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STRUCTURE THEOREMS FOR COMMUTATIVE NOETHERIAN MOORE-PENROSE TWO (MP2) RINGS AND ELEMENTARY DIVISOR RINGS

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Abstract: This paper put forth some original structure theorems for commutative and Noetherian MP2 (Moore-Penrose Two) rings as well as unit MP2 rings. An arbitrary ring R will be called MP2 as follows: Given any nonzero element a in R, there exists a nonzero x in R such that xax = x. Also, an arbitrary ring R will be called MP1 (Moore-Penrose One) if it satisfies the following property: Given any nonzero element a in R, there exists a nonzero x in R such that axa = a. Interestingly enough, MP2 rings appear frequently in atmospheric science isothermal curve estimating problems, and in engineering applications for solving unstable linear systems, or in business demand-supply matrix models with ill-conditioned Leontif matrices.

Key Words: idempotent, Noetherian ring, semisimple ring, Artinian matrix ring

1. Commutative Noetherian MP2 Rings

Definition 1. R is MP2 if for each a $\in$ R, a $\neq 0$, $\exists$ x$\in$ R, x $\neq 0$ such that xax = x.

Definition 2. R is MP1 if for each a $\in$ R, a $\neq 0$, $\exists$ x$\in$ R, x $\neq 0$ such that axa = a.

Theorem 1. *Let R be a commutative, Noetherian MP2 ring with identity, then R is a direct sum of fields.*

Proof. **Claim.** *R is MP1.* If not, then there is an ideal J $\neq$Re for any idempotent e (since all ideals are finitely generated in an Noetherian rng and all finitely generated ideals in an MP1 ring are generated by an idempotent). Now let J be maximal among

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the set of ideals which are not of the form Re. Also, let I be maximal among {I'|I' ⊂ J and I' is of the form Re}so that I = Re. Set f = 1 - e. Then R = I ⊕ Rf and J = (I ∩J) ⊕(J ∩Rf) = I ⊕ (J ∩Rf). Then J ∩Rf ≠ 0 since I ⊂ I ⊕ (J ∩ Rf) ⊆ J. Hence, ∃g ∈ J ∩Rf such that $g^2 = g \neq 0$ because J is MP2. Now I ⊂ I ⊕ Rg = Re + Rg = R(e+g). Moreover, since e = e(e+g), Re ⊂ Re(e+g) ⊂ J. Contradiction! (to the maximality of I) Hence, R is MP1 and since R is Noetherian, R is a direct sum of fields.

Theorem 2. *If R is MP1 and R has ACC then R is semisimple Artinian. In particular, if R is commutative, then R is a direct sum of fields.*

Proof. Since every ideal is finitely generated and hence generated by an idempotent, each ideal is a direct summand of R implying that R has a composition series in the category of R modules. Hence, R has DCC. Thus, since R is semisimple with DCC, R is a direct sum of matrix rings over division rings by Wedderburn's Theorem. But R commutative imples R is a direct sum of fields.

The results of the last two theorems can be applied to commutative, Noetherian rings to obtain:

Theorem 3. *Let R be a commutative Noetherian MP2 ring with identity. Then the following statements are equivalent:*

(1) R is a direct sum of fields.

(2) For some positive integer k, $M_k(R)$ is MP2.

(3) For all positive integers k, $M_k(R)$ is MP2.

PROOF. (3) ⇒ (2) Trivial. (2) ⇒ (1) Since $M_k(R)$ is MP2, R is MP2. By the main theorem proved, R is a direct sum of fields since it is a commutative Noetherian MP2 ring with identity. (1) ⇒ (3) If R = $\oplus\Sigma F_i$ then $M_k(R) = \oplus\Sigma M_k(F_i)$ for all k. But, $M_k(F_i)$ is MP1 for all k; hence, it is MP2 for all k. The MP2 property is preserved by arbitrary direct sums yielding the desired conclusion.

Theorem 4. *Let R be a ring with identity. Then the following statements are equivalent:*

1. *R is MP1.*
2. *For some positive integer k , $M_k(R)$ is MP1.*
3. *For every positive integer k, $M_k(R)$ is MP1.*

Proof. (3) ⇒ (2) is immediate. (2) ⇒ (1) Let A = diag $(a_1, \ldots, a_k)$ where $a_1 = a_2 = \ldots = a_k = a \in R$ which is nonzero. Then $A \in M_k(R)$. Since $M_k(R)$ is MP1, there exist a matrix $X \in M_k(R)$, X nonzero, such that AXA = A. Then for at least one $x_{ii} \in R$, $(1 \leq i \leq k)$, $ax_{ii}a = a$ where x_{ii} is nonzero. Hence, R is MP1. (1) ⇒ (3) is well known since any matrix ring over an MP1 ring is MP1 due to J. Von Neumann [1].

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2. MP2 Ring as an Elementary Divisor Ring

Definition 1. For any positive n, let $M_n(R)$ denote the ring of n x n matrices with entries in R. R is called an elementary divisor ring if for every A in $M_n(R)$, there are units P and Q in $M_n(R)$ such that PAQ is a diagonal matrix.

Definition 2. A ring R with identity is called unit MP2 if $\forall a \in R$ there is a unit u in R such that uau = u.

It has been shown by Melvin Henriksen that if R is unit MP1, then R is an elementary divisor ring [2]. In addition, it is also true that if R is unit MP1 then $M_n(R)$ is unit MP1. Thus, it is natural to investigate if similar properties hold for unit MP2 rings. Interestingly enough, it turns out that unit MP2 rings by definition are elementary divisor rings. However, it will be shown that if R is unit MP2, then it is not necessarily true that $M_n(R)$ is unit MP2.

Now there are MP2 rings which are not unit MP2; for example, let $R = M_2(Z/(6))$ and let $A = \begin{bmatrix} \bar{2}\,\bar{0} \\ \bar{0}\,\bar{0} \end{bmatrix}$, then , after a brief arithmetic excursion, the only X's in R such that XAX = X are $X = \begin{bmatrix} \bar{0} & \bar{0} \\ \bar{0} & \bar{0} \end{bmatrix}$ and $X = \begin{bmatrix} \bar{2} & \bar{0} \\ \bar{0} & \bar{0} \end{bmatrix}$; however, neither X is a unit in R.

Theorem 5. *Let R be a ring with identity and $a \in R$, then the following statements are equivalent:*

(1) There is a unit u in R such that uau = u.

(2) R is a division ring.

Proof. (1) ⇒ (2) Since uau = u then uau- u = 0 and u(au- 1) = 0; multiplying the left-hand side of the equation by u^{-1} , yields u as a right-inverse of a. Similarly, (ua-1)u = 0; multiplying the right-hand side of the equation by u^{-1} , yields u as a left-inverse of a. Since R is not necessarily commutative, R is a division ring. (2) ⇒ (1) Proof is immediately clear.

Theorem 6. *If R is unit MP2 then R is an elementary divisor ring.*

Proof. By Theorem 5, R unit MP2 ⇒ R is a division ring ⇒ R is unit MP1⇒ by Henriksen's result it follows that R is an elementary divisor ring.

Now a weakened version of the previous theoem leads to the following proposition:

Theorem 7. *Let R be a ring with identity possessing no nontrivial zero divisors and $a \in R$, then the following statements are equivalent:*

(1) There is a unit u in R such that uau = u.

(2) There is a unit u in R such that au and ua are idempotents.

(3) There is a unit u in R such that either au or ua is idempotent.

(4) R is a division ring.

PROOF. (1) ⇒ (2) If (1) holds, then $(au)^2 = a(uau) = au$ and $(ua)^2 = (uau)a = ua$. (2) ⇒ (3) Immediately apparent. (3) ⇒ (4) If ua is idempotent, then uaua = ua; hence, (uau - u)a= 0 implies uau = u since R has no nontrivial zero-divisors; thereby ua = 1 and a has a left inverse. Similarly, a has a right inverse and R is a division ring. (4) ⇒ (1) If R is a division ring, then there is a unit y such that ya = 1. Then yay = y.

Remark. Let D be a division ring. Then surely D is unit MP1 as well as unit MP2. However, M_n (D) (n >1) need not be a division ring. Hence, M_n (D) is not unit MP2. Thus, R unit MP2 does not necessarily imply that $M_n(R)$ is unit MP2.

Example 1. Let R be Q the field of rationals. Then R is unit MP2. Now in $M_2(Q)$ let A = $\begin{bmatrix} 1\,0 \\ 1\,0 \end{bmatrix}$ then since A is singular, there cannot be a unit or nonsingular matrix B such that BAB = B, for otherwise A would be invertible ⇒⇐ Contradiction!

Another observation to be made is when an elementary divisor ring R is unit MP1. It should be noted that if R is an elementary divisor ring and $M_n(R)$ is unit MP1 then R is certainly unit MP1. For example, any semisimple Artinian matrix ring is unit MP1 as demonstrated by Gertrude Ehrlich [3].

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