tr[(\mathcal{V}\mathcal{W}+\mathcal{W}\mathcal{V})\mathcal{A}(\mathcal{V}\mathcal{W}+\mathcal{W}\mathcal{V})\mathcal{A}]\\&-2(tr\mathcal{A})tr[(\mathcal{V}\mathcal{W}+\mathcal{W}\mathcal{V})\mathcal{A}(\mathcal{V}\mathcal{W}+\mathcal{W}\mathcal{V})+(\mathcal{V}\mathcal{W}+\mathcal{W}\mathcal{V})^2\mathcal{A}]\\&+2(tr\mathcal{A}^2)[tr(\mathcal{V}\mathcal{W}+\mathcal{W}\mathcal{V})^2]-4[tr(\mathcal{V}\mathcal{W}\mathcal{A}+\mathcal{W}\mathcal{V}\mathcal{A})]^2\\&-(tr\mathcal{A}^2)[tr(\mathcal{V}\mathcal{W}+\mathcal{W}\mathcal{V})]^2+2tr(\mathcal{V}\mathcal{W}\mathcal{A}+\mathcal{W}\mathcal{V}\mathcal{A})(tr\mathcal{A})[tr(\mathcal{V}\mathcal{W}+\mathcal{W}\mathcal{V})].\end{aligned} \tag{8}$$

2.4. Case 4: All the symmetric tensors have no eigenvalue with multiplicity one, for every couple of them $(\mathcal{A}\,,\,\mathcal{B})$ we have that the scalar (1) is zero, however we take three of them $(\mathcal{A}\,,\,\mathcal{B}\,,\,\mathcal{C})$ we have that the scalar (6) is zero, however we take two symmetric tensors $(\mathcal{A}\,,\,\mathcal{B})$ and a skew-symmetric one $\mathcal{W}$ the scalar (7) is zero, but two skew-symmetric tensors $(\mathcal{V}\,,\,\mathcal{W})$ and a symmetric one $\mathcal{A}$ exist such that the scalar (8) is different from zero

We note that the expression (8) is the same of (1), but with $\mathcal{V}\mathcal{W}+\mathcal{W}\mathcal{V}$ instead of $\mathcal{B}$. By repeating the considerations of subsection 2.1, we can apply the results of refs.[7], [8], [9] with $\mathcal{I}_1(\mathcal{V}\mathcal{W}+\mathcal{W}\mathcal{V})\mathcal{I}_1$ instead of $\mathcal{A}$. Obviously the tensors $\mathcal{V}$ and $\mathcal{W}$ intervene when substituting $\mathcal{I}_1(\mathcal{V}\mathcal{W}+\mathcal{W}\mathcal{V})\mathcal{I}_1$, but they have also to be considered as every other tensor. In the results, the tensor $\mathcal{I}_1$ will intervene; it will be necessary to substitute it from eq.$(2)_1$. Every term of the representations, both for scalars and for second order tensors, will become an homogeneous polynomial in the coefficients $\frac{1}{a-b}$ and $\frac{-b}{a-b}$, so that it will be necessary to take separately the terms of the various degrees in such coefficients.

The fifth situation we want to face is expressed in terms of the scalar

$$\begin{aligned}&8tr(\mathcal{W}^2\mathcal{A}\mathcal{W}^2\mathcal{A})-2(tr\mathcal{A})tr[(\mathcal{W}^2\mathcal{A}\mathcal{W}^2+\mathcal{W}^4\mathcal{A})+2(tr\mathcal{A}^2)(tr\mathcal{W}^4)\\&-4[tr(\mathcal{W}^2\mathcal{A})]^2-(tr\mathcal{A}^2)(tr\mathcal{W}^2)^2+2tr(\mathcal{W}^2\mathcal{A})(tr\mathcal{A})(tr\mathcal{W}^2).\end{aligned} \tag{9}$$

2.5. Case 5: All the symmetric tensors have no eigenvalue with multiplicity one, for every couple of them $(\mathcal{A}\,,\,\mathcal{B})$ we have that the scalar (1) is zero, however we take three of them $(\mathcal{A}\,,\,\mathcal{B}\,,\,\mathcal{C})$ we have that the scalar (6) is zero, however we take two symmetric tensors $(\mathcal{A}\,,\,\mathcal{B})$ and a skew-symmetric one $\mathcal{W}$ the scalar (7) is zero, however we take two skew-symmetric tensors $(\mathcal{V}\,,\,\mathcal{W})$ and a symmetric one $\mathcal{A}$ the scalar (8) is zero, but a skew-symmetric tensor $\mathcal{W}$ and a symmetric one $\mathcal{A}$ exist such that the scalar (9) is different from zero

We note that the expression (9) is the same of (1), but with $\mathcal{W}^2$ instead of $\mathcal{B}$. By repeating the considerations of Subsection 2.1, we can apply the results of refs.[7], [8], [9] with $\mathcal{I}_1\mathcal{W}^2\mathcal{I}_1$ instead of $\mathcal{A}$. Obviously the tensor $\mathcal{W}$ intervenes when substituting

$\mathcal{I}_1\mathcal{W}^2\mathcal{I}_1$, but it has also to be considered as every other tensor. In the results, the tensor $\mathcal{I}_1$ will intervene; it will be necessary to substitute it from eq.$(2)_1$. Every term of the representations, both for scalars and for second order tensors, will become an homogeneous polynomial in the coefficients $\frac{1}{a-b}$ and $\frac{-b}{a-b}$, so that it will be necessary to take separately the terms of the various degrees in such coefficients.

2.6. Characteristics of the cases which remain to be considered

In the cases, which remain to be considered, we have that all the symmetric tensors have no eigenvalue with multiplicity one, for every couple of them $(\mathcal{A}\,,\,\mathcal{B})$ we have that the scalar (1) is zero, however we take three of them $(\mathcal{A}\,,\,\mathcal{B}\,,\,\mathcal{C})$ we have that the scalar (6) is zero, however we take two symmetric tensors $(\mathcal{A}\,,\,\mathcal{B})$ and a skew-symmetric one $\mathcal{W}$ the scalar (7) is zero, however we take two skew-symmetric tensors $(\mathcal{V}\,,\,\mathcal{W})$ and a symmetric one $\mathcal{A}$ the scalar (8) is zero, however we take a skew-symmetric tensor $\mathcal{W}$ and a symmetric one $\mathcal{A}$ the scalar (9) is zero. We recall also that the present paper treats the case where there is at least one symmetric tensor $\mathcal{A}$ with two distinct eigenvalues.

Let us work in the reference frames where $\mathcal{A} = diag(a, a, b, b)$ with $a > b$. The eq. (1) may be written as the left hand side of (4); but we have also that

$$\mathcal{B}\mathcal{I}_1 = \begin{pmatrix} B^{11} & B^{12} & 0 & 0 \\ B^{21} & B^{22} & 0 & 0 \\ B^{31} & B^{32} & 0 & 0 \\ B^{41} & B^{42} & 0 & 0 \end{pmatrix} \quad \text{and} \quad \mathcal{B}\mathcal{I}_2 = \begin{pmatrix} 0 & 0 & B^{13} & B^{14} \\ 0 & 0 & B^{23} & B^{24} \\ 0 & 0 & B^{33} & B^{34} \\ 0 & 0 & B^{43} & B^{44} \end{pmatrix},$$

so that the left hand side of (4) is

$$2(B^{11} - B^{22})^2 + 8(B^{12})^2 + 2(B^{33} - B^{44})^2 + 8(B^{34})^2\,.$$

This allows to rewrite the condition that the scalar (1) is zero, for every tensor $\mathcal{B}$ as

$$B^{11} = B^{22} \quad , \quad B^{12} = 0 \quad , \quad B^{33} = B^{44} \quad , \quad B^{34} = 0\,. \tag{10}$$

The left hand side of (6) is the same of (1), but with $\mathcal{BC} + \mathcal{CB}$ instead of $\mathcal{B}$; consequently the condition that the scalar (6) is zero, for every couple of symmetric tensors $\mathcal{B}$ and $\mathcal{C}$ can be rewritten as

$$\begin{aligned} (BC + CB)^{11} &= (BC + CB)^{22} \quad , \quad (BC + CB)^{12} = 0\,, \\ (BC + CB)^{33} &= (BC + CB)^{44} \quad , \quad (BC + CB)^{34} = 0\,, \end{aligned}$$

or

$$\begin{aligned} B^{13}C^{31} + B^{14}C^{41} &= B^{23}C^{32} + B^{24}C^{42}; \\ B^{13}C^{32} + B^{14}C^{42} + C^{13}B^{32} + C^{14}B^{42} &= 0; \end{aligned}$$

$$\begin{aligned} B^{31}C^{13} + B^{32}C^{23} &= B^{41}C^{14} + B^{42}C^{24}; \\ B^{31}C^{14} + B^{32}C^{24} + C^{31}B^{14} + C^{32}B^{24} &= 0\,, \end{aligned} \tag{11}$$

where we have used (10) and also its expression with C instead of B.

Similarly,the condition that the scalar (7) is zero for every couple of tensors $\mathcal{B}$ and $\mathcal{W}$, whose first one is symmetric and the second skew-symmetric, becomes

$$\begin{aligned} (BW - WB)^{11} = (BW - WB)^{22} \quad &, \quad (BW - WB)^{12} = 0\,, \\ (BW - WB)^{33} = (BW - WB)^{44} \quad &, \quad (BW - WB)^{34} = 0\,, \end{aligned}$$

or

$$\begin{aligned} B^{13}W^{31} + B^{14}W^{41} &= B^{23}W^{32} + B^{24}W^{42}; \\ B^{13}W^{32} + B^{14}W^{42} - W^{13}B^{32} - W^{14}B^{42} &= 0\,; \\ B^{31}W^{13} + B^{32}W^{23} &= B^{41}W^{14} + B^{42}W^{24}; \\ B^{31}W^{14} + B^{32}W^{24} - W^{31}B^{14} - W^{32}B^{24} &= 0\,, \end{aligned} \tag{12}$$

where we have used (10).

Similarly,the condition that the scalar (8) is zero for every couple of skew-symmetric tensors $\mathcal{V}$ and $\mathcal{W}$ becomes

$$\begin{aligned} (VW + WV)^{11} = (VW + WV)^{22} \quad &, \quad (VW + WV)^{12} = 0\,, \\ (VW + WV)^{33} = (VW + WV)^{44} \quad &, \quad (VW + WV)^{34} = 0\,, \end{aligned}$$

or

$$\begin{aligned} V^{13}W^{31} + V^{14}W^{41} &= V^{23}W^{32} + V^{24}W^{42}; \\ V^{13}W^{32} + V^{14}W^{42} + W^{13}V^{32} + W^{14}V^{42} &= 0\,; \\ V^{31}W^{13} + V^{32}W^{23} &= V^{41}W^{14} + V^{42}W^{24}; \\ V^{31}W^{14} + V^{32}W^{24} + W^{31}V^{14} + W^{32}V^{24} &= 0\,. \end{aligned} \tag{13}$$

Finally,the condition that the scalar (9) is zero for every skew-symmetric tensor $\mathcal{W}$ becomes

$$\begin{aligned} (W^{13})^2 + (W^{14})^2 = (W^{23})^2 + (W^{24})^2 \quad &; \quad W^{13}W^{23} + W^{14}W^{24} = 0\,; \\ (W^{13})^2 + (W^{23})^2 = (W^{14})^2 + (W^{24})^2 \quad &; \quad W^{13}W^{14} + W^{23}W^{24} = 0\,. \end{aligned} \tag{14}$$

There remains to impose that all the symmetric tensors have no eigenvalue with multiplicity one; by using eq. (10) we find that the characteristic equation of the tensor $\mathcal{B}$ is

$$X^2 - [(B^{13})^2 + (B^{14})^2 + (B^{23})^2 + (B^{24})^2]X$$

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$$+ (B^{13}B^{24} - B^{14}B^{23})^2 = 0\,; \quad (15)$$

$$\text{with} \quad X = (B^{11} - \lambda)(B^{33} - \lambda)\,. \quad (16)$$

If eq. (15) would have two distinct roots X_1 and X_2, then eq. (16) would be splitted in

$$\lambda^2 - (B^{11} + B^{33})\lambda + B^{11}B^{33} - X_1 = 0\,;\ \lambda^2 - (B^{11} + B^{33})\lambda + B^{11}B^{33} - X_2 = 0 \quad (17)$$

which have no common root (otherwise, by subtracting one of these equations from the other, with λ this common root, we would have $X_1 = X_2$).

Consequently, in order to obtain two double roots, both eq. (17) must have a double root, that is

$$(B^{11} - B^{33})^2 + 4X_1 = 0\,;\ (B^{11} - B^{33})^2 + 4X_2 = 0 \quad \text{from which} \quad X_1 = X_2\,.$$

In this way we proved that eq. (15) cannot have two distinct roots, that is its discriminant must be zero; in other words we found

$$[(B^{13})^2 - (B^{24})^2]^2 + [(B^{14})^2 - (B^{23})^2]^2 + 2(B^{13}B^{14} + B^{23}B^{24})^2 + 2(B^{13}B^{23} + B^{14}B^{24})^2 = 0\,, \quad (18)$$

which must hold for whatever symmetric tensor $\mathcal{B}$.

The conditions (10), (11), (12), (13), (14), (18) can be written in a more compact form, as follows.

Eq. (10) says that $\mathcal{B}$ has the form

$$\mathcal{B} = \begin{pmatrix} B^{11} & 0 & B^{13} & B^{14} \\ 0 & B^{11} & B^{23} & B^{24} \\ B^{31} & B^{32} & B^{33} & 0 \\ B^{41} & B^{42} & 0 & B^{33} \end{pmatrix},$$

and this same form has every other symmetric tensor. Moreover we can define the following vectors

$$\vec{b}_1 = \begin{pmatrix} B^{31} \\ B^{41} \end{pmatrix},\ \vec{b}_2 = \begin{pmatrix} B^{32} \\ B^{42} \end{pmatrix},\ \vec{b}_3 = \begin{pmatrix} B^{31} \\ B^{32} \end{pmatrix},\ \vec{b}_4 = \begin{pmatrix} B^{41} \\ B^{42} \end{pmatrix}.$$

After that (11) can be written as

$$\vec{b}_1 \cdot \vec{c}_1 = \vec{b}_2 \cdot \vec{c}_2, \quad \vec{b}_1 \cdot \vec{c}_2 + \vec{b}_2 \cdot \vec{c}_1 = 0, \quad \vec{b}_3 \cdot \vec{c}_3 = \vec{b}_4 \cdot \vec{c}_4 \quad , \quad \vec{b}_3 \cdot \vec{c}_4 + \vec{b}_4 \cdot \vec{c}_3 = 0. \quad (19)$$

Similarly, eq. (12) can be written as

$$\vec{b}_1 \cdot \vec{w}_1 = \vec{b}_2 \cdot \vec{w}_2,\ \vec{b}_1 \cdot \vec{w}_2 + \vec{b}_2 \cdot \vec{w}_1 = 0,\ \vec{b}_3 \cdot \vec{w}_3 = \vec{b}_4 \cdot \vec{w}_4 \quad , \quad \vec{b}_3 \cdot \vec{w}_4 + \vec{b}_4 \cdot \vec{w}_3 = 0, \quad (20)$$

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Eq. (13) can be written as

$$\vec{v}_1 \cdot \vec{w}_1 = \vec{v}_2 \cdot \vec{w}_2, \vec{v}_1 \cdot \vec{w}_2 + \vec{v}_2 \cdot \vec{w}_1 = 0, \;\; \vec{v}_3 \cdot \vec{w}_3 = \vec{v}_4 \cdot \vec{w}_4, \;\; \vec{v}_3 \cdot \vec{w}_4 + \vec{v}_4 \cdot \vec{w}_3 = 0, \tag{21}$$

Eq. (14) can be written as

$$\vec{w}_1 \cdot \vec{w}_1 = \vec{w}_2 \cdot \vec{w}_2 \quad , \quad \vec{w}_1 \cdot \vec{w}_2 = 0, \quad \vec{w}_3 \cdot \vec{w}_3 = \vec{w}_4 \cdot \vec{w}_4 \quad , \quad \vec{w}_3 \cdot \vec{w}_4 = 0, \tag{22}$$

Eq. (18) is

$$(B^{13})^2 = (B^{24})^2 \, ; \quad (B^{14})^2 = (B^{23})^2 \quad , \quad \vec{b}_3 \cdot \vec{b}_4 = 0, \quad \vec{b}_1 \cdot \vec{b}_2 = 0, \tag{23}$$

The conditions (10), (11), (12), (13), (14) and (18) hold from now on for every tensor $\mathcal{B}$, $\mathcal{C}$, $\mathcal{W}$, $\mathcal{V}$.

We can now proceed with a new case. From (10) we note that B^{11}, B^{22}, B^{33}, B^{44} can be obtained from $tr\mathcal{B}$ and $tr\mathcal{AB}$.

$$\text{From} \quad tr\mathcal{B}^2 \quad \text{we obtain} \quad (B^{13})^2 + (B^{14})^2 + (B^{23})^2 + (B^{24})^2 \, . \tag{24}$$

2.7. Case 6: There is a symmetric tensor $\mathcal{B}$, among the independent variables, such that the quantity (24) is different from zero

Through a rotation of the axis 3 and 4, we obtain $B^{14} = 0$. From eq. (23) it follows

$$(B^{13})^2 = (B^{24})^2 \, ; \quad B^{23} = 0 \, . \tag{25}$$

We choose now the versus of the axis 1 and 2 so that $B^{13} > 0$, $B^{24} > 0$ (They cannot be zero, because the quantity (24) is different from zero) and deduce their values from the quantity (24) ; from (25) we have also that $B^{13} = B^{24}$.

After that, eqs. (11), (12) become

$$C^{13} = C^{24} \, ; \quad C^{23} = -C^{14} \, ; \quad W^{13} = W^{24} \, ; \quad W^{23} = -W^{14} \quad . \tag{26}$$

From $\quad tr\mathcal{BC}\, , tr\mathcal{ABW}\, , tr\mathcal{C}^2 \quad$ we obtain $\quad C^{13}\, , C^{24}\, , W^{13}\, , W^{24}\, , (C^{14})^2$. (27)

From $tr\mathcal{WAWA}\, , tr\mathcal{WAW}\, , tr\mathcal{W}^2\mathcal{A}\, , tr\mathcal{W}^2$, we obtain $\quad tr\mathcal{W}\mathcal{I}_1\mathcal{W}\mathcal{I}_1$ (28)
and $tr\mathcal{W}\mathcal{I}_2\mathcal{W}\mathcal{I}_2$,
from which $\quad (W^{12})^2, (W^{34})^2$. After that, from $\quad tr\mathcal{W}^2 \quad$ we find $\quad (W^{14})^2$.

2.8. Subcase 1: There is a symmetric tensor $\mathcal{C}$, among the independent variables, such that $(C^{14})^2 \neq 0$

The reference frame can be chosen such that $C^{14} > 0$; in fact, if we would have $C^{14} < 0$, changing the versus of the axis 1 and 3, we will have $C^{14} > 0$ still maintaining the sign of B^{13}, B^{24}, that is, $B^{13} > 0$, $B^{24} > 0$.

After that, from (24) with $\mathcal{C}$ instead of $\mathcal{B}$, and $(26)_2$ we obtain C^{14}, C^{23}.

For every other symmetric tensor $\mathcal{D}$ we know, from (10) with $\mathcal{D}$ instead of $\mathcal{B}$, that $D^{11} = D^{22}$, $D^{12} = 0$, $D^{33} = D^{44}$, $D^{34} = 0$; after that, we obtain D^{11}, D^{33} from $tr\mathcal{D}$ and $tr\mathcal{AD}$.

From $tr\mathcal{BD}$ we obtain D^{13} (which is also equal to D^{24}) and, from $tr\mathcal{CD}$, we obtain D^{14} (which is also equal to $-D^{23}$).

For every other skew-symmetric tensor $\mathcal{W}$, in (27) we see that we already know W^{13}, W^{24}.

From $tr\mathcal{ACW}$ and (26) we find the values of W^{14}, W^{23}.

From $tr\mathcal{BCW}$ and $tr\mathcal{ABCW}$ we obtain $tr\mathcal{I}_1\mathcal{BCW}$ and $tr\mathcal{I}_2\mathcal{BCW}$ from which W^{12} and W^{34} respectively.

Consequently, all the independent variables have been obtained, in a suitable reference frame, as functions of the scalars of the following table.

The table $1S$

$tr\mathcal{B}$, $tr\mathcal{B}^2$, $tr\mathcal{BC}$, $tr\mathcal{BCW}$, $tr\mathcal{ABCW}$,
for all the symmetric tensors $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and skew-symmetric tensors $\mathcal{W}$.

(The scalars $tr\mathcal{A}$, $tr\mathcal{A}^2$, $tr\mathcal{AB}$, $tr\mathcal{C}^2$, $tr\mathcal{D}$, $tr\mathcal{AD}$, $tr\mathcal{BD}$, $tr\mathcal{CD}$, $tr\mathcal{ABW}$, $tr\mathcal{ACW}$ are included in the previous ones; for example, $tr\mathcal{A}$ is contained in $tr\mathcal{B}$, and so on). It follows that every scalar function of those variables can be expressed, in that reference frame, as a composite function trough the scalars of the table $1S$. But this property cannot depend on the reference frame, so that it holds in every other frame. For this reason, the scalars of the table $1S$ have been included in the set S, which was reported in the Introduction. We have also included the other scalars, already employed in eqs. (1), (6), (7), (8), (9), (27).

Regarding the representation for a second order tensor ϕ^{ij}, we note that through the transformation

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \tag{29}$$

it transforms itself according to the law

$$\phi = \begin{pmatrix} \phi^{11} & \phi^{12} & \phi^{13} & \phi^{14} \\ \phi^{12} & \phi^{22} & \phi^{23} & \phi^{24} \\ \phi^{13} & \phi^{23} & \phi^{33} & \phi^{34} \\ \phi^{14} & \phi^{24} & \phi^{34} & \phi^{44} \end{pmatrix} \to \begin{pmatrix} \phi^{22} & -\phi^{12} & \phi^{24} & -\phi^{23} \\ -\phi^{12} & \phi^{11} & -\phi^{14} & \phi^{13} \\ \phi^{24} & -\phi^{14} & \phi^{44} & -\phi^{34} \\ -\phi^{23} & \phi^{13} & -\phi^{34} & \phi^{33} \end{pmatrix}$$
$$= P^T \phi P \text{ if} \phi \text{ is symmetric}, \tag{30}$$

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$$\phi = \begin{pmatrix} 0 & \phi^{12} & \phi^{13} & \phi^{14} \\ -\phi^{12} & 0 & \phi^{23} & \phi^{24} \\ -\phi^{13} & -\phi^{23} & 0 & \phi^{34} \\ -\phi^{14} & -\phi^{24} & -\phi^{34} & 0 \end{pmatrix} \to \begin{pmatrix} 0 & \phi^{12} & \phi^{24} & -\phi^{23} \\ -\phi^{12} & 0 & -\phi^{14} & \phi^{13} \\ -\phi^{24} & \phi^{14} & 0 & \phi^{34} \\ \phi^{23} & -\phi^{13} & -\phi^{34} & 0 \end{pmatrix}$$
$$= P^T \phi P \qquad \text{if } \phi \text{ is skew-symmetric} . \quad (31)$$

(We note that P is the rotation of the axis 1 and 2 with an angle $\frac{\pi}{2}$, and the rotation of the axis 3 and 4 with an angle $\frac{\pi}{2}$). The independent variables transform themselves like ϕ, but by using eqs. (10), $B^{14} = 0$, $(25)_2$, $B^{13} = B^{24}$ and $(26)_{1,2}$, we find that the independent variables which are symmetric tensors remains unchanged trough this transformation. The same thing can be said for the skew-symmetric tensors, as it can be seen by using eq.$(26)_{3,4}$. Therefore, also ϕ must remain unchanged because it is a function of those variables. Consequently, if ϕ is symmetric, we have

$$\phi^{11} = \phi^{22} , \, \phi^{12} = 0 , \, \phi^{33} = \phi^{44} , \, \phi^{34} = 0 , \, \phi^{13} = \phi^{24} , \, \phi^{14} = -\phi^{23} \quad (32)$$

while if ϕ is skew-symmetric, we have

$$\phi^{13} = \phi^{24} , \, \phi^{14} = -\phi^{23} . \quad (33)$$

After that it is easy to verify that every symmetric tensor ϕ is a linear combination of the tensors reported in the following table 1 Tsy, and every ϕ skew-symmetric is a linear combination of the tensors reported in the following table 1 Tsk.

The table 1 Tsy

$\mathcal{I}$, $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$.

The table 1 Tsk

$\mathcal{BC} - \mathcal{CB}$, $\mathcal{ABC} - \mathcal{CBA}$.

(The first of these tensors includes also $\mathcal{AB} - \mathcal{BA}$ and $\mathcal{AC} - \mathcal{CA}$).

2.9. Subcase 2: For all the symmetric tensors $\mathcal{C}$, among the independent variables, we have $(C^{14})^2 = 0$, but there is a skew-symmetric tensor $\mathcal{W}$, among the independent variables, such that $(W^{14})^2 \neq 0$

Obviously, we have $C^{14} = 0$ and $C^{23} = 0$, thanks to eq. (26), for all the symmetric tensors $\mathcal{C}$.

The reference frame can be chosen such that $W^{14} > 0$; in fact, if we would have $W^{14} < 0$, changing the versus of the axis 1 and 3, we will have $W^{14} > 0$ still maintaining the sign of B^{13}, B^{24}, that is, $B^{13} > 0$, $B^{24} > 0$.

For all the symmetric tensors $\mathcal{C}$, thanks to (10) with $\mathcal{C}$ instead of $\mathcal{B}$, we have that

$$C^{11} = C^{22} , \, C^{12} = 0 , \, C^{33} = C^{44} , \, C^{34} = 0 \quad \text{and, moreover,} \quad (34)$$

$$C^{14} = 0\,,\; C^{23} = 0\,,\; C^{13} = C^{24}\,,$$

whose last one has been deduced from $(11)_1$. After that, we obtain the remaining unknowns of $\mathcal{C}$ from $tr\mathcal{C}$ and $tr\mathcal{AC}$ and $tr\mathcal{BC}$.

Regarding $\mathcal{W}$, in (27) we see that we already know W^{13}, W^{24}.

From $(W^{14})^2$ we obtain W^{14} (and, consequently, W^{23} because of $(26)_4$).

From $tr\mathcal{B}\mathcal{W}^2$ and $tr\mathcal{BWAW}$ we obtain $tr\mathcal{BWI}_1\mathcal{W}$ and $tr\mathcal{BWI}_2\mathcal{W}$ from which W^{12} and W^{34} respectively; so we have finished to obtain the whole tensor $\mathcal{W}$.

For every other skew-symmetric tensor $\mathcal{V}$, from (27) with $\mathcal{V}$ instead of $\mathcal{W}$, we see that we already know V^{13}, V^{24}.

From $tr(\mathcal{WAVA})$, $tr[(\mathcal{WV}+\mathcal{VW})\mathcal{A}]$, $tr\mathcal{WV}$ we obtain $tr\mathcal{WI}_1\mathcal{VI}_1$ and $tr\mathcal{WI}_2\mathcal{VI}_2$ from which $V^{12}W^{12}$ and $V^{34}W^{34}$.

Now we note that $tr\mathcal{BWV}$ can be obtained from $tr(\mathcal{BWV}+\mathcal{BVW})$; in fact,
$tr(\mathcal{BWV}+\mathcal{BVW}) = B^{rs}W^{sa}V^{ar} + B^{rs}V^{sa}W^{ar} = B^{rs}W^{sa}V^{ar} + B^{sr}V^{ra}W^{as}$
$= B^{rs}W^{sa}V^{ar} + B^{rs}(-V^{ar})(-W^{sa}) = 2B^{rs}W^{sa}V^{ar} = 2tr\mathcal{BWV}$,
where, in the second passage we have exchanged the indexes r and s of the second term, while in the subsequent passage we have used the symmetry of $\mathcal{B}$ and the skew-symmetry of $\mathcal{V}$ and $\mathcal{W}$. After that, from $tr\mathcal{BWAVA}$, $tr[\mathcal{BW}(\mathcal{AV}+\mathcal{VA})]$, $tr\mathcal{BWV}$, we obtain $tr\mathcal{BWI}_1\mathcal{VI}_1$ and $tr\mathcal{BWI}_2\mathcal{VI}_2$ from which V^{12} and W^{34}; after that, from the already used scalar $tr(\mathcal{WV})$, we find V^{14} and V^{23}.

Consequently, all the independent variables have been obtained, in a suitable reference frame, as functions of the scalars of the following table.

The table 2 S

$tr\mathcal{WAWA}$, $tr\mathcal{WAW} = tr\mathcal{W}^2\mathcal{A}$, $tr\mathcal{W}^2$, $tr\mathcal{BWAW}$, $tr[(\mathcal{WV}+\mathcal{VW})\mathcal{A}]$, $tr\mathcal{WV}$,
$tr(\mathcal{WAVA})$ $tr\mathcal{BWAVA}$, $tr[\mathcal{BW}(\mathcal{AV}+\mathcal{VA})]$, $tr(\mathcal{BWV}+\mathcal{BVW})$,
for all the symmetric tensors $\mathcal{A}$, $\mathcal{B}$ and skew-symmetric tensors $\mathcal{W}$, $\mathcal{V}$.

(We have not included the scalars already present in Table 1 S). Regarding the representation for second order tensors, we can repeat the procedure of the previous subcase obtaining the condition (32) for every symmetric ϕ^{ij}, and (33) for skew-symmetric ϕ^{ij}. After that the new tensors to be included in the representation are

The table 2 Tsy

$$\mathcal{I},\ \mathcal{A},\ \mathcal{B},\ \mathcal{AW}-\mathcal{WA}.$$

The table 2 Tsk

$$\mathcal{AWA}\,,\ \mathcal{BW}+\mathcal{WB}\,,\ \mathcal{W}\,,\ \mathcal{AB}-\mathcal{BA}\,,\ \mathcal{ABW}+\mathcal{WBA}\,. \tag{35}$$

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In fact, from eq. $(35)_{1,3,4}$ and $(35)_2$ with $\mathcal{A}$ instead of $\mathcal{B}$, we obtain $\mathcal{I}_1\mathcal{W}\mathcal{I}_1$, $\mathcal{I}_2\mathcal{W}\mathcal{I}_2$, $\mathcal{W}$, $\mathcal{AB}-\mathcal{BA}$ of which ϕ^{ij} is certainly a linear combination in the case $W^{12}\neq 0$, $W^{34}\neq 0$.

While, if $W^{12}=0$, $W^{34}\neq 0$, ϕ^{ij} is a linear combination of $\mathcal{BW}+\mathcal{WB}$, $\mathcal{I}_2\mathcal{W}\mathcal{I}_2$, $\mathcal{W}$, $\mathcal{AB}-\mathcal{BA}$ which are linear combinations of $(35)_{1-4}$. (The $(35)_2$ has to be used also with $\mathcal{A}$ instead of $\mathcal{B}$).

Instead of this, if $W^{12}\neq 0$, $W^{34}=0$, then ϕ^{ij} is a linear combination of $\mathcal{BW}+\mathcal{WB}$, $\mathcal{I}_1\mathcal{W}\mathcal{I}_1$, $\mathcal{W}$, $\mathcal{AB}-\mathcal{BA}$.

Finally, if $W^{12}=0$, $W^{34}=0$, then ϕ^{ij} is a linear combination of $\mathcal{BW}+\mathcal{WB}$, $\mathcal{ABW}+\mathcal{WBA}$, $\mathcal{W}$, $\mathcal{AB}-\mathcal{BA}$. (The components 12 and 34 can be obtained from the first two of these tensors).

2.10. Subcase 3: For all the symmetric tensors $\mathcal{C}$, among the independent variables, we have $(C^{14})^2=0$, and for all the skew-symmetric tensors $\mathcal{W}$, among the independent variables, we have $(W^{14})^2=0$, but there is a skew-symmetric tensor $\mathcal{W}$, such that $(W^{12})^2\neq 0$.

Obviously, we have $C^{14}=0$, $C^{23}=0$, $W^{14}=0$ and $W^{23}=0$ thanks to eq. (26), for all the symmetric tensors $\mathcal{C}$ and all the skew-symmetric tensors $\mathcal{W}$.

The reference frame can be chosen such that $W^{12}>0$; in fact, if we would have $W^{12}<0$, changing the versus of the axis 1 and 3, we will have $W^{12}>0$ still maintaining the sign of B^{13}, B^{24}, that is, $B^{13}>0$, $B^{24}>0$.

For all the symmetric tensors $\mathcal{C}$, thanks to (10) with $\mathcal{C}$ instead of $\mathcal{B}$, we have that

$$C^{11}=C^{22}\,,\; C^{12}=0\,,\; C^{33}=C^{44}\,,\; C^{34}=0 \quad \text{and, moreover,} \\ C^{14}=0\,,\; C^{23}=0\,,\; C^{13}=C^{24}\,,$$

whose last one has been deduced from $(11)_1$. After that, we obtain the remaining unknowns of $\mathcal{C}$ from $tr\mathcal{C}$, $tr\mathcal{AC}$ and $tr\mathcal{BC}$.

Regarding $\mathcal{W}$, in (27) we see that we already know W^{13}, W^{24}. Moreover we have $W^{14}=0$ and $W^{23}=0$.

From $(W^{12})^2$ we obtain W^{12}.

From $tr\mathcal{WBWB}$ we obtain W^{34}; so we have finished to obtain all the tensor $\mathcal{W}$.

For every other skew-symmetric tensor $\mathcal{V}$, from (27) with $\mathcal{V}$ instead of $\mathcal{W}$, we see that we already know V^{13}, V^{24}.

Moreover we have $V^{14}=0$ for the hypothesis of this subcase, and $V^{23}=0$ for (26) with $\mathcal{V}$ instead of $\mathcal{W}$.

From $tr(\mathcal{WAVA})$, $tr[(\mathcal{WV}+\mathcal{VW})\mathcal{A}]$, $tr\mathcal{WV}$ we obtain $tr\mathcal{W}\mathcal{I}_1\mathcal{V}\mathcal{I}_1$ and $tr\mathcal{W}\mathcal{I}_2\mathcal{V}\mathcal{I}_2$ from which V^{12} and $V^{34}W^{34}$.

From this last quantity and $tr(\mathcal{WBVB})$we obtain V^{34}.

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Consequently, all the independent variables have been obtained, in a suitable reference frame, as functions of the scalars of the previous tables and of that in the following table

The table 3 S
$tr\mathcal{W}\mathcal{A}\mathcal{V}\mathcal{A}$,

which includes also that with $\mathcal{B}$ instead of $\mathcal{A}$.
Regarding the representation for second order symmetric tensors, we have

The table 3 Tsy

$$\mathcal{I}\,,\,\mathcal{A}\,,\,\mathcal{B}\,,\,\mathcal{B}\mathcal{W}-\mathcal{W}\mathcal{B}\,,\,\mathcal{A}\mathcal{B}\mathcal{W}-\mathcal{W}\mathcal{B}\mathcal{A}\,, \tag{36}$$

of which we use $(36)_{1-4}$ if $W^{34} \neq W^{12}$ and $(36)_{1-3,5}$ if $W^{34} = W^{12}$.

Regarding the representation for second order skew-symmetric tensors, we have that

- If $W^{34} \neq 0$, $W^{34} + W^{12} \neq 0$, then ϕ^{ij} is a linear combination of $\mathcal{B}\mathcal{W} + \mathcal{W}\mathcal{B}$, $\mathcal{I}_2\mathcal{W}\mathcal{I}_2$, $\mathcal{W}$, $\mathcal{A}\mathcal{B} - \mathcal{B}\mathcal{A}$ (which can be obtained from (35), also from $(35)_2$ with $\mathcal{A}$ instead of $\mathcal{B}$).
- If $W^{34} \neq 0$, $W^{34}+W^{12} = 0$, then ϕ^{ij} is a linear combination of $\mathcal{A}\mathcal{B}\mathcal{W}+\mathcal{W}\mathcal{B}\mathcal{A}$, $\mathcal{I}_2\mathcal{W}\mathcal{I}_2$, $\mathcal{W}$, $\mathcal{A}\mathcal{B} - \mathcal{B}\mathcal{A}$.
- If $W^{34} = 0$, then ϕ^{ij} is a linear combination of $\mathcal{B}\mathcal{W}\mathcal{B}$, $\mathcal{B}\mathcal{W}+\mathcal{W}\mathcal{B}$, $\mathcal{W}$, $\mathcal{A}\mathcal{B}-\mathcal{B}\mathcal{A}$. (Their coefficients can be obtained more easily from the components 34, 14, 12, 13 respectively.).

Consequently, there is no new table to insert, because the new tensor $\mathcal{B}\mathcal{W}\mathcal{B}$ is already present in $(35)_1$ with $\mathcal{A}$ instead of $\mathcal{B}$.

2.11. Subcase 4: For all the symmetric tensors $\mathcal{C}$, among the independent variables, we have $(C^{14})^2 = 0$, and for all the skew-symmetric tensors $\mathcal{W}$, among the independent variables, we have $(W^{14})^2 = 0$, $(W^{12})^2 = 0$, but there is a skew-symmetric tensor $\mathcal{W}$, such that $(W^{34})^2 \neq 0$.

Obviously, we have $C^{14} = 0$, $C^{23} = 0$, $W^{14} = 0$ and $W^{23} = 0$ thanks to eq. (26), for all the symmetric tensors $\mathcal{C}$ and all the skew-symmetric tensors $\mathcal{W}$.

The reference frame can be chosen such that $W^{34} > 0$; in fact, if we would have $W^{34} < 0$, changing the versus of the axis 1 and 3, we will have $W^{34} > 0$ still maintaining the sign of B^{13}, B^{24}, that is, $B^{13} > 0$, $B^{24} > 0$.

For all the symmetric tensors $\mathcal{C}$, thanks to (10) with $\mathcal{C}$ instead of $\mathcal{B}$, we have that

$$C^{11} = C^{22}\,,\, C^{12} = 0\,,\, C^{33} = C^{44}\,,\, C^{34} = 0 \quad \text{and, moreover,} \\ C^{14} = 0\,,\, C^{23} = 0\,,\, C^{13} = C^{24}\,,$$

whose last one has been deduced from $(11)_1$. After that, we obtain the remaining unknowns of $\mathcal{C}$ from $tr\mathcal{C}$, $tr\mathcal{AC}$ and $tr\mathcal{BC}$.

Regarding $\mathcal{W}$, in (27) we see that we already know W^{13}, W^{24}. Moreover we have $W^{14} = 0$, $W^{23} = 0$ and $W^{12} = 0$.

From $(W^{34})^2$ we obtain W^{34}; so we have finished to obtain all the tensor $\mathcal{W}$. For every other skew-symmetric tensor $\mathcal{V}$, from (27) with $\mathcal{V}$ instead of $\mathcal{W}$, we see that we already know V^{13}, V^{24}.

Moreover we have $V^{14} = 0$, $V^{12} = 0$ for the hypothesis of this subcase, and $V^{23} = 0$ for (26) with $\mathcal{V}$ instead of $\mathcal{W}$.

From $tr\mathcal{WV}$ we obtain V^{34}.

Consequently there are no new scalars to include in the representation.

Regarding the representation for second order symmetric tensors, we see that eq. (32) still holds; by using it we note that ϕ^{ij} is a linear combination of $(36)_{1-4}$.

Regarding the representation for second order skew-symmetric tensors, we have that eq. (33) still holds. After that we note that ϕ^{ij} is a linear combination of $\mathcal{BWB}$, $\mathcal{BW} + \mathcal{WB}$, $\mathcal{W}$, $\mathcal{AB} - \mathcal{BA}$ (Their coefficients can be obtained more easily from the components 12, 14, 34, 13 respectively), which are already present in the previous subcase.

2.12. Subcase 5: For all the symmetric tensors $\mathcal{C}$, among the independent variables, we have $(C^{14})^2 = 0$, and for all the skew-symmetric tensors $\mathcal{W}$, among the independent variables, we have $(W^{14})^2 = 0$, $(W^{12})^2 = 0$, $(W^{34})^2 = 0$.

Obviously, we have $C^{14} = 0$, $C^{23} = 0$, $W^{14} = 0$ and $W^{23} = 0$ thanks to eq. (26), for all the symmetric tensors $\mathcal{C}$ and all the skew-symmetric tensors $\mathcal{W}$.

Moreover, for all the symmetric tensors $\mathcal{C}$, we have eq. (34). Consequently, they can be obtained from $tr\mathcal{C}$, $tr\mathcal{AC}$ and $tr\mathcal{BC}$.

Regarding $\mathcal{W}$, its components can be obtained from $(27)_{3,4}$ and $W^{12} = 0$, $W^{34} = 0$, $W^{14} = 0$ and $W^{23} = 0$. Consequently there are no new scalars to include in the representation.

Regarding the representation for second order tensors ϕ^{ij} we note that, by changing the versus of the axis 1 and 3, the independent variables remain unchanged so that every function depending on them must also remain unchanged. Instead of this, for the transformation law of tensors, we have that ϕ^{12}, ϕ^{14}, ϕ^{23}, ϕ^{34} transform themselves into $-\phi^{12}$, $-\phi^{14}$, $-\phi^{23}$, $-\phi^{34}$ respectively. It follows that $\phi^{12} = 0$,

$\phi^{14} = 0$, $\phi^{23} = 0$, $\phi^{34} = 0$. We already knew the first two of these equations in the case of a symmetric tensor ϕ^{ij}; now we have proved them also for a skew-symmetric tensor ϕ^{ij}. Moreover, we can take into account also eqs. (32) and (33).

Consequently, if ϕ^{ij} is symmetric, it is a linear combination of $\mathcal{I}$, $\mathcal{A}$, $\mathcal{B}$, which are already present in the correspondent representation.

Instead of this, if ϕ^{ij} is skew-symmetric, it is proportional to $\mathcal{AB} - \mathcal{BA}$, which is already present in the correspondent representation.

In this way, we have finished all the subcases of the case 6. Now, let us treat the other cases; for them, the quantity (24) is zero, so that we have $B^{13} = 0$, $B^{14} = 0$, $B^{23} = 0$, $B^{24} = 0$. Moreover, eq. (10) holds, so that every symmetric tensor $\mathcal{B}$ is such that

$$\mathcal{B} = [4tr\mathcal{A}^2 - (tr\mathcal{A})^2]^{-1}\{[tr\mathcal{B}tr\mathcal{A}^2 - (tr\mathcal{A})tr(\mathcal{AB})]\mathcal{I} + [4tr(\mathcal{AB}) - tr\mathcal{A}tr\mathcal{B}]\mathcal{A}\}\,,$$

so that $\mathcal{B}$ plays a role only through $\mathcal{A}$ and through the scalars present in the above equations, and these have been already included (for example, they are present in (1)).

From now on, we will have only $\mathcal{A}$ as symmetric tensor; moreover, for the skew-symmetric ones, eqs. (13) and (14) still hold.

From the first 4 scalars of Table 2S we deduce $tr\mathcal{I}_1\mathcal{W}\mathcal{I}_1\mathcal{W}$ and $tr\mathcal{I}_2\mathcal{W}\mathcal{I}_2\mathcal{W}$ from which $(W^{12})^2$ and $(W^{34})^2$.

From the 4^{th} scalar of the same table, we obtain

$$(W^{13})^2 + (W^{14})^2 + (W^{23})^2 + (W^{24})^2\,. \tag{37}$$

2.13. Case 7: There is a skew-symmetric tensor $\mathcal{W}$, among the independent variables, such that $(W^{13})^2 + (W^{14})^2 + (W^{23})^2 + (W^{24})^2 \neq 0$

Through a rotation of the axis 3 and 4, we obtain $W^{14} = 0$. After that we have $W^{23} = 0$, otherwise from $(14)_{2,4}$ it would follow $W^{13} = 0$, $W^{24} = 0$ and, consequently, from $(14)_1$ it would follow $W^{23} = 0$ in any case.

From eq. $(14)_5$ it follows $(W^{13})^2 = (W^{24})^2$ and, from the hypothesis of this case, it follows also $(W^{13})^2 \neq 0$, $(W^{24})^2 \neq 0$.

We choose now the versus of the axis 1 and 2 so that $W^{13} > 0$, $W^{24} > 0$ and deduce their values from the quantity (37) because the relations $(W^{13})^2 = (W^{24})^2$ and $W^{13} > 0$, $W^{24} > 0$ imply $W^{13} = W^{24}$.

After that, eq. (13), becomes

$$V^{13} = V^{24}\,, \quad V^{23} = -V^{14}\,. \tag{38}$$

From $tr\mathcal{WAWA}$, $tr\mathcal{WAW} = tr\mathcal{W}^2\mathcal{A}$, $tr\mathcal{W}^2$ we obtain $tr\mathcal{W}\mathcal{I}_1\mathcal{W}\mathcal{I}_1$ and $tr\mathcal{W}\mathcal{I}_2\mathcal{W}\mathcal{I}_2$ from which $(W^{12})^2$ and $(W^{34})^2$.

Similarly, from $tr\mathcal{V}\mathcal{A}\mathcal{V}\mathcal{A}$, $tr\mathcal{V}\mathcal{A}\mathcal{V}$, $tr\mathcal{V}^2\mathcal{A}$, $tr\mathcal{V}^2$ we obtain $(V^{12})^2$ and $(V^{34})^2$.

After that, from $tr\mathcal{V}\mathcal{A}\mathcal{W}\mathcal{A}$, $tr(\mathcal{W}\mathcal{V}\mathcal{A}+\mathcal{V}\mathcal{W}\mathcal{A})$, $tr\mathcal{V}\mathcal{W}$, we obtain $tr\mathcal{V}\mathcal{I}_1\mathcal{W}\mathcal{I}_1$ and $tr\mathcal{V}\mathcal{I}_2\mathcal{W}\mathcal{I}_2$ from which $V^{12}W^{12}$ and $V^{34}W^{34}$.

After that, $tr\mathcal{V}\mathcal{W}$ gives V^{13} and $tr\mathcal{V}^2$ gives $(V^{14})^2$.

The situation, in matrix notation, is expressed by

$$\mathcal{W} = \begin{pmatrix} 0 & W^{12} & W^{13} & 0 \\ -W^{12} & 0 & 0 & W^{13} \\ -W^{13} & 0 & 0 & W^{34} \\ 0 & -W^{13} & -W^{34} & 0 \end{pmatrix},$$

$$\mathcal{V} = \begin{pmatrix} 0 & V^{12} & V^{13} & V^{14} \\ -V^{12} & 0 & -V^{14} & V^{13} \\ -V^{13} & V^{14} & 0 & V^{34} \\ -V^{14} & -V^{13} & -V^{34} & 0 \end{pmatrix},$$

and we know $(W^{12})^2$, $(W^{34})^2$, $W^{13} > 0$, $(V^{12})^2$, $(V^{34})^2$, V^{13} and $(V^{14})^2$.

2.14. Subcase 1: There is a skew-symmetric tensor $\mathcal{V}$, among the independent variables, such that $(V^{14})^2 \neq 0$

The reference frame can be chosen such that $V^{14} > 0$; in fact, if we would have $V^{14} < 0$, changing the versus of the axis 1 and 3, we will have $V^{14} > 0$ still maintaining the sign of W^{13}, W^{24}, that is, $W^{13} > 0$, $W^{24} > 0$.

After that, from the knowledge of $(V^{14})^2$ we obtain V^{14} and, thanks to eq. $(38)_2$, we obtain also V^{23}.

If now $\mathcal{U}$ is any other skew-symmetric tensor, we can repeat with it the same passages done with $\mathcal{V}$ just before starting this subcase; in this way we obtain $U^{13} = U^{24}$, $U^{23} = -U^{14}$ from (13).

After that, from $tr\mathcal{U}\mathcal{A}\mathcal{U}\mathcal{A}$, $tr\mathcal{U}\mathcal{A}\mathcal{U} = tr\mathcal{U}^2\mathcal{A}$, $tr\mathcal{U}^2$, $tr\mathcal{U}\mathcal{A}\mathcal{W}\mathcal{A}$, $tr(\mathcal{W}\mathcal{U}\mathcal{A}+\mathcal{U}\mathcal{W}\mathcal{A})$, $tr\mathcal{U}\mathcal{W}$, we obtain $(U^{12})^2$, $(U^{34})^2$, $U^{12}W^{12}$, $U^{34}W^{34}$, U^{13}, $(U^{14})^2$.

After that, from $tr\mathcal{U}\mathcal{A}\mathcal{V}\mathcal{A}$, $tr(\mathcal{V}\mathcal{U}\mathcal{A} + \mathcal{U}\mathcal{V}\mathcal{A})$,, $tr\mathcal{U}\mathcal{V}$, we obtain $U^{12}V^{12}$ and $U^{34}V^{34}$. Finally, $tr\mathcal{U}\mathcal{V}$ gives U^{14}.

Moreover, from $tr\mathcal{V}\mathcal{W}\mathcal{A}\mathcal{W}\mathcal{A}$, $tr(\mathcal{W}\mathcal{V}\mathcal{W}\mathcal{A}+\mathcal{V}\mathcal{W}^2\mathcal{A})$, $tr\mathcal{V}\mathcal{W}^2$, we obtain $tr\mathcal{V}\mathcal{W}\mathcal{I}_1\mathcal{W}\mathcal{I}_1$ and $tr\mathcal{V}\mathcal{W}\mathcal{I}_2\mathcal{W}\mathcal{I}_2$ from which W^{12} and W^{34} respectively.

From the same scalars, but with $\mathcal{V}$ and $\mathcal{W}$ exchanged, that is $tr\mathcal{W}\mathcal{V}\mathcal{A}\mathcal{V}\mathcal{A}$, $tr(\mathcal{V}\mathcal{W}\mathcal{V}\mathcal{A} + \mathcal{W}\mathcal{V}^2\mathcal{A})$, $tr\mathcal{W}\mathcal{V}^2$, we obtain $tr\mathcal{W}\mathcal{V}\mathcal{I}_1\mathcal{V}\mathcal{I}_1$ and $tr\mathcal{W}\mathcal{V}\mathcal{I}_2\mathcal{V}\mathcal{I}_2$ from which V^{12} and V^{34} respectively.

Finally, from $tr\mathcal{U}\mathcal{W}\mathcal{V}$ and $tr\mathcal{A}\mathcal{U}\mathcal{W}\mathcal{V}$, we obtain $tr\mathcal{I}_1\mathcal{U}\mathcal{W}\mathcal{V}$ and $tr\mathcal{I}_2\mathcal{U}\mathcal{W}\mathcal{V}$ from which U^{12} and U^{34} respectively. But we observe that

$tr\mathcal{V}\mathcal{W}^2 = V^{ab}(W^2)^{ba} = V^{ba}(W^2)^{ab} = -V^{ab}(W^2)^{ba} = -tr\mathcal{V}\mathcal{W}^2$, where, in the second passage we have exchanged the indexes a and b, while in the subsequent

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passage we have used the symmetry of $(\mathcal{W})^2$ and the skew-symmetry of $\mathcal{V}$. It follows that $tr\mathcal{V}\mathcal{W}^2 = 0$.

Therefore, the scalars used are

- the first four scalars of table 2S with $\mathcal{V}$ and $\mathcal{U}$ instead of $\mathcal{W}$,
- the second scalar of table 3S with $\mathcal{A}$ instead of $\mathcal{B}$,
- the sixth and seventh scalar of table 2S,
- the second scalar of table 3S with $\mathcal{A}$ instead of $\mathcal{B}$, $\mathcal{W}$ instead of $\mathcal{V}$ and $\mathcal{U}$ instead of $\mathcal{W}$,
- the sixth and seventh scalar of table 2S with $\mathcal{U}$ instead of $\mathcal{V}$,
- the second scalar of table 3S with $\mathcal{U}$ instead of $\mathcal{W}$, $\mathcal{A}$ instead of $\mathcal{B}$,
- the sixth and seventh scalar of table 2S with $\mathcal{U}$ instead of $\mathcal{W}$,
- the scalars of the following table 4S,

The table $4S$

$tr\mathcal{V}\mathcal{W}\mathcal{A}\mathcal{W}\mathcal{A}$, $tr(\mathcal{W}\mathcal{V}\mathcal{W}\mathcal{A} + \mathcal{V}\mathcal{W}^2\mathcal{A})$, $tr\mathcal{U}\mathcal{W}\mathcal{V}$, $tr\mathcal{A}\mathcal{U}\mathcal{W}\mathcal{V}$.

Regarding the representation for second order tensorial functions, we may adapt to this case the considerations exposed in subcase 1 of case 6; in particular, let us consider the transformation (29) and its consequences (30) and (31) on how symmetric and skew-symmetric tensors transform themselves. We note that the tensor $\mathcal{A}$ and all the skew-symmetric tensors, among the independent variables, remain unchanged. Consequently also every function, depending on them, must remain unchanged. This fact implies that, thanks to eq. (30) every symmetric function ϕ satisfies the conditions

$$\phi^{11} = \phi^{22}\,,\ \phi^{12} = -\phi^{12}\,,\ \phi^{33} = \phi^{44}\,,\ \phi^{34} = -\phi^{34}\,,\ \phi^{13} = \phi^{24}\,,\ \phi^{14} = -\phi^{23}\,,$$

from which $\phi^{12} = 0$, $\phi^{34} = 0$. Consequently, ϕ is a linear combination of the tensors $\mathcal{I}$, $\mathcal{A}$, $\mathcal{A}\mathcal{W} - \mathcal{W}\mathcal{A}$, $\mathcal{A}\mathcal{V} - \mathcal{V}\mathcal{A}$ which are already reported in the previous tables.

Similarly, thanks to eq. (31), we have that if ϕ is skew-symmetric, it satisfies the conditions $\phi^{13} = \phi^{24}$, $\phi^{23} = -\phi^{14}$ so that ϕ is a linear combination of the tensors reported in the following table 4 Tsk.

The table 4 Tsk

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$\mathcal{W}$, $\mathcal{V}$, $\mathcal{AWA}$, $\mathcal{AVA}$, $\mathcal{AW}+\mathcal{WA}$, $\mathcal{AV}+\mathcal{VA}$, $\mathcal{WV}-\mathcal{VW}$, $\mathcal{WAV}-\mathcal{VAW}$.

In fact, it can be easily seen that $\mathcal{W}-\mathcal{I}_1\mathcal{W}\mathcal{I}_1-\mathcal{I}_2\mathcal{W}\mathcal{I}_2$ and $\mathcal{V}-\mathcal{I}_1\mathcal{V}\mathcal{I}_1-\mathcal{I}_2\mathcal{V}\mathcal{I}_2$ are linear combinations of the tensor in Table 4 Tsk; in turns, there exist the scalars α, β, γ, δ such that

$$\phi = \alpha(\mathcal{W}-\mathcal{I}_1\mathcal{W}\mathcal{I}_1-\mathcal{I}_2\mathcal{W}\mathcal{I}_2)+\beta(\mathcal{V}-\mathcal{I}_1\mathcal{V}\mathcal{I}_1-\mathcal{I}_2\mathcal{V}\mathcal{I}_2)+\gamma(\mathcal{WV}-\mathcal{VW}) + \delta(\mathcal{WAV}-\mathcal{VAW})\,.$$

In order to verify this fact, let us begin with the components 12 and 34 obtaining a system with 2 equations for the determination of the two unknowns γ, δ which has an unique solution. After that, the components 14 and 13 give β and α respectively; the components 23 and 24 give these same equations from which β and α have been determined.

2.15. Subcase 2: For all the skew-symmetric tensors $\mathcal{V}$, among the independent variables, we have $V^{14}=0$, but there is a skew-symmetric tensor $\mathcal{V}$, such that $(V^{12})^2 \neq 0$.

Obviously, we have $V^{14}=0$, $V^{23}=0$, $V^{13}=V^{24}$ thanks to eq. (38), for all the skew-symmetric tensors $\mathcal{V}$. Moreover, we already know $V^{13}\ \forall\ \mathcal{V}$ as shown just before the subcase 1.

The reference frame can be chosen such that $V^{12}>0$; in fact, if we would have $V^{12}<0$, changing the versus of the axis 1 and 3, we will have $V^{12}>0$ still maintaining the sign of W^{13}, W^{24}, that is, $W^{13}>0$, $W^{24}>0$.

After that, we obtain V^{12}, W^{12} from $(V^{12})^2$ and $V^{12}W^{12}$ which we already knew just before considering the subcase 1.

From $tr(\mathcal{AVAW}^3)$, $tr[(\mathcal{AV}+\mathcal{VA})\mathcal{W}^3]$, $tr\mathcal{VW}^3$ we obtain $tr\mathcal{I}_1\mathcal{V}\mathcal{I}_1\mathcal{W}^3$ from which W^{34}.

From $tr(\mathcal{AVAWVW})$, $tr[(\mathcal{AV}+\mathcal{VA})\mathcal{WVW}]$, $tr\mathcal{VWVW}$ we obtain $tr\mathcal{I}_1\mathcal{V}\mathcal{I}_1\mathcal{WVW}$ from which V^{34}.

So we have finished of obtaining $\mathcal{V}$ and $\mathcal{W}$. Regarding any other skew- symmetric tensor $\mathcal{U}$, we have

- $U^{14}=0$, for the hypothesis of this subcase,
- $U^{23}=0$, for eq. (38) with $\mathcal{U}$ instead of $\mathcal{V}$,
- $U^{13}=U^{24}$ are already known (See what said between eq. (38) and the beginning of subcase 1).

From $tr(\mathcal{UAVA})$, $tr[(\mathcal{VU}+\mathcal{UV})\mathcal{A}]$, $tr\mathcal{UV}$ we obtain $tr\mathcal{U}\mathcal{I}_1\mathcal{V}\mathcal{I}_1$ from which U^{12}.

From $tr(\mathcal{AVAWUW})$, $tr[(\mathcal{AV}+\mathcal{VA})\mathcal{WUW}]$, $tr\mathcal{VWUW}$ we obtain $tr\mathcal{I}_1\mathcal{V}\mathcal{I}_1\mathcal{WUW}$ from which U^{34}.

Consequently, all the independent variables have been obtained, in a suitable reference frame, as functions of the scalars of the previous tables and of that in the following table.

The table 5 S

$tr(\mathcal{AVAW}^3)$, $tr[(\mathcal{AV}+\mathcal{VA})\mathcal{W}^3]$, $tr\mathcal{VW}^3$, $tr(\mathcal{AVAWVW})$, $tr[(\mathcal{AV}+\mathcal{VA})\mathcal{WVW}]$, $tr\mathcal{VWVW}$, $tr(\mathcal{AVAWUW})$, $tr[(\mathcal{AV}+\mathcal{VA})\mathcal{WUW}]$, $tr\mathcal{VWUW}$.

Regarding the symmetric tensors, with passages like those of subcase 1, we obtain the following elements to include in the corresponding representation.

The table 5 Tsy

$\mathcal{I}$, $\mathcal{A}$, $\mathcal{AW}-\mathcal{WA}$, $\mathcal{WV}+\mathcal{VW}$, $(\mathcal{AVW}+\mathcal{WVA})+(\mathcal{VAW}+\mathcal{WAV})$, $\mathcal{AVAW}+\mathcal{WAVA}$.

In fact, from the last 3 elements of this table, we obtain $\mathcal{I}_1\mathcal{V}\mathcal{I}_1\mathcal{W}+\mathcal{W}\mathcal{I}_1\mathcal{V}\mathcal{I}_1$ whose component 23 is not zero. Consequently, this tensor and the first 3 elements of table 5 Tsy are a set of generators for symmetric tensorial functions.

Regarding the skew-symmetric tensors, with passages like those of subcase 1, we obtain the following elements to include in the corresponding representation.

The table 5 Tsk

$\mathcal{W}$, $\mathcal{AWA}$, $\mathcal{AW}+\mathcal{WA}$, $\mathcal{V}$, $\mathcal{AVA}$, $\mathcal{AV}+\mathcal{VA}$, $\mathcal{WAVA}-\mathcal{AVAW}$, $\mathcal{WV}-\mathcal{VW}$, $\mathcal{WAV}-\mathcal{VAW}$, $\mathcal{WVA}-\mathcal{AVW}$, $\mathcal{WAVAW}$, $\mathcal{WAVW}+\mathcal{WVAW}$, $\mathcal{WVW}$.

In fact,

- $\mathcal{W}-\mathcal{I}_1\mathcal{W}\mathcal{I}_1-\mathcal{I}_2\mathcal{W}\mathcal{I}_2$ is a linear combination of the first 3 elements of the table,
- $\mathcal{I}_1\mathcal{V}\mathcal{I}_1$ is a linear combination of the 4^{th}, 5^{th} and 6^{th} element of the table,
- $\mathcal{W}\mathcal{I}_1\mathcal{V}\mathcal{I}_1-\mathcal{I}_1\mathcal{V}\mathcal{I}_1\mathcal{W}$ is a linear combination of the 7^{th}, 8^{th}, 9^{th} and 10^{th} element of the table,
- $\mathcal{W}\mathcal{I}_1\mathcal{V}\mathcal{I}_1\mathcal{W}$ is a linear combination of the 11^{th}, 12^{th} and 13^{th} element of the table;

moreover, it is easy to verify that every skew-symmetric function ϕ is a linear combination of these 4 tensors obtained from the table. (In fact,

- the component 14 gives the coefficient of $\mathcal{W}\mathcal{I}_1\mathcal{V}\mathcal{I}_1-\mathcal{I}_1\mathcal{V}\mathcal{I}_1\mathcal{W}$,
- the component 34 gives the coefficient of $\mathcal{W}\mathcal{I}_1\mathcal{V}\mathcal{I}_1\mathcal{W}$,
- the component 12 gives the coefficient of $\mathcal{I}_1\mathcal{V}\mathcal{I}_1$,
- the component 13 gives the last coefficient).

2.16. Subcase 3: For all the skew-symmetric tensors $\mathcal{V}$, among the independent variables, we have $V^{14} = 0$, $V^{12} = 0$, but there is a skew-symmetric tensor $\mathcal{V}$, such that $(V^{34})^2 \neq 0$.

Obviously, we have $V^{14} = 0$, $V^{23} = 0$, $V^{13} = V^{24}$ thanks to eq. (38), for all the skew-symmetric tensors $\mathcal{V}$. Moreover, we already know $V^{13}\ \forall\ \mathcal{V}$ as shown just before the subcase 1.

The reference frame can be chosen such that $V^{34} > 0$; in fact, if we would have $V^{34} < 0$, changing the versus of the axis 1 and 3, we will have $V^{34} > 0$ still maintaining the sign of W^{13}, W^{24}, that is, $W^{13} > 0$, $W^{24} > 0$.

After that, we obtain V^{34}, W^{34} from $(V^{34})^2$ and $V^{34}W^{34}$ which we already knew just before considering the subcase 1. So we have finished of obtaining $\mathcal{V}$.

Regarding the tensor $\mathcal{W}$, we need to obtain only W^{12}.

From $tr(\mathcal{AVAW}^3)$, $tr[(\mathcal{AV}+\mathcal{VA})\mathcal{W}^3]$, $tr\mathcal{VW}^3$ (which were already used in the previous subcase) we obtain $tr\mathcal{I}_2\mathcal{V}\mathcal{I}_2\mathcal{W}^3$ from which W^{12}.

Regarding any other skew- symmetric tensor $\mathcal{U}$, we have

- $U^{14} = 0$, $U^{12} = 0$, for the hypothesis of this subcase,
- $U^{23} = 0$, for eq. (38) with $\mathcal{U}$ instead of $\mathcal{V}$,
- $U^{13} = U^{24}$ are already known (See what said between eq. (38) and the beginning of subcase 1, with $\mathcal{U}$ instead of $\mathcal{V}$).

From $tr\mathcal{UV}$ we obtain U^{34}.

From $tr(\mathcal{AVAWUW})$, $tr[(\mathcal{AV}+\mathcal{VA})\mathcal{WUW}]$, $tr\mathcal{VWUW}$ we obtain $tr\mathcal{I}_2\mathcal{V}\mathcal{I}_2\mathcal{WUW}$ from which U^{12}.

There is no new table to insert, because we have not used new scalars.

Regarding the symmetric tensors, with passages like those of subcase 1, we obtain again the generators of the table 5 Tsy.

In fact, from the last 3 elements of this table, we obtain $\mathcal{I}_2\mathcal{V}\mathcal{I}_2\mathcal{W} + \mathcal{W}\mathcal{I}_2\mathcal{V}\mathcal{I}_2$ whose component 23 is not zero. Consequently, this tensor and the first 3 elements of table 5 Tsy are a set of generators for symmetric tensorial functions. Even here there is no new table to insert.

Regarding the skew-symmetric tensors, with passages like those of subcase 1, we obtain again the generators of the table 5 Tsk.

In fact, the tensors $\mathcal{W}-\mathcal{I}_1\mathcal{W}\mathcal{I}_1-\mathcal{I}_2\mathcal{W}\mathcal{I}_2$, $\mathcal{I}_2\mathcal{V}\mathcal{I}_2$, $\mathcal{W}\mathcal{I}_2\mathcal{V}\mathcal{I}_2-\mathcal{I}_2\mathcal{V}\mathcal{I}_2\mathcal{W}$, $\mathcal{W}\mathcal{I}_2\mathcal{V}\mathcal{I}_2\mathcal{W}$ are linear combinations of the elements of table 5 Tsk;

moreover, only the third one of these tensors has the component 14 different from zero; between the remaining ones, only the last one has the component 12 different from zero; between the remaining ones, only the second one has the component 34 different from zero; between the remaining ones, only the first one has the component 13 different from zero. Even here there is no new table to insert.

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2.17. Subcase 4: For all the skew-symmetric tensors $\mathcal{V}$, among the independent variables, we have $V^{14} = 0$, $V^{12} = 0$, $V^{34} = 0$ but we have $(W^{12})^2 \neq 0$.

Obviously, we have $V^{14} = 0$, $V^{23} = 0$, $V^{13} = V^{24}$ thanks to eq. (38), for all the skew-symmetric tensors $\mathcal{V}$. Moreover, we already know $V^{13}\ \forall\ \mathcal{V}$ as shown just before the subcase 1. Consequently, we already know $\forall\ \mathcal{V}$.

Regarding $\mathcal{W}$, the reference frame can be chosen such that $W^{12} > 0$; in fact, if we would have $W^{12} < 0$, changing the versus of the axis 1 and 3, we will have $W^{12} > 0$ still maintaining $W^{13} > 0$, $W^{24} > 0$.

After that, we obtain W^{12} from $(W^{12})^2$ which we knew just before considering the subcase 1.

Now we notice that $tr\mathcal{W}^4 = 2(W^{34})^4 + 8(W^{13})^2(W^{34})^2 + 8(W^{13})^2W^{12}W^{34} +$ terms not depending on W^{34}.

But $(W^{34})^2$ was already known before considering the subcase 1; consequently, by substituting it in $tr\mathcal{W}^4$ we obtain an equation from which to deduce the value of W^{34}. Consequently the unique new scalar used, can be inserted in the following table.

The table 6 S
$tr\mathcal{W}^4$.

Regarding the symmetric tensors, with passages like those of subcase 1, we obtain the following generators.

The table 6 Tsy

$$\mathcal{I},\ \mathcal{A},\ \mathcal{A}\mathcal{W} - \mathcal{W}\mathcal{A},\ \mathcal{W}^2,\ \mathcal{W}\mathcal{A}\mathcal{W}.$$

In fact, from the last 2 elements of this table, we obtain $\mathcal{W}\mathcal{I}_1\mathcal{W}$ whose component 23 is not zero. Consequently, this tensor and the first 3 elements of table 6 Tsy are a set of generators for symmetric tensorial functions.

Regarding the skew-symmetric tensors, with passages like those of subcase 1, we obtain the following generators.

The table 6 Tsk

$$\mathcal{W},\ \mathcal{A}\mathcal{W}\mathcal{A},\ \mathcal{A}\mathcal{W} + \mathcal{W}\mathcal{A},\ \mathcal{A}\mathcal{W}\mathcal{A}\mathcal{W} - \mathcal{W}\mathcal{A}\mathcal{W}\mathcal{A},\ \mathcal{A}\mathcal{W}^2 - \mathcal{W}^2\mathcal{A},\ \mathcal{W}^2\mathcal{A}\mathcal{W} + \mathcal{W}\mathcal{A}\mathcal{W}^2,\ \mathcal{W}^3.$$

In fact, the tensors

$$\mathcal{W} - \mathcal{I}_1\mathcal{W}\mathcal{I}_1 - \mathcal{I}_2\mathcal{W}\mathcal{I}_2\,,\ \mathcal{I}_1\mathcal{W}\mathcal{I}_1\,,\ \mathcal{W}\mathcal{I}_1\mathcal{W}\mathcal{I}_1 - \mathcal{I}_1\mathcal{W}\mathcal{I}_1\mathcal{W}\,,\ \mathcal{W}^2\mathcal{I}_1\mathcal{W} + \mathcal{W}\mathcal{I}_1\mathcal{W}^2\,,$$

are linear combinations of the elements of table 6 Tsk. Well, if $W^{34} \neq 0$, then ϕ is a linear combination of $\mathcal{W}$ and of $(39)_{1-3}$; in fact, these last ones have the component 34 equal to zero; after that, only $(39)_3$ has the component 14 different from zero;

after that, only $(39)_1$ has the component 13 different from zero; finally only $(39)_2$ has the component 12 different from zero.

Instead of this, if $W^{34} = 0$, we can say the same things but with $(39)_4$ instead of $\mathcal{W}$.

2.18. Subcase 5: For all the skew-symmetric tensors $\mathcal{V}$, among the independent variables, we have $V^{14} = 0$, $V^{12} = 0$, $V^{34} = 0$; moreover, we have $W^{12} = 0$, but $(W^{34})^2 \neq 0$.

Obviously, we have $V^{14} = 0$, $V^{23} = 0$, $V^{13} = V^{24}$ thanks to eq. (38), for all the skew-symmetric tensors $\mathcal{V}$. Moreover, we already know $V^{13}\ \forall\ \mathcal{V}$ as shown just before the subcase 1. Consequently, we already know $\forall\ \mathcal{V}$.

Regarding $\mathcal{W}$, the reference frame can be chosen such that $W^{34} > 0$; in fact, if we would have $W^{34} < 0$, changing the versus of the axis 1 and 3, we will have $W^{34} > 0$ still maintaining $W^{13} > 0$, $W^{24} > 0$.

After that, we obtain W^{34} from $(W^{34})^2$ which we already knew just before considering the subcase 1. In this way, also the tensor $\mathcal{W}$ is known and we have not used new scalars; consequently, there is no new table to insert for scalar functions.

Regarding the symmetric tensors, with passages like those of subcase 1, we obtain that they are linear combinations of the first 4 elements of Table 6 Tsy. In fact, the component 23 gives the coefficient of $\mathcal{W}^2$, the component 13 gives the coefficient of $\mathcal{A}\mathcal{W} - \mathcal{W}\mathcal{A}$, the components 11 and 33 give those of $\mathcal{I}$ and $\mathcal{A}$. Consequently, there is no new table to insert.

Regarding the skew-symmetric tensors, with passages like those of subcase 1, it suffices to use the generators of the previous Table 6 Tsk. In fact, the tensors

$$\mathcal{W} - \mathcal{I}_1\mathcal{W}\mathcal{I}_1 - \mathcal{I}_2\mathcal{W}\mathcal{I}_2\,,\ \mathcal{I}_2\mathcal{W}\mathcal{I}_2\,,\ \mathcal{W}\mathcal{I}_2\mathcal{W}\mathcal{I}_2 - \mathcal{I}_2\mathcal{W}\mathcal{I}_2\mathcal{W}\,,\ \mathcal{W}^2\mathcal{I}_2\mathcal{W} + \mathcal{W}\mathcal{I}_2\mathcal{W}^2\,, \quad (39)$$

are linear combinations of the elements of table 6 Tsk; of these,

- only $(39)_4$ has the component 12 different from zero; after that, between the remaining ones,
- only $(39)_3$ has the component 14 different from zero; after that, between the remaining ones,
- only $(39)_1$ has the component 13 different from zero; finally,
- only $(39)_2$ has the component 34 different from zero.

Consequently, there is no need to insert a new table for skew-symmetric tensors.

2.19. Subcase 6: For all the skew-symmetric tensors $\mathcal{V}$, among the independent variables, we have $V^{14} = 0$, $V^{12} = 0$, $V^{34} = 0$; moreover, we have $W^{12} = 0$ and $W^{34} = 0$.

Obviously, we have $V^{14} = 0$, $V^{23} = 0$, $V^{13} = V^{24}$ thanks to eq. (38), for all the skew-symmetric tensors $\mathcal{V}$. Moreover, we already know V^{13} $\forall$ $\mathcal{V}$ as shown just before the subcase 1. Consequently, we already know $\forall$ $\mathcal{V}$. We know also $\mathcal{W}$ and, consequently, there is no new table to insert for scalar functions.

Regarding the representation for second order tensorial functions, we note that by changing the versus of both axis 1 and 3, all the independent variables, remain unchanged. Consequently also every function ϕ^{ij}, depending on them, must remain unchanged. On the other hand, for the transformation rule of tensors, we have that ϕ^{12}, ϕ^{14}, ϕ^{23}, ϕ^{34} transform themselves in $-\phi^{12}$, $-\phi^{14}$, $-\phi^{23}$, $-\phi^{34}$ respectively; it follows that $\phi^{12} = 0$, $\phi^{14} = 0$, $\phi^{23} = 0$, $\phi^{34} = 0$.

Moreover, by using the transformation (29) we find also that $\phi^{11} = \phi^{22}$, $\phi^{33} = \phi^{44}$, $\phi^{13} = \phi^{24}$.

Consequently, every symmetric tensor ϕ is a linear combination of $\mathcal{I}$, $\mathcal{A}$, $\mathcal{AW} - \mathcal{WA}$; but these tensors are already present in table 6 Tsy, so that now there is no new table to insert.

Similarly, every skew-symmetric tensor ϕ is proportional to $\mathcal{W}$ and we don't need to insert a new table for skew-symmetric tensors.

In this way, we have exhausted all the subcases of the case 7. Let us now face the new

2.20. Case 8: For all the skew-symmetric tensors $\mathcal{W}$, among the independent variables, we have $W^{13} = 0$, $W^{14} = 0$, $W^{23} = 0$, $W^{24} = 0$

In this case, we have

$$\mathcal{W} = \begin{pmatrix} 0 & W^{12} & 0 & 0 \\ -W^{12} & 0 & 0 & 0 \\ 0 & 0 & 0 & W^{34} \\ 0 & 0 & -W^{34} & 0 \end{pmatrix} .$$

From $tr\mathcal{W}^2$ and $tr\mathcal{W}^2\mathcal{A}$, we obtain $(W^{12})^2$ and $(W^{34})^2$.

From $tr\mathcal{WV}$, $tr(\mathcal{WVA} + \mathcal{VWA})$ we obtain $V^{12}W^{12}$ and $V^{34}W^{34}$. We note also that these scalars are already present in Table $2S$.

Moreover, we note that by changing the versus of both axis 1 and 2, all the independent variables remain unchanged. Consequently also every function ϕ^{ij}, depending on them, must remain unchanged. On the other hand, for the transformation rule of tensors, we have that ϕ^{13}, ϕ^{14}, ϕ^{23}, ϕ^{24} change sign; it follows that $\phi^{13} = 0$, $\phi^{14} = 0$, $\phi^{23} = 0$, $\phi^{24} = 0$.

More than that, if ϕ^{ij} is skew-symmetric, by using the transformation (29) and its consequence (30) we find also that $\phi^{11} = \phi^{22}$, $\phi^{33} = \phi^{44}$, $\phi^{12} = 0$ and $\phi^{34} = 0$.

Consequently, every symmetric tensor ϕ is a linear combination of $\mathcal{I}$ and $\mathcal{A}$, so that there is no new table to insert for symmetric tensorial functions.

2.21. Subcase 1: There is a skew-symmetric tensors $\mathcal{W}$, among the independent variables, with $(W^{12})^2(W^{34})^2 > 0$.

We choose the versus of the axis 1 and 3 so that $W^{12} > 0$, $W^{34} > 0$ and deduce their values from $(W^{12})^2$ and $(W^{34})^2$ which we already know.

After that, we obtain V^{12} and V^{34} from $V^{12}W^{12}$ and $V^{34}W^{34}$ which we already know. Consequently, there is no new table to insert for scalar functions.

Clearly, every skew-symmetric function ϕ is a linear combination of $\mathcal{W}$ and of $\mathcal{AW} + \mathcal{WA}$ which are already present in previous tables. Consequently, there is no new table to insert for tensorial skew-symmetric functions.

2.22. Subcase 2: For all skew-symmetric tensors $\mathcal{W}$, among the independent variables, we have $(W^{12})^2(W^{34})^2 = 0$, but there are two of them with $(W^{12})^2 > 0$, $(V^{34})^2 > 0$.

We choose the versus of the axis 1 and 3 so that $W^{12} > 0$, $V^{34} > 0$ and deduce their values from $(W^{12})^2$ and $(V^{34})^2$ which we already know. For every other skew-symmetric tensor $\mathcal{U}$ we have $U^{12} = 0$ or $U^{34} = 0$; in the first case we find U^{34} from $tr\mathcal{VU}$, while in the second case we obtain U^{12} from $tr\mathcal{WU}$. Consequently, there is no new table to insert for scalar functions.

Clearly, every skew-symmetric function ϕ is a linear combination of $\mathcal{W}$ and $\mathcal{V}$; then, also in this subcase, there is no new table to insert for tensorial skew-symmetric functions.

2.23. Subcase 3: For all skew-symmetric tensors $\mathcal{W}$, among the independent variables, we have $W^{34} = 0$, but there is at least one of them with $(W^{12})^2 > 0$.

We choose the versus of the axis 1 so that $W^{12} > 0$ and deduce its value from $(W^{12})^2$. After that, we obtain V^{12} from $V^{12}W^{12}$. In this way we have used no new scalar.

Now we note that by changing the versus of the axis 3, all the independent variables remain unchanged. Consequently also every function ϕ^{ij}, depending on them, must remain unchanged. On the other hand, for the transformation rule of tensors, we have that ϕ^{34} changes sign; it follows that $\phi^{34} = 0$.

It follows that every skew-symmetric function ϕ is proportional to $\mathcal{W}$; then, also in this subcase, there is no new table to insert for tensorial skew-symmetric functions.

2.24. Subcase 4: For all skew-symmetric tensors $\mathcal{W}$, among the independent variables, we have $W^{12} = 0$, but there is at least one of them with $(W^{34})^2 > 0$.

It is similar to the previous one. We choose the versus of the axis 3 so that $W^{34} > 0$ and deduce its value from $(W^{34})^2$. After that, we obtain V^{34} from $V^{34}W^{34}$. In this way we have used no new scalar.

Now we note that by changing the versus of the axis 1, all the independent variables remain unchanged. Consequently also every function ϕ^{ij}, depending on them, must remain unchanged. On the other hand, for the transformation rule of tensors, we have that ϕ^{12} changes sign; it follows that $\phi^{12} = 0$.

Consequently, every skew-symmetric function ϕ is proportional to $\mathcal{W}$; then, also in this subcase, there is no new table to insert for tensorial skew-symmetric functions.

2.25. Subcase 5: All the skew-symmetric tensors are zero.

There is no new table to insert for scalar functions. By changing the versus of the axis 1 and 3, all the independent variables remain unchanged. Consequently also every function ϕ^{ij}, depending on them, must remain unchanged. On the other hand, for the transformation rule of tensors, we have that ϕ^{12} and ϕ^{34} change sign; it follows that $\phi^{12} = 0$ and $\phi^{34} = 0$. Joining this result with the previous ones, we have that $\phi^{ij} = 0$ and there is no new table to insert for tensorial skew-symmetric functions.

In this way we have exhausted all the possible cases where no symmetric tensor has eigenvalues whit multiplicity 1, but there is at least one of them with two distinct double eigenvalues a and b.

It remains to consider only the case where all the symmetric tensors $\mathcal{A}$ have a eigenvalue with multiplicity four; in this case we have $\mathcal{A} = \mathcal{I}\,(tr\mathcal{A})$ so that $\mathcal{A}$ plays a role only through the scalar $tr\mathcal{A}$. In other words, it remains to consider only the case where all the independent variables are skew-symmetric tensors. This part will be the subject of a future work.

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