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TWO CIRIC TYPE PROBABILISTIC FIXED POINT THEOREMS FOR DISCONTINUOUS MAPPINGS

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Abstract: In this paper we establish two common fixed point results in probabilistic metric spaces. Our results are established without any continuity assumption on the functions. In one of our theorems we have used the Hadzic type t-norm. In another theorem we have used a control function. Two illustrative examples are given.

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1. Introduction

The theory of probabilistic metric spaces was introduced in 1942 by Menger [14]. The idea was to use distribution functions instead of non-negative real numbers as values of the metric. Thus probabilistic metric spaces have the notion of uncertainty built within the structure of the space. Different aspect of probabilistic metric space theory has developed over the years. Descriptions of several of its aspects have been given in the book by Schweizer and Sklar [22]. Fixed point theory in probabilistic metric spaces was initiated by Sehgal and Bharucha-Reid who had established a probabilistic version of the Banach's contraction mapping principle in such spaces [23]. After that fixed point theory developed greatly over the years. A comprehensive survey of research in this line is given in [11] by Hadzic and Pap. Some more recent references are noted in [1, 2, 10, 13, 17, 18, 19].

In metric fixed point theory a new direction was opened by Khan, Swaleh and Sessa [12]. They introduced a new contraction principle and proved a fixed point result through a control function which they called altering distance function. These

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functions have been used in a number of works in fixed point theory, as, for instances, in [7, 16, 20, 21]. It has been extended to Menger spaces in [3] where a generalization of Sehgal's contraction has been defined with the help of such functions and a unique fixed point result has been established for such contractions. This extension of altering distance function has been called ϕ-function.

2. Mathematical Preliminaries

Definition 2.1. (see [11, 22]) A mapping $F : R \to R^+$ is called a distribution function if it is non-decreasing and left continuous with $\inf_{t\in R} F(t) = 0$ and $\sup_{t\in R} F(t) = 1$, where R is the set of real numbers and R^+ denotes the set of non-negative real numbers.

Definition 2.2. (t-norm, [11, 22] A t-norm is a function $\Delta : [0,1]\times[0,1] \to [0,1]$ which satisfies the following conditions for all $a, b, c, d \in [0,1]$:

1. $\Delta(1, a) = a$,
2. $\Delta(a, b) = \Delta(b, a)$,
3. $\Delta(c, d) \geq \Delta(a, b)$ whenever $c \geq a$ and $d \geq b$,
4. $\Delta(\Delta(a, b), c) = \Delta(a, \Delta(b, c))$.

Definition 2.3. (Hadzic Type t-Norm, see [11]) A t-norm Δ is said to be Hadzic type t-norm if the family $\{\Delta^p\}_{p\in N}$ of its iterates, defined for each $s \in (0,1)$ as:

$$\Delta^0(s) = 1, \quad \Delta^{p+1}(s) = \Delta(\Delta^p(s), s),$$

for all $p \geq 0$, is equi-continuous at $s = 1$, that is, given $\lambda > 0$ there exits $\eta(\lambda) \in (0,1)$ such that

$$1 \geq s > \eta(\lambda) \Rightarrow \Delta^p(s) \geq 1 - \lambda,$$

for all $p \geq 0$.

The following is the definition of Menger space which is a probabilistic metric space of a specific type where the triangular inequality is postulated with the help of a t-norm.

Definition 2.4. (Menger Space, see [11, 22]) A Menger space is a triplet (X, F, Δ), where X is a non empty set, F is a function defined on $X \times X$ to the set of distribution functions and Δ is a t-norm, such that the following are satisfied:

1. $F_{x,y}(0) = 0$ for all $x, y \in X$,
2. $F_{x,y}(s) = 1$ for all $s > 0$ if and only if $x = y$,

3. $F_{x,y}(s) = F_{y,x}(s)$ for all $x, y \in X, s > 0$ and

4. $F_{x,y}(u + v) \geq \Delta\left(F_{x,z}(u), F_{z,y}(v)\right)$ for all $u, v \geq 0$ and $x, y, z \in X$.

An interpretation of $F_{x,y}(t)$ is that it is the probability of the event that the distance between the points x and y is less than t. A metric space becomes a Menger space if we write $F_{x,y}(t) = H(t - d(x, y))$ where H is the Heaviside function given by:
$H(t) = 1$, if $t > 0$,
$H(t) = 0$, if $t \leq 0$.

Definition 2.5. (see [11, 22]) A sequence $\{x_n\} \subset X$ is said to be converge to some point $x \in X$ if given $\epsilon > 0, \lambda > 0$ we can find a positive integer $N_{\epsilon,\lambda}$ such that for all $n > N_{\epsilon,\lambda}$

$$F_{x_n,x}(\epsilon) \geq 1 - \lambda. \tag{2.1}$$

Definition 2.6. (see [11, 22]) A sequence $\{x_n\}$ is said to be a Cauchy sequence in X if given $\epsilon > 0, \lambda > 0$ there exists a positive integer $N_{\epsilon,\lambda}$ such that
$F_{x_n,x_m}(\epsilon) \geq 1 - \lambda$ for all

$$m, n > N_{\epsilon,\lambda}. \tag{2.2}$$

Definition 2.5 and 2.6 can be equivalently written by replacing '$\geq$' with '$>$' in (2.1) and (2.2) respectively. More often than not, they are written in that way. We have given them in the present form for our convenience in the proofs of our theorems.

Definition 2.7. (see [11, 22]) A Menger space (X, F, Δ) is said to be complete if every Cauchy sequence is convergent in X.

The following is the extension of the control function introduced by Khan et. al [12] to probabilistic metric spaces. It has been shown that a ϕ-function can generate an altering distance function [3].

Definition 2.8. (Φ-function, see [3]) A function $\phi : R \to R^+$ is said to be a Φ-function if it satisfies the following conditions:

1. $\phi(t) = 0$ if and only if $t = 0$,

2. $\phi(t)$ is strictly monotone increasing and $\phi(t) \to \infty$ as $t \to \infty$,

3. ϕ is left continuous in $(0, \infty)$,

4. ϕ is continuous at 0.

Several uses of the ϕ-function in probabilistic fixed point and coincidence point problems have been made as, for instances, in [4, 5, 6, 9, 10, 15].

We will make use of the following function in our theorems.

Definition 2.9. (Ψ-function) A function $\psi : [0, 1] \times [0, 1] \times [0, 1] \to [0, 1]$ is said to be a Ψ-function if

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1. ψ-is monotone increasing and continuous,
2. $\psi(x, x, x) \geq x$ for all $0 < x < 1$,
3. $\psi(1, 1, 1) = 1, \psi(0, 0, 0) = 0$.

The purpose of this paper is to prove common fixed point theorems in Menger spaces. The inequalities we have used are motivated by a recent result of Ciric et. al [8]. We have also two examples illustrating the basic features of our theorems. Our results are derived without any assumption of continuity.

3. Main Results

In this section we have two theorems and two examples. The first theorem is motivated by the form of the inequality used by Ciric et. al [8].

Theorem 3.1. *Let (X, F, Δ) be a complete Menger space with a Hadzic type t-norm such that whenever $x_n \to x$ and $y_n \to y$, $F_{x_n,y_n}(t) \to F_{x,y}(t)$. Let $S, T : X \to X$ be two self mappings on X which satisfy the following inequality:*

$$F_{Sx,Ty}(t) + q(1 - \max\{F_{x,Ty}(t), F_{y,Sx}(t)\}) > \psi(F_{x,y}(\frac{t}{k}), F_{x,Sx}(\frac{t}{k}), F_{y,Ty}(\frac{t}{k})), \quad (3.1)$$

for all $x, y \in X$, $t > 0$, where $0 < k < 1$, $q \geq 0$ and ψ is a Ψ-function. Then S and T have a common fixed point in X. The fixed point is unique if q=0.

Proof. Let $x_0 \in X$ be arbitrary. We define a sequence $\{x_n\}_{n=0}^{\infty}$ in X as follows:

$$x_{2n+1} = Sx_{2n}, \quad x_{2n+2} = Tx_{2n+1} \text{ for all } n \geq 0. \quad (3.2)$$

Putting $x = x_{2n}, y = x_{2n+1}$ in (3.1), for all $t > 0$, we have

$$F_{Sx_{2n},Tx_{2n+1}}(t) + q(1 - \max\{F_{x_{2n},Tx_{2n+1}}(t), F_{x_{2n+1},Sx_{2n}}(t)\}) > \psi(F_{x_{2n},x_{2n+1}}(\frac{t}{k}), F_{x_{2n},Sx_{2n}}(\frac{t}{k}), F_{x_{2n+1},Tx_{2n+1}}(\frac{t}{k})),$$

that is,

$$F_{x_{2n+1},x_{2n+2}}(t) + q(1 - \max\{F_{x_{2n},x_{2n+2}}(t), F_{x_{2n+1},x_{2n+1}}(t)\}) > \psi(F_{x_{2n},x_{2n+1}}(\frac{t}{k}), F_{x_{2n},x_{2n+1}}(\frac{t}{k}), F_{x_{2n+1},x_{2n+2}}(\frac{t}{k})).$$

Now, for all $t > 0$ and $n \geq 0$,

$$\max\{F_{x_{2n},x_{2n+2}}(t), F_{x_{2n+1},x_{2n+1}}(t)\} = \max\{F_{x_{2n},x_{2n+2}}(t), 1\} = 1.$$

Therefore, for all $t > 0$, $n \geq 0$, we have

$$F_{x_{2n+1},x_{2n+2}}(t) > \psi(F_{x_{2n},x_{2n+1}}(\frac{t}{k}), F_{x_{2n},x_{2n+1}}(\frac{t}{k}), F_{x_{2n+1},x_{2n+2}}(\frac{t}{k})). \quad (3.3)$$

We now claim that for all $t > 0$, $n \geq 0$,

$$F_{x_{2n+1},x_{2n+2}}(\frac{t}{k}) \geq F_{x_{2n},x_{2n+1}}(\frac{t}{k}). \quad (3.4)$$

If possible, let for some $s > 0$ and some $n \geq 0$,

$$F_{x_{2n+1},x_{2n+2}}(\frac{s}{k}) < F_{x_{2n},x_{2n+1}}(\frac{s}{k}).$$

Then, from (3.3), using the properties of ψ, we have

$$\begin{aligned} F_{x_{2n+1},x_{2n+2}}(s) &> \psi(F_{x_{2n},x_{2n+1}}(\frac{s}{k}), F_{x_{2n},x_{2n+1}}(\frac{s}{k}), F_{x_{2n+1},x_{2n+2}}(\frac{s}{k})) \\ &\geq \psi(F_{x_{2n+1},x_{2n+2}}(\frac{s}{k}), F_{x_{2n+1},x_{2n+2}}(\frac{s}{k}), F_{x_{2n+1},x_{2n+2}}(\frac{s}{k})) \\ &\geq F_{x_{2n+1},x_{2n+2}}(\frac{s}{k}) \geq F_{x_{2n+1},x_{2n+2}}(s), \end{aligned}$$

which is a contradiction.

Therefore (3.4) holds for all $t > 0$ and $n \geq 0$.

Using (3.4) in (3.3), and by the properties of ψ, for all $t > 0$, $n \geq 0$, we have

$$\begin{aligned} F_{x_{2n+1},x_{2n+2}}(t) &> \psi(F_{x_{2n},x_{2n+1}}(\frac{t}{k}), F_{x_{2n},x_{2n+1}}(\frac{t}{k}), F_{x_{2n+1},x_{2n+2}}(\frac{t}{k})) \\ &\geq \psi(F_{x_{2n},x_{2n+1}}(\frac{t}{k}), F_{x_{2n},x_{2n+1}}(\frac{t}{k}), F_{x_{2n},x_{2n+1}}(\frac{t}{k})) \geq F_{x_{2n},x_{2n+1}}(\frac{t}{k}). \quad (3.5) \end{aligned}$$

Similarly, for all $t > 0$ and $n \geq 0$, we can prove that

$$F_{x_{2n},x_{2n+1}}(t) > F_{x_{2n-1},x_{2n}}(\frac{t}{k}). \quad (3.6)$$

Combining (3.5) and (3.6), for all $n \geq 1$, $t > 0$, we get

$$F_{x_n,x_{n+1}}(t) > F_{x_{n-1},x_n}(\frac{t}{k}). \quad (3.7)$$

By repeated applications of this inequality, for all $t > 0$, $n \geq 0$, we obtain

$$F_{x_n,x_{n+1}}(t) > F_{x_0,x_1}(\frac{t}{k^n}). \quad (3.8)$$

Taking limit as $n \to \infty$ on both sides, for all $t > 0$, we have

$$\lim_{n\to\infty} F_{x_{n+1},x_n}(t) = 1. \quad (3.9)$$

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Again, by repeated applications of (3.7), it follows that for all $t > 0$, $n \geq 0$ and each $i \geq 1$,

$$F_{x_{n+i},x_{n+i+1}}(t) > F_{x_n,x_{n+1}}(\frac{t}{k^i}). \tag{3.10}$$

We next prove that $\{x_n\}$ is a Cauchy sequence (see Definition 2.6), that is, we prove that for arbitrary $\epsilon > 0$ and $0 < \lambda < 1$, there exists $N(\epsilon, \lambda)$ such that

$$F_{x_n,x_m}(\epsilon) \geq 1 - \lambda \text{ for all } n, m \geq N(\epsilon, \lambda).$$

Without loss of generality we can assume that $m > n$.

Now,

$$\epsilon = \epsilon\frac{1-k}{1-k} > \epsilon(1-k)(1 + k + k^2 + ... + k^{m-n-1}).$$

Then, by the monotone increasing property of F, we have

$$F_{x_n,x_m}(\epsilon) \geq F_{x_n,x_m}(\epsilon(1-k)(1 + k + k^2 + ... + k^{m-n-1})),$$

that is,

$$F_{x_n,x_m}(\epsilon) \geq \Delta(F_{x_n,x_{n+1}}(\epsilon(1-k)), \Delta(F_{x_{n+1},x_{n+2}}(\epsilon k(1-k)), \Delta(..., \\ \Delta(F_{x_{m-2},x_{m-1}}(\epsilon k^{m-n-2}(1-k)), F_{x_{m-1},x_m}(\epsilon k^{m-n-1}(1-k)))...))). \tag{3.11}$$

Putting $t = (1-k)\epsilon k^i$ in (3.10), we get

$$F_{x_{n+i},x_{n+i+1}}((1-k)\epsilon k^i) > F_{x_n,x_{n+1}}((1-k)\epsilon).$$

Then, by (3.11), we have

$$F_{x_n,x_m}(\epsilon) \geq \Delta(F_{x_n,x_{n+1}}(\epsilon(1-k)), \Delta(F_{x_n,x_{n+1}}(\epsilon(1-k)), \Delta(..., \\ \Delta(F_{x_n,x_{n+1}}(\epsilon(1-k)), F_{x_n,x_{n+1}}(\epsilon(1-k)))......))),$$

that is,

$$F_{x_n,x_m}(\epsilon) \geq \Delta^{(m-n)} F_{x_n,x_{n+1}}(\epsilon(1-k)). \tag{3.12}$$

Since the t-norm Δ is a Hadzic type t-norm, the family $\{\Delta^p\}$ of its iterates is equicontinuous at the point $s = 1$, that is, there exists $\eta(\lambda) \in (0, 1)$ such that for all $m > n$,

$$\Delta^{(m-n)}(s) \geq 1 - \lambda \text{ whenever } \eta(\lambda) < s \leq 1. \tag{3.12}$$

Since, $F_{x_0,x_1}(t) \to 1$ as $t \to \infty$ and $0 < k < 1$, there exists an positive integer $N(\epsilon, \lambda)$ such that

$$F_{x_0,x_1}(\frac{(1-k)\epsilon}{k^n}) > \eta(\lambda) \text{ for all } n \geq N(\epsilon, \lambda). \tag{3.14}$$

From (3.14) and (3.10), with $n = 0$, $i = n$ and $t = (1-k)\epsilon$, we get

$$F_{x_n,x_{n+1}}(\epsilon(1-k)) \geq F_{x_0,x_1}(\frac{(1-k)\epsilon}{k^n}) > \eta(\lambda),$$

for all $n \geq N(\epsilon, \lambda)$.

Then, from (3.13) with $s = F_{x_n,x_{n+1}}(\epsilon(1-k))$, we have

$$\Delta^{(m-n)}(F_{x_n,x_{n+1}}(\epsilon(1-k))) \geq 1 - \lambda.$$

It then follows from (3.12) that

$$F_{x_n,x_m}(\epsilon) \geq 1 - \lambda \text{ for all } m, n \geq N(\epsilon, \lambda).$$

Thus $\{x_n\}$ is a Cauchy sequence.

Since X is complete, there is some $z \in X$ such that $\lim_{n\to\infty} x_n = z$.

Then,

$$\lim_{n\to\infty} Sx_{2n} = \lim_{n\to\infty} x_{2n+1} = z \text{ and } \lim_{n\to\infty} Tx_{2n+1} = \lim_{n\to\infty} x_{2n+2} = z. \tag{3.15}$$

Now, we prove that $Tz = z$.

Putting $x = x_{2n}$, $y = z$ in the inequality (3.1), for all $t > 0$, we have

$$\begin{aligned} F_{Sx_{2n},Tz}(t) &+ q(1 - \max\{F_{x_{2n},Tz}(t), F_{z,Sx_{2n}}(t)\}) \\ &> \psi(F_{x_{2n},z}(\frac{t}{k}), F_{x_{2n},Sx_{2n}}(\frac{t}{k}), F_{z,Tz}(\frac{t}{k})). \end{aligned} \tag{3.16}$$

Taking limit as $n \to \infty$ in (3.16) for all $t > 0$, we have

$$\begin{aligned} F_{z,Tz}(t) &+ q(1 - \max\{F_{z,Tz}(t), F_{z,z}(t)\}) \\ &\geq \psi(F_{z,z}(\tfrac{t}{k}), F_{z,z}(\tfrac{t}{k}), F_{z,Tz}(\tfrac{t}{k})), \end{aligned} \tag{3.17}$$

(since by our assumption $x_n \to x, y_n \to y$ implies $F_{x_n,y_n} \to F_{x,y}$)

that is,

$$F_{z,Tz}(t) \geq \psi(1, 1, F_{z,Tz}(\tfrac{t}{k})) \geq F_{z,Tz}(\tfrac{t}{k}). \tag{3.18}$$

(by the properties of ψ)

By repeated applications of (3.18), for all $t > 0$, we obtain

$$F_{z,Tz}(t) \geq F_{z,Tz}(\tfrac{t}{k^n}).$$

Taking limit as $n \to \infty$ on both sides, for all $t > 0$,

$$F_{z,Tz}(t) \geq \lim_{n\to\infty} F_{z,Tz}(\frac{t}{k^n}) = 1,$$

which implies

$$F_{z,Tz}(t) = 1.$$

Thus $z = Tz$.

Similarly we can prove that $Sz = z$.

Now, we prove the uniqueness of the fixed point for the case where $q = 0$. Let z and

w be two distinct common fixed points of S and T. Then, we have $0 < F_{z,w}(t) < 1$ for some $t > 0$.
Then, by the inequality (3.1), for $t > 0$, we get

$$F_{Sz,Tw}(t) > \psi(F_{z,w}(\tfrac{t}{k}), F_{z,Sz}(\tfrac{t}{k}), F_{w,Tw}(\tfrac{t}{k})),$$

that is,

$$\begin{aligned} F_{z,w}(t) &> \psi(F_{z,w}(\tfrac{t}{k}), F_{z,z}(\tfrac{t}{k}), F_{w,w}(\tfrac{t}{k})) \\ &= \psi(F_{z,w}(\tfrac{t}{k}), 1, 1) \\ &\geq F_{z,w}(\tfrac{t}{k}) \text{ (by a property of } \psi) \\ &\geq F_{z,w}(t), \text{ which is a contradiction.} \end{aligned}$$

Hence $z = w$.
Now we give the following example in support of Theorem 3.1.

Example 3.1. Let $X = \{x_1, x_2, x_3, x_4\}$, where the t-norm $\Delta(a, b) = min(a, b)$ and $F_{x,y}(t)$ is defined as:
$F_{x_1,x_2}(t) = F_{x_1,x_3}(t) = F_{x_1,x_4}(t) =$

$$F_{x_2,x_4}(t) = F_{x_3,x_4}(t) = \begin{cases} 0, & \text{if } t \leq 0, \\ 0.40, & \text{if } 0 < t \leq 7, \\ 1, & \text{if } t > 7, \end{cases}$$

$$F_{x_2,x_3}(t) = \begin{cases} 0, & \text{if } t \leq 0, \\ 0.80, & \text{if } 0 < t \leq 4, \\ 1, & \text{if } t > 4. \end{cases}$$

Then (X, F, Δ) be a complete Menger space. If we define the mappings S and T as follows: $Sx_1 = x_3, Sx_2 = x_2, Sx_3 = x_2, Sx_4 = x_2$ and $Tx_1 = x_2, Tx_2 = x_2, Tx_3 = x_2, Tx_4 = x_3$ then it satisfies all the conditions required in the Theorem 3.1 where $q = 0$, $k = 0.75$ and $\psi(x, y, z) = \min\{x, y, z\}$. Here x_2 is the unique common fixed point of S and T.

In our next theorem we use the control function 'ϕ' (Definition 2.8) in the inequality (3.1) with $q = 0$. Then the result of the previous theorem can not be established by following the same argument. Instead, we have a fixed point result in this case where the t-norm is the minimum t-norm.

Theorem 3.2 *Let (X, F, Δ) be a complete Menger space with $\Delta(a, b) = \min\{a, b\}$. Let $S, T : X \to X$ be two self mappings on X which satisfy the following inequality:*

$$F_{Sx,Ty}(\phi(t)) > \psi(F_{x,y}(\phi(\tfrac{t}{c})), F_{x,Sx}(\phi(\tfrac{t}{c})), F_{y,Ty}(\phi(\tfrac{t}{c}))) \quad (3.19)$$

for all $x, y \in X$, $t > 0$ where $0 < c < 1$, ϕ is a Φ-function and ψ is a Ψ-function. Then S and T have a unique common fixed point in X.

Proof: Let $x_0 \in X$ be arbitrary. We define a sequence $\{x_n\}_{n=0}^{\infty}$ in X as follows:

$$x_{2n+1} = Sx_{2n},\ x_{2n+2} = Tx_{2n+1} \text{ for all } n \geq 0. \quad (3.20)$$

Putting $x = x_{2n}, y = x_{2n+1}$ in (3.19), for all $t > 0$ and $n \geq 0$, we have
$F_{Sx_{2n},Tx_{2n+1}}(\phi(t)) > \psi(F_{x_{2n},x_{2n+1}}(\phi(\frac{t}{c})), F_{x_{2n},Sx_{2n}}(\phi(\frac{t}{c})), F_{x_{2n+1},Tx_{2n+1}}(\phi(\frac{t}{c}))),$
that is,

$$F_{x_{2n+1},x_{2n+2}}(\phi(t)) > \psi(F_{x_{2n},x_{2n+1}}(\phi(\tfrac{t}{c})), F_{x_{2n},x_{2n+1}}(\phi(\tfrac{t}{c})), F_{x_{2n+1},x_{2n+2}}(\phi(\tfrac{t}{c}))). \quad (3.21)$$

We now claim that for all $t > 0$, $n \geq 0$,

$$F_{x_{2n+1},x_{2n+2}}(\phi(\tfrac{t}{c})) \geq F_{x_{2n},x_{2n+1}}(\phi(\tfrac{t}{c})). \quad (3.22)$$

If possible, let for some $s > 0$ and some $n \geq 0$,

$$F_{x_{2n+1},x_{2n+2}}(\phi(\tfrac{s}{c})) < F_{x_{2n},x_{2n+1}}(\phi(\tfrac{s}{c})).$$

Then, from (3.21), using the properties of ψ, we have for $s > 0$ and $n \geq 0$,

$$\begin{aligned} F_{x_{2n+1},x_{2n+2}}(\phi(s)) &> \psi(F_{x_{2n},x_{2n+1}}(\phi(\tfrac{s}{c})), F_{x_{2n},x_{2n+1}}(\phi(\tfrac{s}{c})), F_{x_{2n+1},x_{2n+2}}(\phi(\tfrac{s}{c}))) \\ &\geq \psi(F_{x_{2n+1},x_{2n+2}}(\phi(\tfrac{s}{c})), F_{x_{2n+1},x_{2n+2}}(\phi(\tfrac{s}{c})), F_{x_{2n+1},x_{2n+2}}(\phi(\tfrac{s}{c}))) \\ &\geq F_{x_{2n+1},x_{2n+2}}(\phi(\tfrac{s}{c})) \\ &\geq F_{x_{2n+1},x_{2n+2}}(\phi(s)), \end{aligned}$$

which is a contradiction.

Therefore (3.22) holds for all $t > 0$ and $n \geq 0$.

Using (3.22) in (3.21), and by the properties of ψ, for all $t > 0$, $n \geq 0$, we have

$$\begin{aligned} F_{x_{2n+1},x_{2n+2}}(\phi(t)) &> \psi(F_{x_{2n},x_{2n+1}}(\phi(\tfrac{t}{c})), F_{x_{2n},x_{2n+1}}(\phi(\tfrac{t}{c})), F_{x_{2n+1},x_{2n+2}}(\phi(\tfrac{t}{c}))) \\ &\geq \psi(F_{x_{2n},x_{2n+1}}(\phi(\tfrac{t}{c})), F_{x_{2n},x_{2n+1}}(\phi(\tfrac{t}{c})), F_{x_{2n},x_{2n+1}}(\phi(\tfrac{t}{c}))) \\ &\geq F_{x_{2n},x_{2n+1}}(\phi(\tfrac{t}{c})). \end{aligned} \quad (3.23)$$

Similarly, for all $t > 0$ and $n > 0$, we can prove that

$$F_{x_{2n},x_{2n+1}}(\phi(t)) > F_{x_{2n-1},x_{2n}}(\phi(\tfrac{t}{c})). \quad (3.24)$$

Combining (3.23) and (3.24), for all $n \geq 1$ and $t > 0$, we get

$$F_{x_n,x_{n+1}}(\phi(t)) > F_{x_{n-1},x_n}(\phi(\tfrac{t}{c})).$$

By repeated applications of this inequality, for all $t > 0$, $n \geq 0$, we have

$$F_{x_n,x_{n+1}}(\phi(t)) > F_{x_0,x_1}(\phi(\tfrac{t}{c^n})). \quad (3.25)$$

Taking limit as $n \to \infty$ on both sides of (3.25), for all $t > 0$, we obtain

$$\lim_{n\to\infty} F_{x_{n+1},x_n}(\phi(t)) = 1. \quad (3.26)$$

Again, by virtue of a property of ϕ, given $s > 0$ we can find $t > 0$ such that $s > \phi(t)$. Thus the above limit implies that for all $s > 0$,

$$\lim_{n\to\infty} F_{x_n,x_{n+1}}(s) = 1. \quad (3.27)$$

We next prove that $\{x_n\}$ is a Cauchy sequence. If possible, let $\{x_n\}$ be not a Cauchy sequence. Then there exist $\epsilon > 0$ and $0 < \lambda < 1$ for which we can find subsequences $\{x_{m(k)}\}$ and $\{x_{n(k)}\}$ of $\{x_n\}$ with $n(k) > m(k) > k$ such that

$$F_{x_{m(k)},x_{n(k)}}(\epsilon) < 1 - \lambda. \quad (3.28)$$

We take $n(k)$ corresponding to $m(k)$ to be the smallest integer satisfying (3.28) so that

$$F_{x_{m(k)},x_{n(k)-1}}(\epsilon) \geq 1 - \lambda. \quad (3.29)$$

If $\epsilon_1 < \epsilon$, then we have

$$F_{x_{m(k)},x_{n(k)}}(\epsilon_1) \leq F_{x_{m(k)},x_{n(k)}}(\epsilon).$$

We conclude that it is possible to construct $\{x_{m(k)}\}$ and $\{x_{n(k)}\}$ with $n(k) > m(k) > k$ and satisfying (3.28) and (3.29) whenever ϵ is replaced by a smaller positive value. As ϕ is continuous at 0 and strictly monotone increasing with $\phi(0) = 0$, it is possible to obtain $\epsilon_2 > 0$ such that $\phi(\epsilon_2) < \epsilon$.

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Then, by the above argument, it is possible to obtain an increasing sequence of integers $\{m(k)\}$ and $\{n(k)\}$ with $n(k) > m(k) > k$ such that

$$F_{x_{m(k)},x_{n(k)}}(\phi(\epsilon_2)) < 1-\lambda \tag{3.30}$$

and

$$F_{x_{m(k)},x_{n(k)-1}}(\phi(\epsilon_2)) \geq 1-\lambda. \tag{3.31}$$

Now, we get the following possible cases:

Case-I: $m(k)$ is odd and $n(k)$ is even for an infinite number of values of k. Then there exist $\{m(l)\} \subset \{m(k)\}$ and $\{n(l)\} \subset \{n(k)\}$ with $n(l) > m(l) > l$ such that

$$F_{x_{m(l)},x_{n(l)}}(\phi(\epsilon_2)) < 1-\lambda \tag{3.32}$$

and

$$F_{x_{m(l)},x_{n(l)-1}}(\phi(\epsilon_2)) \geq 1-\lambda. \tag{3.33}$$

Then,

$$x_{m(l)} = Sx_{m(l)-1} \text{ and } x_{n(l)} = Tx_{n(l)-1}.$$

By (3.32), we have

$$\begin{aligned} 1-\lambda &> F_{x_{m(l)},x_{n(l)}}(\phi(\epsilon_2)) \\ &= F_{Sx_{m(l)-1},Tx_{n(l)-1}}(\phi(\epsilon_2)) \\ &> \psi(F_{x_{m(l)-1},x_{n(l)-1}}(\phi(\tfrac{\epsilon_2}{c})), F_{x_{m(l)-1},Sx_{m(l)-1}}(\phi(\tfrac{\epsilon_2}{c})), F_{x_{n(l)-1},Tx_{n(l)-1}}(\phi(\tfrac{\epsilon_2}{c}))), \end{aligned}$$

that is,

$$1-\lambda > \psi(F_{x_{m(l)-1},x_{n(l)-1}}(\phi(\tfrac{\epsilon_2}{c})), F_{x_{m(l)-1},x_{m(l)}}(\phi(\tfrac{\epsilon_2}{c})), F_{x_{n(l)-1},x_{n(l)}}(\phi(\tfrac{\epsilon_2}{c}))). \tag{3.34}$$

Since ϕ is strictly increasing and $0 < c < 1$, we can choose $\eta > 0$ such that $\phi(\frac{\epsilon_2}{c}) = \phi(\epsilon_2) + \eta$.

Therefore,

$$F_{x_{m(l)-1},x_{n(l)-1}}(\phi(\tfrac{\epsilon_2}{c})) \geq \Delta(F_{x_{m(l)-1},x_{m(l)}}(\eta), F_{x_{m(l)},x_{n(l)-1}}(\phi(\epsilon_2))). \tag{3.35}$$

Again, by (3.27) we have for sufficiently large l and by the property of ϕ,

$$F_{x_{m(l)-1},x_{m(l)}}(\eta) \geq 1-\lambda. \tag{3.36}$$

Using (3.33) and (3.36) in (3.35) we have

$$\begin{aligned} F_{x_{m(l)-1},x_{n(l)-1}}(\phi(\tfrac{\epsilon_2}{c})) &\geq \Delta(F_{x_{m(l)-1},x_{m(l)}}(\eta), F_{x_{m(l)},x_{n(l)-1}}(\phi(\epsilon_2))) \\ &\geq \Delta(1-\lambda, 1-\lambda) \\ &= 1-\lambda. \end{aligned} \tag{3.37}$$

Again, by (3.27) we have for sufficiently large l,

$$F_{x_{m(l)-1},x_{m(l)}}(\phi(\tfrac{\epsilon_2}{c})) \geq 1-\lambda \tag{3.38}$$

and

$$F_{x_{n(l)-1},x_{n(l)}}(\phi(\tfrac{\epsilon_2}{c})) \geq 1-\lambda. \tag{3.39}$$

Using (3.37), (3.38) and (3.39) in (3.34) for $\epsilon_2 > 0$, $0 < c < 1$ and by the property of ψ, we have

$$\begin{aligned} 1-\lambda &> \psi(F_{x_{m(l)-1},x_{n(l)-1}}(\phi(\tfrac{\epsilon_2}{c})), F_{x_{m(l)-1},x_{m(l)}}(\phi(\tfrac{\epsilon_2}{c})), F_{x_{n(l)-1},x_{n(l)}}(\phi(\tfrac{\epsilon_2}{c}))) \\ &\geq \psi(1-\lambda, 1-\lambda, 1-\lambda) \geq 1-\lambda. \end{aligned}$$

Which is a contradiction.

Case-II: $m(k)$ is even and $n(k)$ is odd for an infinite number of values of k. Then there exist $\{m(l)\} \subset \{m(k)\}$ and $\{n(l)\} \subset \{n(k)\}$ such that (3.32) and (3.33) hold. This case is similar to Case-I and we can get a contradiction.

Case-III: $m(k)$ and $n(k)$ both are even for an infinite number of values of k. Then there exist $\{m(l)\} \subset \{m(k)\}$ and $\{n(l)\} \subset \{n(k)\}$ with $n(l) > m(l) > l$ such that (3.32) and (3.33) hold.

As $0 < c < 1$ we can choose $\epsilon_3 < \epsilon_2$ such that $\frac{\epsilon_3}{c} \geq \epsilon_2$. Therefore by the property of ϕ we can take $\phi(\frac{\epsilon_3}{c}) \geq \phi(\epsilon_2)$.

Now, by (3.32), we have

$$\begin{aligned} 1-\lambda &> F_{x_{m(l)},x_{n(l)}}(\phi(\epsilon_2)) \\ &\geq \Delta(F_{x_{m(l)},x_{m(l)+1}}(\phi(\epsilon_2)-\phi(\epsilon_3)), F_{x_{m(l)+1},x_{n(l)}}(\phi(\epsilon_3))). \end{aligned} \tag{3.40}$$

Now by the inequality (3.19), we have

$$\begin{aligned} F_{x_{m(l)+1},x_{n(l)}}(\phi(\epsilon_3)) &> \psi(F_{x_{m(l)},x_{n(l)-1}}(\phi(\tfrac{\epsilon_3}{c})), F_{x_{m(l)},x_{m(l)+1}}(\phi(\tfrac{\epsilon_3}{c})), \\ &\qquad F_{x_{n(l)-1},x_{n(l)}}(\phi(\tfrac{\epsilon_3}{c}))) \\ &\geq \psi(F_{x_{m(l)},x_{n(l)-1}}(\phi(\epsilon_2)), F_{x_{m(l)},x_{m(l)+1}}(\phi(\tfrac{\epsilon_3}{c})), \\ &\qquad F_{x_{n(l)-1},x_{n(l)}}(\phi(\tfrac{\epsilon_3}{c}))). \end{aligned} \tag{3.41}$$

By (3.27) we have for sufficiently large l,

$$F_{x_{m(l)},x_{m(l)+1}}(\phi(\tfrac{\epsilon_3}{c})) \geq 1-\lambda, \tag{3.42}$$

$$F_{x_{n(l)-1},x_{n(l)}}(\phi(\tfrac{\epsilon_3}{c})) \geq 1-\lambda \tag{3.43}$$

and

$$F_{x_{m(l)},x_{m(l)+1}}(\phi(\epsilon_2)-\phi(\epsilon_3)) \geq 1-\lambda. \tag{3.44}$$

Using (3.33), (3.42), (3.43) in (3.41) we have

$$\begin{aligned} F_{x_{m(l)+1},x_{n(l)}}(\phi(\epsilon_3)) &> \psi(F_{x_{m(l)},x_{n(l)-1}}(\phi(\epsilon_2)), F_{x_{m(l)},x_{m(l)+1}}(\phi(\tfrac{\epsilon_3}{c})), \\ &\qquad F_{x_{n(l)-1},x_{n(l)}}(\phi(\tfrac{\epsilon_3}{c}))) \\ &\geq \psi(1-\lambda, 1-\lambda, 1-\lambda) \geq 1-\lambda. \end{aligned} \tag{3.45}$$

Now, using (3.44) and (3.45) in (3.40) we have

$$\begin{aligned} 1-\lambda &> F_{x_{m(l)},x_{n(l)}}(\phi(\epsilon_2)) \\ &\geq \Delta(1-\lambda, 1-\lambda) = 1-\lambda. \end{aligned}$$

Which is a contradiction.

Case-IV: $m(k)$ and $n(k)$ both are odd for an infinite number of values of k. Then there exist $\{m(l)\} \subset \{m(k)\}$ and $\{n(l)\} \subset \{n(k)\}$ such that (3.32) and (3.33) hold.

This case is similar to Case-III and we can get a contradiction.

Combining all the above four cases we conclude that $\{x_n\}$ is a Cauchy sequence.

Since X is complete, there is some $z \in X$ such that $\lim_{n\to\infty} x_n = z$.

Then,

$$\lim_{n\to\infty} Sx_{2n} = \lim_{n\to\infty} x_{2n+1} = z \text{ and } \lim_{n\to\infty} Tx_{2n+1} = \lim_{n\to\infty} x_{2n+2} = z. \tag{3.46}$$

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Now we claim that $Tz = z$.
Let us choose c_0 such that $0 < c < c_0 < 1$. (3.47)
Now, for all $t > 0$, we have
$$F_{z,Tz}(\phi(t)) \geq \Delta(F_{z,Sx_{2n}}(\phi(t) - \phi(c_0 t)), F_{Sx_{2n},Tz}(\phi(c_0 t))). \tag{3.48}$$

As Δ is continuous, taking liminf as $n \to \infty$ on both sides of the above inequality, for all $t > 0$, we have
$$\begin{aligned} F_{z,Tz}(\phi(t)) &\geq \Delta(\liminf_{n\to\infty} F_{z,Sx_{2n}}(\phi(t) - \phi(c_0 t)), \liminf_{n\to\infty} F_{Sx_{2n},Tz}(\phi(c_0 t))) \\ &= \Delta(1, \liminf_{n\to\infty} F_{Sx_{2n},Tz}(\phi(c_0 t))). \quad (3.49) \\ &\text{(by (3.46))} \end{aligned}$$
Now, for all $t > 0$ and $n \geq 0$, from (3.19), we have
$$F_{Sx_{2n},Tz}(\phi(c_0 t)) > \psi(F_{x_{2n},z}(\phi(\tfrac{c_0 t}{c})), F_{x_{2n},Sx_{2n}}(\phi(\tfrac{c_0 t}{c})), F_{z,Tz}(\phi(\tfrac{c_0 t}{c}))). \tag{3.50}$$
Taking liminf as $n \to \infty$ on both sides of (3.50), we have for all $t > 0$,
$$\begin{aligned} \liminf_{n\to\infty} F_{Sx_{2n},Tz}(\phi(c_0 t)) \geq \psi(&\liminf_{n\to\infty} F_{x_{2n},z}(\phi(\frac{c_0 t}{c})), \liminf_{n\to\infty} F_{x_{2n},Sx_{2n}}(\phi(\frac{c_0 t}{c})), \\ &F_{z,Tz}(\phi(\frac{c_0 t}{c}))), \end{aligned}$$
that is,
$$\liminf_{n\to\infty} F_{Sx_{2n},Tz}(\phi(c_0 t)) \geq \psi(1, 1, F_{z,Tz}(\phi(\frac{c_0 t}{c}))),$$
(by (3.46))
that is,
$$\liminf_{n\to\infty} F_{Sx_{2n},Tz}(\phi(c_0 t)) \geq F_{z,Tz}(\phi(\frac{t}{(\frac{c}{c_0})})). \tag{3.51}$$
(since ψ is monotone increasing)
We now take $\frac{c}{c_0} = p$. Then by (3.47), $0 < p < 1$. Hence we get from (3.51) that for all $t > 0$,
$$\liminf_{n\to\infty} F_{Sx_{2n},Tz}(\phi(c_0 t)) \geq F_{z,Tz}(\phi(\frac{t}{p})). \tag{3.52}$$
Combining (3.49) and (3.52) for all $t > 0$, we get
$$F_{z,Tz}(\phi(t)) \geq F_{z,Tz}(\phi(\tfrac{t}{p})). \ (0 < p < 1)$$
By repeated applications of this inequality, for all $t > 0$ we obtain
$$F_{z,Tz}(\phi(t)) \geq F_{z,Tz}(\phi(\tfrac{t}{p^n})). \tag{3.53}$$
Taking limit as $n \to \infty$ on both sides of (3.53), for all $t > 0$, we get
$$F_{z,Tz}(\phi(t)) \geq \lim_{n\to\infty} F_{z,Tz}(\phi(\frac{t}{p^n})) = 1.$$
Therefore by a property of ϕ we get, $z = Tz$.
Similarly we can prove that $z = Sz$.
Thus z is a common fixed point of S and T.

Now we prove the uniqueness of the common fixed point. Let z and w be two distinct common fixed points of S and T. Then the properties of ϕ imply $0 < F_{z,w}(\phi(t)) < 1$ for some $t > 0$.

Then, by the inequality (3.19) for $t > 0$, we get

$$F_{Sz,Tw}(\phi(t)) > \psi(F_{z,w}(\phi(\tfrac{t}{c})), F_{z,Sz}(\phi(\tfrac{t}{c})), F_{w,Tw}(\phi(\tfrac{t}{c}))),$$

that is,

$$\begin{aligned} F_{z,w}(\phi(t)) &> \psi(F_{z,w}(\phi(\tfrac{t}{c})), F_{z,z}(\phi(\tfrac{t}{c})), F_{w,w}(\phi(\tfrac{t}{c}))) \\ &= \psi(F_{z,w}(\phi(\tfrac{t}{c})), 1, 1) \\ &\geq F_{z,w}(\phi(\tfrac{t}{c})) \\ &\geq F_{z,w}(\phi(t)), \text{ which is a contradiction.} \end{aligned}$$

Hence $z = w$.

Remark: In Theorem 3.1 the '$>$' sign in equality (3.1) can be replaced by '$\geq$' provided that in the definition of ψ are taken $\psi(t,t,t) > t$ for all $0 < t < 1$. The structure of the proof remains unaltered. The same is the case with Theorem 3.2.

Taking $S = T$ in the Theorem 3.2 we get the following Corollary.

Corollary 3.1. *Let (X, F, Δ) be a complete Menger space with $\Delta(a,b) = \min\{a,b\}$. Let $T : X \to X$ be a self mapping on X. Assume that T satisfies the following inequality:*

$$F_{Tx,Ty}(\phi(t)) > \psi(F_{x,y}(\phi(\tfrac{t}{c})), F_{x,Tx}(\phi(\tfrac{t}{c})), F_{y,Ty}(\phi(\tfrac{t}{c})))$$

for all $x, y \in X$, $t > 0$, where $0 < c < 1$, ϕ is a Φ-function and ψ is a Ψ-function. Then T has a unique fixed point in X.

Example 3.2. Let $X = [0,1]$, for all $t > 0$, $F_{x,y}(t) = e^{-\frac{|x-y|}{t}}$ where $x, y \in X$ and $\Delta(a,b) = \min\{a,b\}$. Then (X, F, Δ) is a complete Menger space. Let $S, T : X \to X$ be defined as follows:

$$Sx = Tx = \begin{cases} \frac{x}{4} & \text{if } x \in [0, \frac{1}{2}), \\ \frac{x}{5} & \text{if } x \in [\frac{1}{2}, 1], \end{cases}$$

for all $x, y \in X$ and $\psi(x,y,z) = \min\{x,y,z\}$. Let $\phi(t) = \sqrt{t}$, $c = \frac{4}{9}$.

We show that the inequality (3.19) is satisfied with the quantities described above. We have following three cases.

Case-I $x, y \in [0, \frac{1}{2})$. Without loss of generality we can assume that $x \geq y$. Here $Tx = \frac{x}{4}$, $Ty = \frac{y}{4}$.

We have to prove

$$e^{-\frac{|x-y|}{4\sqrt{t}}} > \min\{e^{-\frac{|x-y|\sqrt{c}}{\sqrt{t}}}, e^{-\frac{3x\sqrt{c}}{4\sqrt{t}}}, e^{-\frac{3y\sqrt{c}}{4\sqrt{t}}}\}.$$

Since $x > y$, $c = \frac{4}{9}$, we have $\frac{|x-y|}{4} < \frac{3x\sqrt{c}}{4}$.

Hence $e^{-\frac{|x-y|}{4\sqrt{t}}} > e^{-\frac{3x\sqrt{c}}{4\sqrt{t}}} > \min\{e^{-\frac{|x-y|\sqrt{c}}{\sqrt{t}}}, e^{-\frac{3x\sqrt{c}}{4\sqrt{t}}}, e^{-\frac{3y\sqrt{c}}{4\sqrt{t}}}\}$.

Case-II $x \in [0, \frac{1}{2})$, $y \in [\frac{1}{2}, 1]$. Here $y > x$, $Tx = \frac{x}{4}$, $Ty = \frac{y}{5}$.

We have to prove

$$e^{-\frac{|5x-4y|}{20\sqrt{t}}} > \min\{e^{-\frac{|x-y|\sqrt{c}}{\sqrt{t}}}, e^{-\frac{3x\sqrt{c}}{4\sqrt{t}}}, e^{-\frac{4y\sqrt{c}}{5\sqrt{t}}}\}.$$

Since $y > x$, $c = \frac{4}{9}$, we have, $|\frac{5x-4y}{20}| < \frac{4y\sqrt{c}}{5}$.

Hence,

$$e^{-\frac{|5x-4y|}{20\sqrt{t}}} > e^{-\frac{4y\sqrt{c}}{5\sqrt{t}}} > \min\{e^{-\frac{|x-y|\sqrt{c}}{\sqrt{t}}}, e^{-\frac{3x\sqrt{c}}{4\sqrt{t}}}, e^{-\frac{4y\sqrt{c}}{5\sqrt{t}}}\}.$$

Case-III $x, y \in [\frac{1}{2}, 1]$. Without loss of generality we can assume that $x \geq y$. Here $Tx = \frac{x}{5}$, $Ty = \frac{y}{5}$.
We have to prove

$$e^{-\frac{|x-y|}{5\sqrt{t}}} > \min\{e^{-\frac{|x-y|\sqrt{c}}{\sqrt{t}}}, e^{-\frac{4x\sqrt{c}}{5\sqrt{t}}}, e^{-\frac{4y\sqrt{c}}{5\sqrt{t}}}\}.$$

Since $x > y$, $c = \frac{4}{9}$, we have, $\frac{|x-y|}{5} < \frac{4x\sqrt{c}}{5}$.
Hence,

$$e^{-\frac{|x-y|}{5\sqrt{t}}} > e^{-\frac{4x\sqrt{c}}{5\sqrt{t}}} > \min\{e^{-\frac{|x-y|\sqrt{c}}{\sqrt{t}}}, e^{-\frac{4x\sqrt{c}}{5\sqrt{t}}}, e^{-\frac{4y\sqrt{c}}{5\sqrt{t}}}\}.$$

Combining all the above three cases, we conclude that the mappings S and T satisfy the inequality (3.19). Thus all the conditions of Theorem 3.2 are satisfied. Hence by an application of Theorem 3.2, there is a unique fixed point of T. Here 0 is the unique common fixed point of S and T.
Remark: If we take $\phi(t) = t$ and $q = 0$ in the above example, then Theorem 3.1 is also satisfied. Further it is noted that the function in the above example is not continuous.

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