International Electronic Journal of Pure and Applied Mathematics
Volume 5 No. 3 2012, 105-110

SPECTRAL SOLUTION OF THE STOKES EQUATION PAST AN ELLIPSOID OF REVOLUTION

Yoshihiro Mochimaru
Department of International Development Engineering
Tokyo Institute of Technology
2-12-1, Ookayama, Meguro-Ku, Tokyo, 152−8550, JAPAN

Abstract: Steady-state Stokes flow past an ellipsoid of revolution placed parallel to a uniform flow is analyzed using a spectral scheme based on a boundary fitted conformal mapping to give at least the first two significant terms of Fourier series in some combined variables of vorticity and a stream function, which leads to estimation of drag force acting on the body.

AMS Subject Classification: 30C20, 65M70, 65N35, 76D07, 76M22
Key Words: spectral solution, conformal mapping, Stokes flow

1. Introduction

No steady-state solution exists for the Stokes equation to a two-dimensional field past a single body in an infinite extension of a uniform flow field, although a closed form of steady-state solution past a sphere (in a three-dimensional case) in an infinite extension exists. Happel and Brenner tried to find an analytical solution past an ellipsoid of revolution[1], although their estimated solution is limited only to the first term of the series expansion regarding the deviation from a shape of a sphere. The effects of the difference between shapes in [1] should come from at least the first and the second term in their expansion, so that accuracy of the said solution remains only within a known solution past a sphere (i.e. a constant value).

In the following steady-state Stokes flow past an ellipsoid of revolution placed parallel to a uniform flow is treated, and introduced is a boundary fitted conformal

Received: August 15, 2012

mapping system to give a series solution asymptotically valid as long as the ellipsoid of revolution is nearly a sphere.

2. Analysis

2.1. Formulation of the Problem

The steady-state Stokes equation for an incompressible Newtonian fluid is given by

$$\boldsymbol{O} = -\nabla p + \mu \Delta \boldsymbol{V}, \tag{1}$$

where p: hydrodynamic pressure, μ: viscosity (assumed to be constant), $\boldsymbol{V}$: velocity vector. Incompressibility gives

$$\nabla \cdot \boldsymbol{V} = 0. \tag{2}$$

Since p is a scalar quantity, the necessary and sufficient condition to equation (1) becomes

$$\boldsymbol{O} = \Delta \left(\nabla \times \boldsymbol{V} \right). \tag{3}$$

Under the condition equation (2), $\boldsymbol{V}$ is expressible only in terms of curl-components. In an axisymmetric flow field, equation (3) becomes

$$\left\{ \frac{\partial}{\partial r} \left(\frac{\partial}{\partial r} + \frac{1}{r} \right) + \frac{\partial^2}{\partial z^2} \right\} \left(\frac{\partial U}{\partial z} - \frac{\partial w}{\partial r} \right) = 0, \tag{4}$$

where z is a coordinate parallel to the uniform flow direction (the positive direction being downstream), r is a coordinate normal to the z-axis, the origin being located at the center of the body of an ellipsoid; U: velocity component normal to the z-axis, w: velocity component parallel to the z-axis. Thus

$$w = (1/r)(\partial \psi / \partial r), \tag{5}$$

$$U = -\frac{1}{r} \frac{\partial \psi}{\partial z}, \tag{6}$$

where ψ: axisymmetric stream function. Since vorticity $\nabla \times \boldsymbol{V}$ is normal to the rz-plane, a non-zero component of vorticity, ζ, is defined as

$$\zeta \equiv \frac{\partial U}{\partial z} - \frac{\partial w}{\partial r}. \tag{7}$$

Thus Eqs.(4) and (7) become

$$L\zeta = 0, \tag{8}$$

$$\zeta + L\left(\psi / r \right) = 0, \tag{9}$$

$$L \equiv \frac{\partial}{\partial r}\left(\frac{\partial}{\partial r}+\frac{1}{r}\right)+\frac{\partial^2}{\partial z^2} \tag{10}$$

respectively. Boundary conditions at the surface of the ellipsoid of revolution are

$$U = w = 0, \text{(no slip flow condition.)} \tag{11}$$

Far away ($\sqrt{r^2+z^2} \to +\infty$) conditions are

$$w = 1, \text{(normalized with respect to the uniform flow speed, } U_\infty.) \tag{12}$$

$$U = 0. \tag{13}$$

2.2 Boundary Fitted Conformal Mapping System

Conformal mapping is assumed to be given by

$$z + ir = f(\alpha + i\beta), \alpha, \beta : \text{real}, \tag{14}$$

where f : an analytic function. Coordinates are assumed to be made dimensionless with respect to the semi-axis parallel to the z-axis, and the surface is assumed to be given by $\alpha = \alpha_0 (> 0)$ and the flow region by $\alpha > \alpha_0, 0 \le \beta \le \pi$. For a prolate spheroid

$$f = \cosh(\alpha + i\beta)/\cosh\alpha_0. \tag{15}$$

For an oblate spheroid

$$f = \sinh(\alpha + i\beta)/\sinh\alpha_0. \tag{16}$$

Thus Eqs.(8) and (9) become

$$\left[\left(\frac{\partial^2}{\partial\alpha^2}+\frac{\partial^2}{\partial\beta^2}\right)+3\left(\frac{1}{r}\frac{\partial r}{\partial\alpha}\frac{\partial}{\partial\alpha}+\frac{1}{r}\frac{\partial r}{\partial\beta}\frac{\partial}{\partial\beta}\right)\right]\left(\frac{\zeta}{r}\right)=0, \tag{17}$$

$$J\frac{\zeta}{r}+\left[\left(\frac{\partial^2}{\partial\alpha^2}+\frac{\partial^2}{\partial\beta^2}\right)+3\left(\frac{1}{r}\frac{\partial r}{\partial\alpha}\frac{\partial}{\partial\alpha}+\frac{1}{r}\frac{\partial r}{\partial\beta}\frac{\partial}{\partial\beta}\right)\right]\left(\frac{\psi}{r^2}\right)=0,$$
$$J \equiv |f'|^2 \tag{18}$$

respectively. Then equation (11) becomes

$$\left(\frac{\psi}{r^2}\right)_{\alpha=\alpha_0}=0, \quad \frac{\partial}{\partial\alpha}\left(\frac{\psi}{r^2}\right)_{\alpha=\alpha_0}=0. \tag{19}$$

Equations (12) and (13) become

$$\frac{\psi}{r^2}\to\frac{1}{2} \quad \text{as} \quad \alpha\to+\infty \quad \text{(necessary condition.)} \tag{20}$$

2.3 Spectral Solution

For a prolate spheroid

$$J = \{\cosh 2\alpha - \cos 2\beta\} / \{2\cosh^2 \alpha_0\} . \tag{21}$$

For an oblate spheroid

$$J = \{\cosh 2\alpha + \cos 2\beta\} / \{2\sinh^2 \alpha_0\} . \tag{22}$$

In these cases

$$J \equiv 2J_0 \cosh 2\alpha + J_2 \cos 2\beta. \tag{23}$$

Let

$$\frac{\zeta}{r} \equiv \sum_{n=0}^{\infty} A_n \left\{ \sum_{m=0}^{n} F_{nm}(\alpha) \cos 2(n-m)\beta \right\}, \tag{24}$$

$$\frac{\psi}{r^2} \equiv \frac{1}{2} + \sum_{n=0}^{\infty} B_n \left\{ \sum_{m=0}^{n} G_{nm} \cos 2(n-m)\beta \right\}, \tag{25}$$

where A_n's ($n \geq 0$) and B_n's ($n \geq 0$) are constants to be determined. Since $\frac{1}{r}\frac{\partial r}{\partial \beta} = \cot\beta$, from equation (17) up to the $\cos 2\beta$ term,

$$\frac{d^2}{d\alpha^2}F_{00} + \frac{3}{r}\frac{dr}{d\alpha}F_{00} = 0, F_{00}(+\infty) = 0 (\text{necessary condition,}) \tag{26}$$

$$\frac{d^2}{d\alpha^2}F_{10} + \frac{3}{r}\frac{dr}{d\alpha}\frac{d}{d\alpha}F_{10} - 10F_{10} = 0, \quad F_{10}(+\infty) = 0 \quad (\text{necessary condition,}) \tag{27}$$

$$\frac{d^2}{d\alpha^2}F_{11} + \frac{3}{r}\frac{dr}{d\alpha}\frac{d}{d\alpha}F_{11} = 6F_{10}, F_{11}(+\infty) = 0 \quad (\text{necessary condition.}) \tag{28}$$

For the current cases $\frac{1}{r}\frac{dr}{d\alpha} = \coth\alpha$ or $\tanh\alpha$ depending on the configuration (a prolate spheroid or an oblate spheroid), and so if $\alpha \geq \alpha_0$ (assuming a relatively large value of α_0), $\frac{1}{r}\frac{dr}{d\alpha} \sim 1$. Under these circumstances, in an asymptotic sense

$$F_{00} = \mathrm{e}^{-3\alpha}, \quad F_{10} = \mathrm{e}^{-5\alpha}, \quad F_{11} = \frac{3}{5}\mathrm{e}^{-5\alpha}. \tag{29}$$

Thus from equation (18) through the far away boundary condition, equation (20), we get (neglecting $\mathrm{e}^{-7\alpha}$ or smaller terms)

$$B_0 G_{00} = A_0 \frac{J_0}{2}\left(\mathrm{e}^{-\alpha} - \frac{1}{5}\mathrm{e}^{-5\alpha}\right) + A_1\left(\frac{J_0}{5}\alpha\mathrm{e}^{-3\alpha} - \frac{J_2}{20}\mathrm{e}^{-5\alpha}\right) + b_0\mathrm{e}^{-3\alpha}, \tag{30}$$

$$B_1G_{10} = A_0\frac{J_2}{10}\mathrm{e}^{-3\alpha} + A_1\left(\frac{J_0}{10}\mathrm{e}^{-3\alpha} + \frac{3J_2}{35}\alpha\mathrm{e}^{-5\alpha}\right) + b_1\mathrm{e}^{-5\alpha}, \tag{31}$$

$$\begin{aligned} B_1G_{11} &= -A_0\frac{J_2}{5}\alpha\mathrm{e}^{-3\alpha} + A_1\left\{-\frac{J_0}{5}\alpha\mathrm{e}^{-4\alpha}\right. \\ &\quad \left.+\frac{9J_2}{175}\left(\alpha+\frac{7}{10}\right)\mathrm{e}^{-5\alpha}\right\} + \frac{3}{5}b_1\mathrm{e}^{-5\alpha}, \end{aligned} \tag{32}$$

where b_0, b_1 are integral constants to be determined. Taking into the surface boundary conditions, equation (19), independent of β gives

$$A_0J_0 = \frac{-\frac{3}{2}\left(1\mp\frac{6}{7}\mathrm{e}^{-2\alpha_0}\right)}{\mathrm{e}^{-\alpha_0}\mp\frac{16}{35}\mathrm{e}^{-3\alpha_0}-\frac{223}{875}\mathrm{e}^{-5\alpha_0}\mp\frac{6}{35}\mathrm{e}^{-7\alpha_0}}, \tag{33}$$

$$\frac{A_1}{A_0} = 2\Big/\left\{\pm1-\frac{6}{7}\mathrm{e}^{-2\alpha_0}\right\}, \tag{34}$$

$$\begin{aligned} &\frac{b_0}{A_0J_0}\times\left\{3\mathrm{e}^{-2\alpha_0}\left(1\mp\frac{6}{7}\mathrm{e}^{-2\alpha_0}\right)\right\} \\ &= -\frac{1}{2}\pm\left(-\frac{6}{5}\alpha_0+\frac{29}{35}\right)\mathrm{e}^{-2\alpha_0}+\left(\frac{36}{35}\alpha_0+\frac{13}{70}\right)\mathrm{e}^{-4\alpha_0}\mp\frac{3}{7}\mathrm{e}^{-6\alpha_0}, \end{aligned} \tag{35}$$

$$\frac{b_1}{A_0J_0} = \frac{12}{35}\left(\alpha_0-\frac{1}{2}\right)\Big/\left\{1\mp\frac{6}{7}\mathrm{e}^{-2\alpha_0}\right\}, \tag{36}$$

where upper double signs for a prolate and lower double signs for an oblate spheroid. Drag force, F, acting on the ellipsoid is given by

$$\begin{aligned} \frac{F}{6\pi\mu bU_\infty} &= \frac{a}{6b}\left[\int_0^\pi r^3\frac{\partial}{\partial\alpha}\left(\frac{\zeta}{r}\right)d\beta\right]_{\alpha=\alpha_0} \\ &= -\frac{2}{3}\mathrm{e}^{-3\alpha_0}\left(\frac{b}{a}\right)^2A_0, \end{aligned} \tag{37}$$

where a : semi axis along the z-axis, b : semi axis normal to the z-axis, $b/a = \tanh\alpha_0$ for a prolate spheroid, and $\coth\alpha_0$ for an oblate one. Let $\omega \equiv b/a$, i.e. $\mathrm{e}^{-2\alpha_0} = |\omega-1|/(\omega+1)$, and assuming $|\omega-1| \ll 1$, then equation (37) becomes

$$\frac{F}{6\pi\mu bU_\infty} \approx 1-\frac{6}{5}(1-\omega)-\frac{33}{250}(1-\omega)^2, \tag{38}$$

where $\omega < 1$ for a prolate spheroid, and $\omega > 1$ for an oblate spheroid.

International Electronic Journal of Pure and Applied Mathematics – IEJPAM, Volume 5, No. 3 (2012)

3. Conclusion

Stokes flow solution past an ellipsoid of revolution (except a sphere) is shown within the first two terms in spectral forms to show an expression of drag force.

References

[1] J. Happel, H. Brenner, *Low Reynolds Number Hydrodynamics with Special Applications to Particulate Media*, Prentice Hall (1965).