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A PERSPECTIVE ON COLLATZ $3x + 1$ CONJECTURE, II

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Abstract: In this paper we provide the some of the development of the results obtained in the previous paper (3) of the same authors and some essential results towards the convergence of the trajectories of $3x+1$ mapping. We define and discuss the properties of the transition point set Y* on the positive odd integer of the form $4n+3$ where n is non negative integer and invariant subsets Y** of Y*. Our results lead us to conclude Collatz conjecture is enough to verify for a subset of odd integers with the property that (odd integer+1) is divisible by just 4. Also we have shown that Collatz sequence *cannot* increase indefinitely.

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1. Introduction

Let us recall the Collatz $3x + 1$ conjecture. On the positive integers define the function $F(x) = 3x + 1$ if x is odd and $F(x) = x/2$ if x is even. The $3x + 1$ conjecture, stated in 1937 by Lothar Collatz, is, "for each integer x, applying successive iterations of F, eventually yields 1".

Currently, the conjecture has been verified for all numbers up to $5.6.10^{13}$ [1]. For a brief history of the development of $3x + 1$ problem and its generalizations, see [2].

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International Electronic Journal of Pure and Applied Mathematics – IEJPAM, Volume 5, No. 4 (2012)

2. Background and Notations

We need the following notion in our discussion:

The orbit of x_o is the set of points

$$x_o,\ f(x_o),\ f^2(x_o),\ f^3(x_o), \dots,$$

where f^n denotes composition of real valued function f with itself.

Throughout our discussion, x_o denotes an odd positive integer greater than or equal to 1.

To begin with, the set $\mathbb{O} = 2\mathbb{N} - 1$ of odd positive integers is partitioned into two parts

$$\begin{aligned}(a) : \quad &= \left\{ x_o \in \mathbb{O} : \frac{x_o + 1}{2} \text{ is odd} \right\} = \{1 \bmod 4\}\\ (b) : \quad &= \left\{ x_o \in \mathbb{O} : \frac{x_o + 1}{2} \text{ is even} \right\} = \{-1 \bmod 4\}.\end{aligned}$$

The first part, the residue class 1 mod 4, is called the *mother orbit*, whereas the residue class -1 mod 4 is called the *secondary orbit*.

Given an odd x_o. The iterate of x_o,

$$F(x_o) = 3x_o + 1 \tag{1}$$

is, clearly, an even integer.

In general, when

$$x_o = \frac{2^{2n} - 1}{3},\ n = 1, 2, 3, ... \tag{2}$$

the iterations of x_o, $F(x_o)$ give 1 for all $n \geq 1$.

Further,

$$x_o + 1 = \frac{2^{2n} - 1}{3} + 1 = \frac{2^{2n} + 2}{3} = \frac{2\left(2^{2n-1} + 1\right)}{3}$$

and this implies that $\frac{x_o+1}{2}$ is odd and hence (2) belongs to the subset *(a)*. Thus we see that the odd integers (2) with $\frac{x_o+1}{2}$ is odd, the iterations of $F(x_o)$ directly lead to 1.

We remark that (2) is the solution of the recurrence equation $a_n = 4a_{n-1} + 1$ with $a_o = 1$. Also note that $3a_n + 1 = 4(3a_{n-1} + 1)$.

3. Previous Results

We provide the results obtained in paper (3) as follows.

Lemma 1. *For every odd integer y_o, if y_o is in (b) then $\frac{y_o-1}{2}$ is odd.*

International Electronic Journal of Pure and Applied Mathematics – IEJPAM, Volume 5, No. 4 (2012)

In order to apply successive iterations of F, we rewrite F as:

$$F(x) = 3x + 1 = 3(x-1) + 4 \quad \text{when } x \text{ is odd.} \tag{3}$$

The first iteration of y_o in (3), $F(y_o) = 3(y_o - 1) + 4$ is even, and hence by Lemma 1, we get

$$\frac{3(y_o-1)}{2} + 2 \tag{4}$$

Denote (4) by $H(y_o)$. Then, by Lemma 1, $\forall y \equiv -1 \bmod 4$ the successive iterations of

$$H(y) = \frac{3(y-1)}{2} + 2 = \frac{F(y)}{2} = F^2(y) \tag{5}$$

leads to a sequence of *odd numbers*. In other words, the secondary orbit of y_o

$$y_o,\ H(y_o),\ H^2(y_o),\ H^3(y_o), \ldots \tag{6}$$

consists of only *odd numbers* and hence there is a possibility of returning (6) to the mother orbit. More precisely,

Theorem 1. *If $y \equiv -1 \bmod 4$ and $y+1$ is divisible by 2^n but not by 2^{n+1}, then $H^{n-1}(y) \equiv 1 \bmod 4$.*

Lemma 2. *If $x_o \geq 5$ is in (a) then $x_o - 1$ is divisible by at least 4.*

In order to apply successive iterations of F, we rewrite F as:

$$F(x) = 3x + 1 = 3(x-1) + 4 \quad \text{when } x \text{ is odd.}$$

Let $x_o \equiv 1 \bmod 4$. Then by Lemma 2, we get

$$\frac{3(x_o-1)}{4} + 1 = \frac{F(x_o)}{4} = F^3(x_o) \tag{7}$$

which we denote by $G(x_o)$.

Case 1. Further, if $G(x_o), G^2(x_o), \cdots, G^{i-1}(x_o)$ are also in *(a)*, then

$$G^i(x_o) = \frac{3^i(x_o-1)}{4^i} + 1, \quad i \geq 1. \tag{8}$$

Note that the integers (10) are of the form $3k+1$, where $k = \frac{3^{i-1}(x_o-1)}{4^i}$. Evidently, $G^i(x_o)$ may be an odd or an even integer as x_o in *(a)*.

Case 2. $G^i(x_o) \equiv -1 \bmod 4$ and $G^i(x_o) = 3k+1$ for an even integer k. Then the iterations of H lead to

$$H^{n-1}G^i(x_o) = \frac{3^{n-1}\left(\frac{3^i(x_o-1)}{4^i} + 2\right)}{2^{n-1}} - 1 \tag{9}$$

provided $G^i(x_o)+1$ is divisible by 2^n (see Theorem 1).

Case 3. $G^i(x_o) = 3k+1$ for an odd integer $k \equiv -1 \bmod 4$. Then by Lemma 1, $k-1$ is divisible by 2, hence the iteration of F gives

$$FG^i(x_o) = H\left(\frac{G^i(x_o)-1}{3}\right) \tag{10}$$

since

$$FG^i(x_o) = \frac{\left(\frac{3^i(x_o-1)}{4^i}+1\right)}{2} = 3\left(\frac{3^i(x_o-1)}{2.3.4^i} - \frac{1}{2}\right) + 2.$$

Case 4. Lastly, $G^i(x_o) = 3k+1$ for an odd integer $k \equiv 1 \bmod 4$. As $G^i(x_o) = 3k+1$ by Example 1, the iterations of F directly lead to 1 if $k = \frac{2^{2n}-1}{3}$. Otherwise, by Lemma 2, it follows that $G^i(x_o) = 3k+1 = 3(k-1)+4$ is divisible by at least 4. Thus the iterations of F (twice) yield

$$F^2G^i(x_o) = G\left(\frac{G^i(x_o)-1}{3}\right) \tag{11}$$

since $FH = G$ from (9) and (5).

We remark that in *Case 3 and Case 4*, by iterations of F we mean the iterations of the function $F(x) = \frac{x}{2}$, when x is even. Summing up, we have

Theorem 2. *Let $x_o \equiv 1 \bmod 4$. Then the successive iterations of $G(x_o)$, where $G(x)$ is given by (9), lead to any of the four Cases given above.*

Now it is worth studying the properties of the functions $G(x)$ and $H(x)$.

Property 1. If $x_o \equiv 1 \bmod 4$ then $G(x_o) < x_o$, since the derivative of $G(x_o)$ with respect to x_o is $\frac{3}{4} < 1$.

Property 2. On the contrary, if $x_o \equiv -1 \bmod 4$ then $H(x_o) > x_o$, since the derivative of $H(x_o)$ with respect to x_o is $\frac{3}{2} > 1$.

4. Main Results

The results contained in this section are the development of results in paper (3) and claims the convergence of trajectories of $3x+1$ mapping on positive odd integers in (b). To begin with, Theorem 1, for any y from the subset (b), we have

$$H^{n-1}(y) = \frac{3^{n-1}}{2^{n-1}}(y+1) - 1$$

if $y+1$ is divisible by $2^n, n \geq 2$. Further,

$$GH^{n-1}(y) = \frac{3(\frac{3^{n-1}}{2^{n-1}}(y+1)-2)}{4} + 1 = \frac{3(3^{n-1}\frac{(y+1)}{2^n} - 1)}{2} + 1$$

International Electronic Journal of Pure and Applied Mathematics – IEJPAM, Volume 5, No. 4 (2012)

$$= \frac{3^n y' - 3}{2} + 1 = \frac{3^n y' - 1}{2}, \; y' = \frac{y+1}{2^n}.$$

Note that $\frac{3^n y' - 1}{2}$ is of the form $3k+1$. Thus,

$$GH^{n-1}(y) = \frac{3^n y' - 1}{2} \tag{12}$$

Theorem 3. *Let* $y \equiv -1 \bmod 4$ *and* $y+1$ *be divisible by* $2^n, n \geq 2$. *Then*

$$GH^{n-1}(y) = \frac{3^n y' - 1}{2} = GH(2^2 3^{n-2} y' - 1), \; y' = \frac{y+1}{2^n}. \tag{13}$$

Proof.

$$\begin{aligned} GH(2^2 3^{n-2} y' - 1) &= G\left(\frac{3(2^2 3^{n-2} y' - 2)}{2} + 2\right) \\ &= G(3(2.3^{n-2} y' - 1) + 2) \\ &= G(2.3^{n-1} y' - 1) \\ &= 3\left(\frac{2.3^{n-1} y' - 1 - 1}{4}\right) + 1 \\ &= \frac{3(3^{n-1} y' - 1)}{2} + 1 = \frac{3^n y' - 3}{2} + 1 = \frac{3^n y' - 1}{2}. \end{aligned}$$

Example 1. When $y = 79$, $\frac{79+1}{2^4} = 5$ with $n = 4$ and $y' = 5$, by Theorem 3,

$$GH^3(79) = GH(2^2 3^2 5 - 1) = GH(179) = \frac{3^4 5 - 1}{2}.$$

Note that $179+1$ is just divisible by 2^2.

Hence, we conclude: $\forall y \in (b)$ with $y' = \frac{y+1}{2^n}, n \geq 2$ there is an element $y^* \in (b)$ such that $y^* + 1$ is just divisible by 2^2 and

$$y^* = 2^2 3^{n-2} y' - 1 \tag{14}$$

Thus, by Theorem 3, the successive iterations of $y \in (b)$ lead to :

$$\{y^* \equiv -1 \bmod 4 \text{ and } y^* + 1 \text{ is just divisible by } 2^2\} \tag{15}$$

We call the set of integers (15) which is a subset of (b) as transition point set of any $y \in (b)$. Denote the set (15) by Y^*.

Hence by Theorem 3, it is "sufficient" to iterate only integers from Y^*, otherwise we can use (14). Now we are on Y^*.

5. Properties of Y^*

Every $y \in Y^*$ satisfies :

$$\frac{y-1}{2} \text{ is in the subset } (a) \tag{16}$$

since $\frac{\frac{y-1}{2}+1}{2} = \frac{y+1}{4}$ is an odd by definition of Y^*. Next,

$$\frac{\frac{y-1}{2}-1}{2} = \frac{y-3}{4} \text{ is an even integer} \tag{17}$$

since $\frac{y-1}{2} \in (a)$, by (16).

6. Orbit of Y^*

Following the orbit of $y \in (b)$, we found that every element in Y^* can be given by (14) with the remainder $y' = \frac{y+1}{2^n}, n \geq 2$. To study the orbit of Y^* consider an $y^* \in Y^*$.

Following the orbit of y^* , by Algorithm, (14) gives

$$H(y^*) = (2^2 3^{n-2} y' - 1) + 2.3^{n-2} y', \tag{18}$$

since by definition $H(y) = \frac{3(y-1)}{2} + 2 = y + \frac{y+1}{2}$.
Further, applying $G(x) = \frac{1}{2}H(x)$ to (7), we get

$$GH(y^*) = (2^2 3^{n-2} y' - 1) + \frac{3^{n-2}y' + 1}{2}. \tag{19}$$

At this stage we need : **Observation 1.** If an element $y \in (a)$ then $3y \in (b)$ and if an element $y \in (b)$ then $3y \in (a)$. In particular, if $y^* \in Y^*$ then $3y^* \in (a)$.

The proof is immediate from the definitions of the subsets (a) and (b).

$\mathbf{3^{n-2}y' \equiv 1}$ mod $\mathbf{4}$

Consider now (19) with $3^{n-2}y' \in (a)$. Clearly, (19) is even, since the last term on the right is odd. Using $-1 = -2 + 1$, addition of 1 to the last term in (19) yields

$$GH(y^*) = (2^2 3^{n-2} y' - 2) + \frac{3^{n-2}y' + 3}{2} \tag{20}$$

By Algorithm, apply $\frac{x-1}{3}$ to (20) to get

$$\frac{GH(y^*) - 1}{3} = (2^2 3^{n-3} y' - 1) + \frac{3^{n-3}y' + 1}{2}$$

International Electronic Journal of Pure and Applied Mathematics – IEJPAM, Volume 5, No. 4 (2012)

$$\begin{aligned} &= (3^{n-2}y' - 1) + \frac{3^{n-2}y' + 1}{2} \\ &= 2G(3^{n-2}y') - 1 = G(2.3^{n-2}y' - 1). \end{aligned} \tag{21}$$

From (21) it follows that

$$\frac{GH(y^*) - 1}{3} = \begin{cases} G(2y' - 1) = G(\frac{y^*-1}{2}), & \text{when } n = 2 \\ G(2.3^{n-2}y' - 1) = G(6.3^{n-3}y' - 1) = GH\left(\frac{y^*-2}{3}\right), & \text{when } n > 2, \end{cases}$$

by using (18). Clearly, by (14), $\frac{y^*-2}{3} \in Y^*$.

$\mathbf{3^{n-2}y' \in Y^*}$

If $3^{n-2}y' \in Y^*$ then (19)+1 is divisible by just 2, due to last term on the right side of (19). Hence (19) is in (a). In this case it is convenient to write (19) in the form

$$GH(y^*) = 4GH(3^{n-2}y') - 3 = 2(2GH(3^{n-2}y') - 1) - 1. \tag{22}$$

$\mathbf{3^{n-2}y' \notin Y^*}$

If $3^{n-2}y' \notin Y^*$ then $\frac{3^{n-2}y'+1}{2^2}$ is even. Now either

$$\frac{3^{n-2}y' + 1}{2^3} \text{ is odd} \tag{23}$$

or

$$\frac{3^{n-2}y' + 1}{2^3} \text{ is even.} \tag{24}$$

When (24) holds, (19) is in Y^*, since (19)+1 is just divisible by 2^2 due to first term on the right side of (19). Then $GH(y^*)$ can be given by

$$GH(y^*) = 2^2H^2\left(\frac{y^* - 3}{8}\right) - 1, \tag{25}$$

since $\frac{y^*-3}{2^3} \notin Y^*$ in the subset (b).

When (23) holds (19) $\notin Y^*$, since $\frac{(19)+1}{2^2}$ is even. Let $(19) + 1$ be divisible by, say 2^p with $p \geq 3$. In order to determine p, consider

$$\frac{GH(y^*) + 1}{4} = 3^{n-2}y' + \frac{3^{n-2}y' + 1}{2^3}. \tag{26}$$

Note that the last term on the right side of (26) is (23). Recalling that $3^{n-2}y'$ is the remainder of y^*, rewrite (23) as follows:

$$\frac{3^{n-2}y' + 1}{2^3} = \frac{\frac{y^*+1}{4} + 1}{2^3} = \frac{\frac{y^*-3}{2^3} + 1}{4}. \tag{27}$$

International Electronic Journal of Pure and Applied Mathematics – IEJPAM, Volume 5, No. 4 (2012)

As (23) is odd it follows from (27) that $\frac{y^*-3}{2^3} \in Y^*$. Thus, the remainder $3^{n-2}y'$ of y^* satisfying (23) is equivalent to $\frac{y^*-3}{2^3} \in Y^*$.

To make use of (27), rewrite (26) as

$$\frac{GH(y^*)+1}{4} = (3^{n-2}y'+1) - 2 + \frac{3^{n-2}y'+1}{2^3} + 1. \tag{28}$$

As the RHS of (28) is even, we get

$$\frac{GH(y^*)+1}{2^3} = \frac{(3^{n-2}y'+1)}{2} - 1 + \frac{\frac{3^{n-2}y'+1}{2^3}+1}{2}. \tag{29}$$

A simple comparison of (29) with (19) finally, reveals that

$$\frac{GH(y^*)+1}{2^3} = GH\left(\frac{3^{n-2}y'-1}{2}\right) = GH\left(\frac{y^*-3}{2^3}\right), \tag{30}$$

by using (27). Thus, it follows from (30) that the value of p depends on the oddness or evenness of $GH(\frac{y^*-3}{2^3})$. Further, the recurrence relation (30) shows that the value of p is determined by the "position" of y^* in the set Y^*.

For every y^* whose remainder satisfying (23), there exists a position function given by:

$$\begin{array}{cccccc} 27 & 91 & 155 & 219 & \cdots & (y^*) \\ \downarrow & \downarrow & \downarrow & \downarrow & & \\ 3 & 11 & 19 & 27 & \cdots & (\frac{y^*-3}{8}). \end{array} \tag{31}$$

We also know from (19) that

$$GH(y^*) = y^* + \frac{\frac{y^*+1}{4}+1}{2} = y^* + \frac{y^*-3}{2^3} + 1 = y^* + \text{ position value} + 1. \tag{32}$$

It is clear from (32) that

$$GH(y^*) = \begin{cases} even, & \text{if the position value is even} \\ odd, & \text{otherwise.} \end{cases}$$

Further,

$$\frac{3^{n-2}y'-1}{2} = \text{position value}, \tag{33}$$

where $3^{n-2}y' = \frac{y^*+1}{4}$ is the remainder of y^*.

7. Invariant Subset Y^{**} in Y^*

Consider Y^{**}={$y^* \in Y^*$ and its position value, $\frac{y^*-3}{2^3} \notin Y^*$ in the subset (b)}.

Obviously, $Y^{**} \subset Y^*$. Further, Y^{**} is invariant with respect to GH in Y^*. This result is immediate from (32). In particular, when $\frac{y^*-3}{2^3} = 8^i - 1, (i = 1, 2, 3, ...)$ which is not in Y^*, we obtain

$$\begin{aligned}(GH)^i(y^*) &= (GH)^i(2^3(8^i-1)+3) \\ &= (GH)^{i-1}(2^3(8^i+8^{i-1}-1)+3),\end{aligned}$$

by using (32).

Hence applying GH successively $i - times$, we get

$$(GH)^i(y^*) = (2^3((1+8)^i-1)+3) = 2^3(3^{2i}-1)+3.$$

As $(GH)^i(y^*) \in Y^*$, applying once more GH, finally, we get

$$(GH)^{i+1}(y^*) = 2^3(3^{2i}-1)+3+3^{2i} = 3^{2i+2}-5.$$

Note that $(GH)^{i+1}(y^*)$ is *even*.

Theorem 4. *Let $y^* \in Y^*$ and $3^{n-2}y'$ be the remainder of y^*. Then, we have:*
$GH(y^*)$ is even when $3^{n-2}y' \equiv 1 \mod 4$,
$GH(y^*) \equiv 1 \mod 4$ when $3^{n-2}y' \in Y^*$,
$GH(y^*) \in Y^*$ when $3^{n-2}y'$ satisfies (24), an
$GH(y^*)$ satisfies (30) when $3^{n-2}y'$ satisfies (23).

Thus, Theorem 4 was "compelling" to conclude that Collatz sequence *cannot* increase indefinitely. It also follows from Theorem 4 that every odd integer is the reminder of some $y^* \in Y^*$ and it is given by $3^{n-2}y'$, $n \geq 2$.

Consider now (21) with $n = 2$. We observe that

$$G\left(\frac{y^*-1}{2}\right) = G(3y_1^*) = 2GH(y_1^*) - 1 \tag{34}$$

for some $y_1^* \in Y^*$ when y^* is of the form $(6k+1)$ with $k \in Y^*$.

Further, when y^* is of the form $(6k+3)$ with $k = 8m, m = 0, 1, 2, ...$ we have

$$G\left(\frac{y^*-1}{2}\right) = G(3y_2^*-8) = 2GH(y_2^*) - 7 \tag{35}$$

for some $y_2^* \in Y^*$, where $y_2^* = \frac{y^*+15}{6}$.

Thus, in view of (35), (34), finally (21) becomes

$$\frac{GH(y^*)-1}{3} = \begin{cases} 2GH(y_2^*)-7, & \text{when } n=2 \\ 2GH(y_1^*)-1, & \text{when } n=2 \\ GH\left(\frac{y^*-2}{3}\right), & \text{when } n>2. \end{cases}$$

Notice that when y^* is of the form $3k$ with $k=8m+1, m=0,1,2,...$ there is an $x_1 \in (a)$ such that

$$y^* = \frac{G(x_1)-1}{3}, \tag{36}$$

with $G(x_1)$ is even.

References

[1] Scott W. Williams, Million-buck problems, *The Mathematical Intelligencer*, **24**, No .3 (2002), 17-20.

[2] Jeffrey C. Lagarias, The $3x+1$ problem and its generalizations, *Amer. Math. Monthly*, **92**, No. 1 (1985), 3-23; Reprinted in: *Conference on Organic Mathematics*, Canadian Math. Society, Conference Proceedings, Volume 20 (1997), 305-331; www.cecm.sfu.ca/organics/papers.

[3] E.S. Lakshminarayanan, R. Rammohan, A perspective on Collatz $3x+1$ conjecture, I, *Int. Elect. J. Pure and Applied Mathematics*, **5**, No. 4 (2012).