International Electronic Journal of Pure and Applied Mathematics
Volume 5 No. 4 2012, 127-133

A PERSPECTIVE ON COLLATZ $3x + 1$ CONJECTURE, I

E.S. Lakshminarayanan[1], R. Rammohan[2 §]
[1]School of Mathematics
Madurai Kamaraj University
Madurai, Tamilnadu, INDIA
[2]Department of Mathematics
Thiagarajar College of Engineering
Madurai, Tamilnadu, INDIA

Abstract: In this article we provide an elegant method which is based on investigating the divisors of integers $3x + 1$ when x is odd. This method employs the convergence of orbits on the partitions of the odd integers of the form $4n+1$ or $4n+3$ where n is a positive integer. We hope that this method may provide a proof for the Collatz $3x + 1$ conjecture. An algorithmic analysis of this method is discussed.

AMS Subject Classification: 11Y55
Key Words: Collatz, partition, orbit, iteration, algorithm

1. Introduction

Let us recall the Collatz $3x + 1$ conjecture. On the positive integers define the function $F(x) = 3x + 1$ if x is odd and $F(x) = x/2$ if x is even. The $3x + 1$ conjecture, stated in 1937 by Lothar Collatz, is, "for each integer x, applying successive iterations of F, eventually yields 1".

Currently, the conjecture has been verified for all numbers up to $5.6.10^{13}$ [1]. For a brief history of the development of $3x + 1$ problem and its generalizations, see [2].

Received: September 4, 2012

§Correspondence author

2. Partition of Odd Integers

We need the following notion in our discussion:

The orbit of x_o is the set of points

$$x_o,\ f(x_o),\ f^2(x_o),\ f^3(x_o),\dots,$$

where f^n denotes composition of real valued function f with itself.

Throughout our discussion, x_o denotes an odd positive integer greater than or equal to 1.

To begin with, the set $\mathbb{O} = 2\mathbb{N} - 1$ of odd positive integers is partitioned into two parts

$$(a):\ = \left\{x_o \in \mathbb{O} : \frac{x_o+1}{2} \text{ is odd}\right\} = \{1 \bmod 4\}$$

$$(b):\ = \left\{x_o \in \mathbb{O} : \frac{x_o+1}{2} \text{ is even}\right\} = \{-1 \bmod 4\}.$$

The first part, the residue class 1 mod 4, is called the *mother orbit*, whereas the residue class -1 mod 4 is called the *secondary orbit*.

Given an odd x_o. The iterate of x_o,

$$F(x_o) = 3x_o + 1 \tag{1}$$

is, clearly, an even integer. To perform next iteration it is necessary to know as to when does the even integer $3x_o + 1$ become odd? In other words, what are the divisors of $3x_o + 1$? The following example provides an insight into the problem.

Example 1. When $x_o = (1, 5, 21, 85, ...)$ the divisor of $3x_o + 1$ is respectively, $(2^2, 2^4, 2^6, 2^8, ...)$ and hence the iteration of each x_o, $F(x_o)$ leads to 1. A close observation of the sequence reveals that, in general, when

$$x_o = \frac{2^{2n}-1}{3},\ n = 1, 2, 3, ... \tag{2}$$

the iterations of x_o, $F(x_o)$ give 1 for all
$n \geq 1$.

Further,

$$x_o + 1 = \frac{2^{2n}-1}{3} + 1 = \frac{2^{2n}+2}{3} = \frac{2\left(2^{2n-1}+1\right)}{3}$$

and this implies that $\frac{x_o+1}{2}$ is odd and hence (2) belongs to the subset *(a)*. Thus we see that the odd integers (2) with $\frac{x_o+1}{2}$ is odd, the iterations of $F(x_o)$ directly lead to 1.

We remark that (2) is the solution of the recurrence equation $a_n = 4a_{n-1} + 1$ with $a_o = 1$. Also note that $3a_n + 1 = 4(3a_{n-1} + 1)$.

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The above example provides a key to our method. If the conjecture was true, then the orbit of every odd integer would have landed at any one of the point (odd integer) x_o given by (2). However, the orbit of an integer, after a certain number of iterations, may escape from the subset *(a)*. Indeed, when $x_o = 9$ the iterations of F lead to the sequence $\{9, 28, 14, 7, \ldots\}$, in which the odd integer 7 escapes from the subset *(a)*, since $\frac{(7+1)}{2}$ is even, and thus lands in the subset *(b)*. Hence it is necessary to study the "secondary orbit", the orbit of an integer from the subset *(b)*. Nevertheless the secondary orbit eventually returns to the "mother orbit", the orbit of an odd integer from the subset *(a)*.

3. Secondary Orbit: Case (b)

Lemma 1. *For every odd integer y_o, if y_o is in (b) then $\frac{y_o-1}{2}$ is odd.*

The proof is elementary. As $\frac{y_o+1}{2}$ is even, we have $\frac{y_o+1}{2} = 2\,m$, and it follows that $\frac{y_o-1}{2} = 2m - 1$ is odd for $m = 1, 2, \ldots$

In order to apply successive iterations of F, we rewrite F as:

$$F(x) = 3x + 1 = 3(x - 1) + 4 \quad \text{when } x \text{ is odd.} \tag{3}$$

The first iteration of y_o in (3), $F(y_o) = 3(y_o - 1) + 4$ is even, and hence by Lemma 1, we get

$$\frac{3(y_o - 1)}{2} + 2 \tag{4}$$

Denote (4) by $H(y_o)$. Then, by Lemma 1, $\forall y \equiv\ -1 \bmod 4$ the successive iterations of

$$H(y) = \frac{3(y-1)}{2} + 2 = \frac{F(y)}{2} = F^2(y) \tag{5}$$

leads to a sequence of *odd numbers*. In other words, the secondary orbit of y_o

$$y_o,\ H(y_o),\ H^2(y_o),\ H^3(y_o), \ldots \tag{6}$$

consists of only *odd numbers* and hence there is a possibility of returning (6) to the mother orbit. More precisely,

Theorem 1. *If $y \equiv\ -1 \bmod 4$ and $y + 1$ is divisible by 2^n but not by 2^{n+1}, then $H^{n-1}(y) \equiv\ 1 \bmod 4$.*

Proof. From (5), we obtain the second iteration $H^2(y) = \frac{3^2(y-1)}{2^2} + \frac{3}{2} + 2$.

In general, after $n - 1$ iterations, we obtain

$$H^{n-1}(y) = \frac{3^{n-1}(y-1)}{2^{n-1}} + \left(\frac{3}{2}\right)^{n-2} + \left(\frac{3}{2}\right)^{n-3} + \ldots + \frac{3}{2} + 2. \tag{7}$$

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To check whether (7) is in *(a)*, we consider

$$H^{n-1}(y)+1=\frac{3^{n-1}(y-1)}{2^{n-1}}+\left(\frac{3}{2}\right)^{n-2}+\left(\frac{3}{2}\right)^{n-3}+...+\frac{3}{2}+2+1$$

$$=\frac{3^{n-1}(y-1)}{2^{n-1}}+\frac{3^{n-1}}{2^{n-2}}=\frac{3^{n-1}(y+1)}{2^{n-1}} \tag{8}$$

Setting $y=y_o$ in (8), we get $\frac{H^{n-1}(y_o)+1}{2}$ is odd. The proof is complete.

4. Mother Orbit Case(a)

Lemma 2. *If $x_o \geq 5$ is in (a) then x_o-1 is divisible by at least 4.*

Proof. The proof is elementary. As $\frac{x_o+1}{2}$ is odd, we have $\frac{x_o+1}{2}=2\,m+1$, and it follows that $x_o-1=4m$, for $m=1,2,\cdots$. The proof is complete.

In order to apply successive iterations of F, we rewrite F as:

$$F(x)=3x+1=3(x-1)+4 \quad \text{when } x \text{ is odd.}$$

Let $x_o \equiv\ 1 \bmod 4$. Then by Lemma 2, we get

$$\frac{3(x_o-1)}{4}+1=\frac{F(x_o)}{4}=F^3(x_o) \tag{9}$$

which we denote by $G(x_o)$.

Case 1. Further, if $G(x_o),G^2(x_o),\cdots,G^{i-1}(x_o)$ are also in *(a)*, then

$$G^i(x_o)=\frac{3^i(x_o-1)}{4^i}+1, \quad i\geq 1. \tag{10}$$

Note that the integers (10) are of the form $3k+1$, where $k=\frac{3^{i-1}(x_o-1)}{4^i}$. Evidently, $G^i(x_o)$ may be an odd or an even integer as x_o in *(a)*.

Case 2. $G^i(x_o)\equiv\ -1 \bmod 4$ and $G^i(x_o)=3k+1$ for an even integer k. Then the iterations of H (see Section 1) lead to

$$H^{n-1}G^i(x_o)=\frac{3^{n-1}\left(\frac{3^i(x_o-1)}{4^i}+2\right)}{2^{n-1}}-1 \tag{11}$$

(see (8)) provided $G^i(x_o)+1$ is divisible by 2^n (see Theorem 1).

Case 3. $G^i(x_o)=3k+1$ for an odd integer $k\equiv\ -1 \bmod 4$. Then by Lemma 1, $k-1$ is divisible by 2, hence the iteration of F gives

$$FG^i(x_o)=H\left(\frac{G^i(x_o)-1}{3}\right) \tag{12}$$

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since

$$FG^i(x_o) = \frac{\left(\frac{3^i(x_o-1)}{4^i}+1\right)}{2} = 3\left(\frac{3^i(x_o-1)}{2.3.4^i}-\frac{1}{2}\right)+2.$$

Case 4. Lastly, $G^i(x_o) = 3k+1$ for an odd integer $k \equiv 1 \bmod 4$. As $G^i(x_o) = 3k+1$ by Example 1, the iterations of F directly lead to 1 if $k = \frac{2^{2n}-1}{3}$. Otherwise, by Lemma 2, it follows that $G^i(x_o) = 3k+1 = 3(k-1)+4$ is divisible by at least 4. Thus the iterations of F (twice) yield

$$F^2G^i(x_o) = G\left(\frac{G^i(x_o)-1}{3}\right) \tag{13}$$

since $FH = G$ from (9) and (5).

We remark that in *Case 3 and Case 4*, by iterations of F we mean the iterations of the function $F(x) = \frac{x}{2}$, when x is even. Summing up, we have

Theorem 2. *Let $x_o \equiv 1 \bmod 4$. Then the successive iterations of $G(x_o)$, where $G(x)$ is given by (9), lead to any of the four Cases given above.*

Now it is worth studying the properties of the functions $G(x)$ and $H(x)$.

Property 1. If $x_o \equiv 1 \bmod 4$ then $G(x_o) < x_o$, since the derivative of $G(x_o)$ with respect to x_o is $\frac{3}{4} < 1$.

Property 2. On the contrary, if $x_o \equiv -1 \bmod 4$ then $H(x_o) > x_o$, since the derivative of $H(x_o)$ with respect to x_o is $\frac{3}{2} > 1$.

5. Algorithm

In order to apply Theorem 2 we introduce a function

$$L(x) = \begin{cases} G(x), & \text{for } x \equiv 1 \bmod 4, \\ H(x), & \text{for } x \equiv -1 \bmod 4, \\ \frac{x-1}{3}, & \text{when } x \text{ is even,} \end{cases}$$

where $G(x), H(x)$ are given by (9), (5) respectively.

To apply $L(x)$ successively denote

$$L^j(x_o) = L(L^{j-1}(x_o)), j \geq 1 \text{ and } L^o(x_o) = x_o.$$

Then by Theorem 1 and 2, the successive iterations of L yields

$$L^j(x_o) = \begin{cases} GL^{j-1}(x_o), & \text{if } L^{j-1}(x_o) \text{ is odd but in } (a)...(\alpha) \\ HL^{j-1}(x_o), & \text{if } L^{j-1}(x_o) \text{ is odd but in } (b)...(\beta) \\ \frac{L^{j-1}(x_o)-1}{3}, & \text{if } L^{j-1}(x_o) \text{ is even } ...(\gamma) \end{cases}$$

for $j \geq 1$. By Theorem 2, (α) and (β) follow from *Case 1* and *Case 2*, respectively. In (γ) if $L^{j-1}(x_o)$ is even and divisible by just 2 then by *Case 3*, $L^{j+1} = H$ (see (12)). If $L^{j-1}(x_o)$ is even and divisible by at least 4 then by *Case 4* , $L^{j+1} = G$ (see(13)). Remarkably, indeed the odd integers x_o given by (2) in Example 1 belong to the range of the function $\frac{x-1}{3}$.

To justify the Algorithm we prove that $\{L^j(x_o) : j \geq 0\} \subset \{F^j(x_o) : j \geq 0\}$.

Let $x \in \{L^j(x_o)\}$ for some $j > 1$ and $x \equiv 1 \bmod 3$. Then, by Theorem 2, there $\exists\ x_1 \equiv 1 \bmod 4$ such that $x = (F^3)^i(x_1)$ for some $i > 1$. When $x_1 \equiv 1$ *mod* 4,in particular 5 mod 12, and $x_1 \equiv 2$ *mod* 3, by Theorem 1, there $\exists\ y_1 \equiv -1 \bmod 4$ such that $x_1 \equiv (F^2)^{i_1}(y_1)$ for some i_1 and hence $x = (F_3)^i(F^2)^{i_1}(y_1)$. When $x_1 \equiv 1 \bmod 4$ and $x_1 \equiv$ mod 3, by Theorem 2, there $\exists\ z_1 \equiv 1$ mod 3,in particular 4 mod 6, even, such that $\frac{z_1-1}{3} = x_1$. Also by Theorem 2, $\exists\ z_2 \equiv 1$ *mod* 4 such that $(F^3)^{i_2}z_2 = z_1$ for some i_2. Hence $x = (F^3)^i(\frac{(F^3)^{i_2}(z_2)-1}{3})$

Next, let $x^* \in \{L^j(x_0)\}$ for some $j > 1$ and $x^* \equiv 2 \bmod 3$. Then, by Theorem 2, $\exists\ x_1^* \equiv -1$ *mod* 4 such that $x^* = (F^2)^{i_3}(x_1^*)$ for some i_3.

When $x_1^* \equiv -1$ *mod* 4,in particular 7 mod 12, and $x_1^* \equiv 1$ *mod* 3, by Theorem 2, $\exists$ $y_1^* \equiv 1$ *mod* 4 such that $x_1^* = (F^3)^{i_4}(y_1^*)$ for some i_4 and hence $x^* = (F^2)^{i_3}(F^3)^{i_4}(y_1^*)$.

When $x_1^* \equiv -1$ *mod* 4 and $x_1^* \equiv$ *mod* 3, by Theorem 2, $\exists\ z_1^* \equiv 1$ *mod* 3, in particular 4 mod 6, even, such that $\frac{z_1^*-1}{3} = x_1^*$. Also, by theorem 2, $\exists$, $z_2^* \equiv 1$ *mod* 4 such that $(F^3)^{i_4}(z_2^*) = z_1^*$ for some i_4. Hence $x^* = (F^3)^i(\frac{(F^3)^{i_4}(z_2^*)-1}{3})$

Finally, let $x^{**} \in \{L^j(x_o)\}$ for some $j > 1$ and $x^{**} \equiv$ mod 3. Then, by Theorem 2, there $\exists\ y^{**} \equiv 1$ mod 3, even, such that $x^{**} = F(y^{**})$, where $H = (\frac{y^{**}-1}{3}) = F(y^{**})$ if $\frac{y^{**}-1}{3} \equiv -1$ *mod* 4, otherwise $x^{**} = F^2(y^{**})$ where $G(\frac{y^{**}-1}{3}) = F^2(y^{**})$ if $\frac{y^{**}-1}{3} \equiv 1$ *mod* 4.

If we follow the orbit of x_o it is not hard to see that on account of Property 2, the secondary orbit of x_o under L given by (β) forms an increasing sequence of odd integers. But by Theorem 1, this secondary orbit will return to the mother orbit *Case (a)* and hence we have to apply L given by (α) to this mother orbit. On account of Property 1 and (9), the mother orbit forms a decreasing sequence, giving the integers of the form $3k + 1$, $k \geq 1$. If $3k + 1$ is odd but in *(b)* we return to the secondary orbit. Also if $3k + 1$ is even but divisible by 2 we return to the secondary orbit. Thus the transition from secondary orbit to mother orbit and vice versa can take place many a time. Finally, if $3k + 1$ is even and divisible by 4 an application of L given by (γ) to $3k + 1$ may yield 1, or we may have to return to the mother orbit and repeat iterating. But by Theorem 1 and 2, on returning each time to the mother orbit we get a sequence of integers of the form $3k + 1$ but each time with different k, say $k^{'}, k^{''}, k^{'''}, \ldots k^{\overbrace{''' \cdots '}^{n\ times}}$, and hence when $3k^{\overbrace{''' \cdots '}^{n\ times}} + 1 = 4$ the iteration of L given by (γ) yields 1. In other words, by Theorem 1 and 2 there is a positive integer $m > 1$ such that

$$L^m(x_o) = \frac{L^{m-1}(x_o) - 1}{3} = 1 \quad \text{and hence} \quad L^{m-1}(x_o) = 4.$$

Moreover, by Theorem 2, $L^m(x_o) \neq HL^{m-1}(x_o)$, since $L^{m-1}(x_o)$ is even. Also $L^m(x_o) \neq GL^{m-1}(x_o)$, since $L^{m-1}(x_o) > 1$. Thus the successive iterations of x_o ends with one application of $\frac{x-1}{3}$.

As an application of theorem 1 and 2, let us complete with the instance when $x_o = 9$. Note that $9 \in (a)$ and hence $L^1(9) = 7$. The successive iterations of 7 gives: $L^2(7) = HL^1 = 11$. Note that $11 > 7$ by Property 2 in Section 2. $L^3(11) = 17, L^4(17) = G(17) = 13$. Note that $13 < 17$ by Property 1 in Section 2. $L^5(13) = 10, L^6(10) = 3, L^7(3) = 5, L^8(5) = 4$, and $L^9(4) = 1$.

Thus, by theorem 1 and 2, with the aid of Algorithm the orbit of 9 is given by

$$\{9, 7, 11, 17, 13, 10, 3, 5, 4, 1\}. \tag{14}$$

It is worth comparing (14) with Collatz sequence, that is, applying successive iterations of F, when $x = 9$ $\{9, 28, 14, 7, 22, 11, 34, 17, 52, 26, 13, 40, 20, 10, 5, 16, 8, 4, 2, 1\}$.

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