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ON FUZZY IDEALS OF PO-Γ-SEMIGROUPS

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Abstract: In this paper the notions of fuzzy ideals of a po-Γ-semigroup have been introduced and some of their important properties have been investigated.

A characterization of regular po-Γ-semigroup has been obtained. Various relationships between fuzzy ideals of a po-Γ-semigroup and those of its operator po-semigroups have also been obtained.

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1. Introduction

A semigroup is an algebraic structure consisting of a non-empty set S together with an associative binary operation [14]. The formal study of semigroups began in the early 20th century. Semigroups are important in many areas of mathematics, for example, coding and language theory, automata theory, combinatorics and mathematical analysis. The concept of fuzzy sets was introduced by Lofti Zadeh [41] in his classic paper in 1965. Azirel Rosenfeld [26] used the idea of fuzzy set to introduce the notions of fuzzy subgroups. Nobuaki Kuroki [20, 21, 22] is the pioneer of fuzzy ideal theory of semigroups. Kuroki [21] characterized several classes of semigroups in terms of fuzzy left, fuzzy right and fuzzy bi-ideals. X.Y. Xie [39, 40] introduced the idea of extensions of fuzzy ideals in semigroups. N. Kehayopulu and M. Tsingelis [16, 17], S.K Lee [25] worked on po-semigroups. Others who worked on fuzzy

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po-semigroup theory are N. Kehayopulu and M. Tsingelis [18, 19], X.Y. Xie and F. Yan [40], M. Shabir and A. Khan [37].

The notion of a Γ-semigroup was introduced by *Sen* and Saha [34] as a generalization of semigroups and ternary semigroup. Γ-semigroup has been analyzed by a good many mathematicians, for instance by Chattopadhyay [2, 3], Dutta and Adhikari [1, 7, 8], Hila [12, 13], Chinram [4], Sen et al. [31, 32, 33, 34, 35], Seth [36], N.K. Saha [27]. S.K. Sardar and S.K. Majumder [28, 29, 30, 10, 11] have introduced the notion of fuzzification of ideals, prime ideals, semiprime ideals and ideal extensions of Γ-semigroups and studied them via its operator semigroups.

Y.I. Kwon and S.K. Lee [23, 24], T.K. Dutta and N.C. Adhikari [9], Chinram and K. Tinpun [5], P. Dheena and B. Elavarasan [6], M. Siripitukdet and A. Iampan [15, 38] are some authors who worked on po-Γ-semigroup theory. T.K. Dutta and N.C. Adhikari [9] have studied different properties of po-Γ-semigroup by defining operator po-semigroups of po-Γ-semigroups.

The paper consists of four sections. In Section 2 we recall some elementary definitions for their use in the sequel. In Section 3 we introduce the notion of fuzzy ideals in a po-Γ-semigroup. Among the other results, we obtain a characterization of regular po-Γ-semigroup in terms of fuzzy ideals. In Section 4 we study corresponding fuzzy ideals. Here among the other results, we obtain an inclusion preserving bijection between the set of all fuzzy ideals of a po-Γ-semigroup and that of its operator semigroup.

2. Preliminaries

Definition 2.1. (see [34]) Let $S = \{x, y, z, ...\}$ and $\Gamma = \{\alpha, \beta, \gamma....\}$ be two non-empty sets. Then S is called a Γ-semigroup if there exists a mapping $S \times \Gamma \times S \to S$(images to be denoted by $a\alpha b$) satisfying (1) $x\gamma y \in S$, (2) $(x\beta y)\gamma z = x\beta(y\gamma z)$ for all $x, y, z \in S, \alpha, \beta, \gamma \in \Gamma$.

Definition 2.2. (see [6]) A Γ-semigroup S is said to be po-Γ-semigroup(partially ordered Γ-semigroup) if (1) S is a poset, (2) $a \leq b$ in S implies that $a\alpha c \leq b\alpha c$, $c\alpha a \leq c\alpha b$ in S for all $c \in S$ and for all $\alpha \in \Gamma$.

Remark 1. Definition 2.1 and 2.2 are the definitions of one sided Γ-semigroup and one sided po-Γ-semigroup respectively. The following are the definitions of both sided Γ-semigroup [7] and both sided po-Γ-semigroup [9] given by T.K. Dutta and N.C. Adhikari. Throughout this paper unless or otherwise mentioned S stands for both sided po-Γ-semigroup.

Definition 2.3. (see [7]) Let S and Γ be two non-empty sets. S is called a Γ-semigroup if there exist mappings from $S \times \Gamma \times S$ to S, written as $(a, \alpha, b) \longrightarrow a\alpha b$,

International Electronic Journal of Pure and Applied Mathematics – IEJPAM, Volume 5, No. 2 (2012)

and from $\Gamma \times S \times \Gamma$ to Γ, written as $(\alpha, a, \beta) \longrightarrow \alpha a\beta$ satisfying the following associative laws $(a\alpha b)\beta c = a(\alpha b\beta)c = a\alpha(b\beta c)$ and $\alpha(a\beta b)\gamma = (\alpha a\beta)b\gamma = \alpha a(\beta b\gamma)$ for all $a, b, c \in S$ and for all $\alpha, \beta, \gamma \in \Gamma$.

Definition 2.4. (see [9]) A Γ-semigroup S is said to be a po-Γ-semigroup if (1) S and Γ are posets, (2) $a \leq b$ in S implies that $a\alpha c \leq b\alpha c, c\alpha a \leq c\alpha b$ in S and $\gamma a\alpha \leq \gamma b\alpha$ in Γ for all $c \in S$ and for all $\alpha, \gamma \in \Gamma$, (3) $\alpha \leq \beta$ in Γ implies that $\alpha a\gamma \leq \beta a\gamma, \gamma a\alpha \leq \gamma a\beta$ in Γ and $a\alpha b \leq a\beta b$ in S for all $\gamma \in \Gamma$ and for all $a, b \in S$.

Remark 2. The partial order relations on S and Γ are denoted by same symbol $\leq$.

Example 1. (see [9]) Let S be the set of all 2×3 matrices over the set of positive integers and Γ be the set of all 3×2 matrices over same set. Then S is a Γ-semigroup with respect to the usual matrix multiplication. Also S and Γ are posets with respect to $\leq$ defined by $(a_{ik}) \leq (b_{ik})$ if and only if $a_{ik} \leq b_{ik}$ for all i, k. Then S is a po-Γ-semigroup.

Definition 2.5. (see [41]) A fuzzy subset μ of a non-empty set X is a function $\mu : X \to [0, 1]$.

Definition 2.6. (see [28]) Let μ be a fuzzy subset of a non-empty set X. Then the set $\mu_t = \{x \in X : \mu(x) \geq t\}$ for $t \in [0, 1]$, is called the level subset or t-level subset of μ.

3. Fuzzy Ideals

Definition 3.1. (see [9]) Let S be a po-Γ-semigroup. A non-empty subset I of S is said to be a right ideal(left ideal) of S if (1) $I\Gamma S \subseteq I$(resp. $S\Gamma I \subseteq I$), (2) $a \in I$ and $b \leq a$ imply $b \in I$. I is said to be an ideal of S if it is a right ideal as well as a left ideal of S.

Definition 3.2. A non-empty fuzzy subset μ of a po-Γ-semigroup S is called the fuzzy left ideal(right ideal) of S if (1) $x \leq y$ implies $\mu(x) \geq \mu(y) \forall x, y \in S$, (2) $\mu(x\alpha y) \geq \mu(y)(\mu(x\alpha y) \geq \mu(x)) \forall x, y \in S, \alpha \in \Gamma$. μ is said to be a fuzzy ideal of S if it is a fuzzy right ideal as well as a fuzzy left ideal of S.

Example 2. Let S be the set of all non positive integers without zero and Γ be the set of all non positive even integers without zero. Then S is a Γ-semigroup if $a\gamma b$ denotes the usual multiplication of integers a, γ, b where $a, b \in S$ and $\gamma \in \Gamma$. Again with respect to usual $\leq$ of $\mathbb{Z}$, S becomes a po-Γ-semigroup. Let μ be a fuzzy subset of S, defined as follows

$$\mu(y) = \begin{cases} 0.1 & \text{if } x = -1, \\ 0.3 & \text{if } x = -2, \\ 0.5 & \text{if } x < -2. \end{cases}$$

Then μ becomes a fuzzy ideal of S.

Example 3. Let S be the set of all non positive integers without zero and Γ be the set of all non positive even integers without zero. Then S is a Γ-semigroup if $a\gamma b$ denotes the usual multiplication of integers a, γ, b where $a, b \in S$ and $\gamma \in \Gamma$. Again with respect to usual $\leq$ of $\mathbb{Z}$, S becomes a po-Γ-semigroup. Let μ be a fuzzy subset of S, defined as follows

$$\mu(x) = \tfrac{-x}{10},\ -10 < x < 0$$

x	-1	-2	-3	-4	-5	-6	-7	-8	-9
$\mu(x)$	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9

$$\mu(x) = \mu(-10+1) + \tfrac{-x}{10^{2+1}},\ -10^2 < x \leq -10$$

x	-10	-11	-12	-13	-14	-15	..	-98	-99
$\mu(x)$	0.910	0.911	0.912	0.913	0.914	0.915	..	0.998	0.999

$$\mu(x) = \mu(-10^2+1) + \tfrac{-x}{10^{3+2+1}},\ -10^3 < x \leq -10^2$$

x	-100	-101	-102	-103	-104	..	..	-998	-999
$\mu(x)$	0.999100	0.999101	0.999102	0.999103	0.999104	..	..	0.999998	0.999999

$$\cdots$$

$$\mu(x) = \mu(-10^{n-1}+1) + \tfrac{-x}{10^{n+(n-1)+..+2+1}},\ -10^n < x \leq -10^{n-1}$$

It is easy to verify that μ is a fuzzy ideal of po-Γ-semigroup S.

Theorem 3.3. *Let I be a non-empty subset of a po-Γ-semigroup S and for any two elements α, β in $[0, 1]$ with $\alpha \leq \beta \neq 0$, μ be a fuzzy subset of S which is defined by*

$$\mu(x) = \begin{cases} \beta, & \text{if } x \in I \\ \alpha, & \text{if } x \notin I. \end{cases}$$

Then I is a left ideal(right ideal, ideal) of S if and only if μ is a fuzzy left ideal(resp. fuzzy right ideal, fuzzy ideal) of S.

Proof. Let I be a left ideal of a po-Γ-semigroup S. Let $s, x \in S$ and $\gamma \in \Gamma$. If $x \in I$, then $s\gamma x \in I$ and $\mu(x) = \beta = \mu(s\gamma x)$. If $x \notin I$, then $\mu(x) = \alpha \leq \mu(s\gamma x)$. Thus $\mu(s\gamma x) \geq \mu(x)$ $\forall s, x \in S$ and $\gamma \in \Gamma$.

Let $x, y \in S$ be such that $x \leq y$. If $y \in I$, then $\mu(y) = \beta$. Since I is a left ideal of S, so $x \in I$. Then $\mu(x) = \beta = \mu(y)$. If $y \notin I$, then $\mu(y) = \alpha \leq \mu(x)$. Hence $\mu(x) \geq \mu(y)$. Consequently, μ is a fuzzy left ideal of S.

Conversely, let μ be a fuzzy left ideal of S. Let $x \in I$, $s \in S$ and $\gamma \in \Gamma$, then $\mu(x) = \beta$. Also $\mu(s\gamma x) \geq \mu(x) = \beta$. Hence $s\gamma x \in I$.

Let $x, y \in S$ be such that $y \leq x$. Let $x \in I$. Then $\mu(x) = \beta$. Since μ is a fuzzy left ideal of S, $\mu(y) \geq \mu(x) = \beta$. Consequently, $y \in I$. Hence I is a left ideal of S. Similarly we can prove the other cases also. □

If we take $\alpha = 0$ and $\beta = 1$ then from the above theorem we obtain the following result.

Corollary 3.4. *Let I be a non-empty subset of a po-Γ-semigroup S and χ_I be the characteristic function of I. Then I is a left ideal(right ideal, ideal) of S if and only if χ_I is a fuzzy left ideal(resp. fuzzy right ideal, fuzzy ideal) of S.*

Theorem 3.5. *Let S be a po-Γ-semigroup and μ be a non-empty fuzzy subset of S. Then μ is a fuzzy left ideal(fuzzy right ideal, fuzzy ideal) of S if and only if μ_t is a left ideal(resp. right ideal, ideal) of S for all $t \in \operatorname{Im}\mu$, where $\mu_t = \{x \in S : \mu(x) \geq t\}$.*

Proof. Let μ be a fuzzy left ideal of S. Let $t \in \operatorname{Im}\mu$. Then there exist some $\alpha \in S$ such that $\mu(\alpha) = t$ and so $\alpha \in \mu_t$. Thus $\mu_t \neq \phi$. Let $s \in S, x \in \mu_t$ and $\gamma \in \Gamma$. Now $\mu(s\gamma x) \geq \mu(x) \geq t$. Hence $s\gamma x \in \mu_t$.

Let $x, y \in S$ be such that $y \leq x$. Let $x \in \mu_t$. Then $\mu(x) \geq t$. Since μ is a fuzzy left ideal of S, so $\mu(y) \geq \mu(x) \geq t$. Consequently, $y \in \mu_t$. Hence μ_t is a left ideal of S.

Conversely, suppose μ_t is a left ideal of S, for all $t \in \operatorname{Im}\mu$. Let $x, s \in S$ and $\gamma \in \Gamma$. Now let $\mu(x) = t$. Then $x \in \mu_t$. Since μ_t is a left ideal of S, so $s\gamma x \in \mu_t$. Consequently, $\mu(s\gamma x) \geq t = \mu(x)$.

Let $x, y \in S$ be such that $x \leq y$. Let $\mu(y) = t$, then $y \in \mu_t$. Since μ_t is a left ideal of S, so $x \in \mu_t$. Then $\mu(x) \geq t = \mu(y)$. Consequently, μ is a fuzzy left ideal of S. Similarly we can prove the other cases also. □

Proposition 3.6. *Let $\{\mu_i : i \in I\}$ be a family of fuzzy left ideals(fuzzy right ideals, fuzzy ideals) of a po-Γ-semigroup S then $\bigcap_{i \in I} \mu_i$, if it is non-empty, is a fuzzy left ideal(resp. fuzzy right ideal, fuzzy ideal) of S.*

Proof. Let $\{\mu_i : i \in I\}$ be a family of fuzzy left ideals of S and $x, y \in S$. If $x \leq y$,

International Electronic Journal of Pure and Applied Mathematics – IEJPAM, Volume 5, No. 2 (2012)

then

$$\begin{aligned}(\bigcap_{i\in I}\mu_i)(x) &= \bigwedge_{i\in I}(\mu_i(x))\\ &\geq \bigwedge_{i\in I}(\mu_i(y))(\text{since } x\leq y\Rightarrow \mu_i(x)\geq\mu_i(y))\\ &= (\bigcap_{i\in I}\mu_i)(y).\end{aligned}$$

Now let $x, y \in S$ and $\alpha \in \Gamma$. Then

$$\begin{aligned}(\bigcap_{i\in I}\mu_i)(x\gamma y) &= \bigwedge_{i\in I}(\mu_i(x\gamma y))\\ &\geq \bigwedge_{i\in I}(\mu_i(y))(\text{since } \mu_i(x\gamma y)\geq\mu_i(y))\\ &= (\bigcap_{i\in I}\mu_i)(y).\end{aligned}$$

Hence $\bigcap_{i\in I}\mu_i$ is a fuzzy left ideal of S. Similarly we can prove the other cases also. □

Definition 3.7. Let S be a po-Γ-semigroup and μ_1, μ_2 are two fuzzy subsets of S. Then the product $\mu_1 \circ \mu_2$ of μ_1 and μ_2 is defined as

$$(\mu_1\circ\mu_2)(x) = \begin{cases}\sup\limits_{x\leq u\gamma v}[\min\{\mu_1(u),\mu_2(v)\} : u,v\in S;\gamma\in\Gamma]\\ 0,\ \text{if for any } u,v\in S \text{ and for any } \gamma\in\Gamma, x\nleq u\gamma v\end{cases}$$

Theorem 3.8. *In a po-Γ-semigroup S for a non-empty fuzzy subset μ of S the following are equivalent:* (1) *μ is a fuzzy left ideal(fuzzy right ideal) of S,* (2) *$\chi\circ\mu\subseteq\mu$(resp. $\mu\circ\chi\subseteq\mu$) and $x\leq y$ implies $\mu(x)\geq\mu(y)\forall x,y\in S$, where χ is the characteristic function of S.*

Proof. (1) $\Rightarrow$ (2) : Let μ be a fuzzy left ideal of S. Since μ is a fuzzy left ideal of S, so from definition $x \leq y$ implies that $\mu(x) \geq \mu(y)\forall x, y \in S$. Let $a \in S$. Suppose there exists $u, v \in S$ and $\delta \in \Gamma$ such that $a \leq u\delta v$. Since μ is a fuzzy left ideal of S, we have

$$\begin{aligned}(\chi\circ\mu)(a) &= \sup_{a\leq x\gamma y}[\min\{\chi(x),\mu(y)\}]\\ &= \sup_{a\leq x\gamma y}[\min\{1,\mu(y)\}] = \sup_{a\leq x\gamma y}\mu(y).\end{aligned}$$

Now since μ is a fuzzy left ideal, $\mu(x\gamma y) \geq \mu(y) \forall x, y \in S$ and $\forall \gamma \in \Gamma$. So in particular, $\mu(y) \leq \mu(a) \forall a \leq x\gamma y$. Hence $\sup\limits_{a \leq x\gamma y} \mu(y) \leq \mu(a)$. Thus $\mu(a) \geq (\chi \circ \mu)(a)$. If there do not exist $x, y \in S$ and $\gamma \in \Gamma$ such that $a \leq x\gamma y$ then $(\chi \circ \mu)(a) = 0 \leq \mu(a)$. Hence $\chi \circ \mu \subseteq \mu$. By similar argument we can show that $\mu \circ \chi \subseteq \mu$ when μ is a fuzzy right ideal.

(2) $\Rightarrow$ (1) : Let $\chi \circ \mu \subseteq \mu$. Let $x, y \in S, \gamma \in \Gamma$ and $a := x\gamma y$. Then clearly $a \leq x\gamma y$. Now $\mu(x\gamma y) = \mu(a) \geq (\chi \circ \mu)(a)$. Now , we have

$$\begin{aligned}(\chi \circ \mu)(a) &= \sup_{a \leq u\alpha v} [\min\{\chi(u), \mu(v)\}] \\ &\geq \min\{\chi(x), \mu(y)\} = \min\{1, \mu(y)\} = \mu(y).\end{aligned}$$

Hence $\mu(x\gamma y) \geq \mu(y)$. Hence μ is a fuzzy left ideal of S. By a similar argument we can show that if $\mu \circ \chi \subseteq \mu$, then μ is a fuzzy right ideal of S. □

In view of above theorem we can have the following theorem.

Theorem 3.9. *In a po-Γ-semigroup S for a non-empty fuzzy subset μ of S the following are equivalent:* (1) *μ is a fuzzy ideal of S,* (2) *$\chi \circ \mu \subseteq \mu$, $\mu \circ \chi \subseteq \mu$ and $x \leq y$ implies $\mu(x) \geq \mu(y) \forall x, y \in S$, where χ is the characteristic function of S.*

Definition 3.10. (see [9]) A po-Γ semigroup S is called regular if for any $x \in S$ there exist $a \in S, \alpha, \beta \in \Gamma$ such that $x \leq x\alpha a\beta x$.

Definition 3.11. Let A be a subset of a po-Γ semigroup S. Then we define $(A] := \{x \in S : x \leq y$ for some $y \in A\}$.

We now obtain the following result whose proof is a matter of routine verification.

Proposition 3.12. *In a po-Γ-semigroup S, if A and B are any two non-empty subsets of S, then $(A]\Gamma(B] \subseteq (A\Gamma B]$.*

Moreover, if A and B are any two ideals(left, right or both sided) of S, then (1) $(A] = A, (B] = B$, (2) $(A \cap B] = (A] \cap (B]$.

Theorem 3.13. *A po-Γ semigroup S is regular if and only if $A \cap B = (A\Gamma B]$ for any right ideal A and for any left ideal B of S.*

Proof. Let S be regular. Let $x \in (A\Gamma B]$. Then $x \leq a\alpha b$ for some $a \in A, b \in B, \alpha \in \Gamma$. Since A is a right ideal of S, $A\Gamma S \subseteq A$ and so $a\alpha b \in A$. Similarly B being a left ideal of S, $a\alpha b \in B$. So $a\alpha b \in A \cap B$. Hence $x \in (A \cap B] = (A] \cap (B]$(using Prop.3.12(2)) $= A \cap B$(using Prop.3.12(1)). Hence $(A\Gamma B] \subseteq A \cap B$. Again, let

$x \in A \cap B$. Then$x \in A$ and $x \in B$. Since S is regular, there exist $a \in S$ and $\alpha, \beta \in \Gamma$ such that $x \leq x\alpha a\beta x$. Now since $x \in A$ and A is a right ideal, $x\alpha a \in A$ whence $x\alpha a\beta x \in A\Gamma B$. Consequently, $x \in (A\Gamma B]$. So $A \cap B \subseteq (A\Gamma B]$. Hence $A \cap B = (A\Gamma B]$.

Conversely, suppose for any right ideal A and any left ideal B of S, $A \cap B = (A\Gamma B]$ and let $x \in S$. Let $R(x) = (\{x\} \cup x\Gamma S]$ and $L(x) = (\{x\} \cup S\Gamma x]$. Clearly $R(x)$ is a right ideal and $L(x)$ is a left ideal of S. So by hypothesis $R(x) \cap L(x) = (R(x)\Gamma L(x)]$. Since $x \in R(x) \cap L(x)$, $x \in (R(x)\Gamma L(x)]$. Then $x \leq r\alpha l$ for some $r \in R(x), \alpha \in \Gamma, l \in L(x)$. Now $r \in R(x) \Rightarrow r \leq x$ or $r \leq x\beta s_1$ for some $\beta \in \Gamma, s_1 \in S$. Similarly $l \in L(x) \Rightarrow l \leq x$ or $l \leq s_2\gamma x$ for some $\gamma \in \Gamma, s_2 \in S$. In each of these cases there will be some $s \in S, \gamma_1, \gamma_2 \in \Gamma$ such that $x \leq x\gamma_1 s\gamma_2 x$. Hence S is regular. □

Proposition 3.14. *Let μ_1 be a fuzzy right ideal and μ_2 be a fuzzy left ideal of a po-Γ semigroup S. Then $\mu_1 \circ \mu_2 \subseteq \mu_1 \cap \mu_2$.*

Proof. Let $x \in S$. Suppose there exist $u_1, v_1 \in S$ and $\gamma_1 \in \Gamma$ such that $x \leq u_1\gamma_1 v_1$. Then

$$\begin{aligned}(\mu_1 \circ \mu_2)(x) &= \sup_{x \leq u\gamma v} \min\{\mu_1(u), \mu_2(v)\} \\ &\leq \sup_{x \leq u\gamma v} \min\{\mu_1(u\gamma v), \mu_2(u\gamma v)\} \\ &\leq \min\{\mu_1(x), \mu_2(x)\} = (\mu_1 \cap \mu_2)(x).\end{aligned}$$

Suppose, there do not exist $u, v \in S$ and $\gamma \in \Gamma$ such that $x \leq u\gamma v$. Then $(\mu_1 \circ \mu_2)(x) = 0 \leq (\mu_1 \cap \mu_2)(x)$. Thus $\mu_1 \circ \mu_2 \subseteq \mu_1 \cap \mu_2$. □

Proposition 3.15. *Let S be a regular po-Γ semigroup and μ_1, μ_2 be two fuzzy subsets of S. Then $\mu_1 \circ \mu_2 \supseteq \mu_1 \cap \mu_2$.*

Proof. Let $c \in S$. Since S is regular, then there exist $x \in S$ and $\gamma_1, \gamma_2 \in \Gamma$ such that $c \leq c\gamma_1 x\gamma_2 c = c\gamma c$ where $\gamma := \gamma_1 x\gamma_2 \in \Gamma$. Then

$$\begin{aligned}(\mu_1 \circ \mu_2)(c) &= \sup_{c \leq u\alpha v} \min\{\mu_1(u), \mu_2(v)\} \\ &\geq \min\{\mu_1(c), \mu_2(c)\} = (\mu_1 \cap \mu_2)(c).\end{aligned}$$

Hence $\mu_1 \circ \mu_2 \supseteq \mu_1 \cap \mu_2$. □

We obtain the following result by routine calculation.

Proposition 3.16. *Let S be a po-Γ-semigroup and $A, B \subseteq S$. Then (1) $(A] \subseteq (B]$ if and only if $\mu_{(A]} \leq \mu_{(B]}$, (2) $\mu_{(A \cap B]} \leq \mu_{(A]} \cap \mu_{(B]}$, (3) $\mu_{(A]} \circ \mu_{(B]} = \mu_{(A\Gamma B]}$, where $\mu_{(A]}$, $\mu_{(B]}$, $\mu_{(A \cap B]}$ and $\mu_{(A\Gamma B]}$ are characteristic functions of $(A]$, $(B]$, $(A \cap B]$ and $(A\Gamma B]$ respectively.*

Proof. (1) Let $(A] \subseteq (B]$. Let $x \in S$. If $x \in (A]$, then $x \in (B]$. So $\mu_{(A]}(x) = 1 = \mu_{(B]}(x)$. If $x \notin (A]$, then $\mu_{(A]}(x) = 0 \leq \mu_{(B]}(x)$. So $\mu_{(A]} \leq \mu_{(B]}$.

Conversely, let $\mu_{(A]} \leq \mu_{(B]}$. Let $x \in (A]$. Then $\mu_{(A]}(x) = 1 \leq \mu_{(B]}(x)$. So $\mu_{(B]}(x) = 1$. Hence $x \in (B]$. So $(A] \subseteq (B]$.

(2) Let $x \in S$. If $\mu_{(A\cap B]}(x) = 1$ then $x \in (A \cap B]$. So $x \leq y$ for some $y \in A \cap B$. Since $y \in A$ and $y \in B$, by Definition 3.11, $x \in (A]$ and $x \in (B]$. Hence $(\mu_{(A]} \cap \mu_{(B]})(x) = 1$. Now if $\mu_{(A\cap B]}(x) = 0$ then clearly $\mu_{(A\cap B]}(x) \leq (\mu_{(A]} \cap \mu_{(B]})(x)$. Hence $\mu_{(A\cap B]} \leq \mu_{(A]} \cap \mu_{(B]}$.

(3) Let $x \in S$. Now if $(\mu_{(A]} \circ \mu_{(B]})(x) = 1$ then

$$\sup_{x \leq u\gamma v} [\min\{\mu_{(A]}(u), \mu_{(B]}(v)\} : u, v \in S; \gamma \in \Gamma] = 1.$$

So there exist $a \in (A]$ and $b \in (B]$ and $\alpha \in \Gamma$ such that $x \leq a\alpha b$. Then by Definition 3.11, $x \in ((A]\Gamma(B]] \subseteq ((A\Gamma B]] = (A\Gamma B]$. Hence $\mu_{(A\Gamma B]}(x) = 1$. Now if $(\mu_{(A]} \circ \mu_{(B]})(x) = 0$ then if possible, suppose $(\mu_{(A\Gamma B]})(x) = 1$. Then $x \in (A\Gamma B]$. So $x \leq a\alpha b$ for some $a \in A, b \in B$ and $\alpha \in \Gamma$. Since $a \in (A], b \in (B]$, $\mu_{(A]}(a) = 1$, $\mu_{(B]}(b) = 1$ whence $\min\{\mu_{(A]}(a), \mu_{(B]}(b)\} = 1$. Now by Definition 3.7, $(\mu_{(A]} \circ \mu_{(B]})(x) \geq \min\{\mu_{(A]}(a), \mu_{(B]}(b)\} = 1$ which is a contradiction. So $(\mu_{(A\Gamma B]})(x) = 0$. Hence $\mu_{(A]} \circ \mu_{(B]} = \mu_{(A\Gamma B]}$. □

The following example shows that equality in Prop. 3.16(2) need not hold.

Example 4. Let S be the set of all 2×3 matrices over the set of positive integers with zero and Γ be the set of all 3×2 matrices over same set. Then S is a Γ-semigroup with respect to the usual matrix multiplication. Also S and Γ are posets with respect to $\leq$ defined by $(a_{ik}) \leq (b_{ik})$ if and only if $a_{ik} \leq b_{ik}$ for all i, k. Then S is a po-Γ-semigroup. Let
$y = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$, $w = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$, $z = \begin{pmatrix} 1 & 3 & 4 \\ 5 & 6 & 7 \end{pmatrix}$.
Consider $A = \{y, w\}$ and $B = \{z, w\}$, two subsets of S.Now let $x = \begin{pmatrix} 0 & 1 & 2 \\ 3 & 4 & 5 \end{pmatrix} \in S$. Clearly $x \leq y$ and $x \leq z$. So $x \in (A], (B]$. Then $(\mu_{(A]} \cap \mu_{(B]})(x) = 1$. But there exists no $t \in A \cap B$ such that $x \leq t$ as $A \cap B = \{w\}$. So $\mu_{(A\cap B]}(x) = 0$. Hence $\mu_{(A]} \cap \mu_{(B]} \nleq \mu_{(A\cap B]}$.

Now we obtain the following characterization of a regular po-Γ-semigroup in terms of fuzzy ideals.

Theorem 3.17. *In a po-Γ semigroup S, the following conditions are equivalent: (1) S is regular, (2) $\mu_1 \circ \mu_2 = \mu_1 \cap \mu_2$ for every fuzzy right ideal μ_1 and every fuzzy* *left ideal μ_2 of S.*

Proof. (1) $\Rightarrow$ (2) : Let S be a regular. Then by Propositions 3.14 and 3.15 we obtain $\mu_1 \circ \mu_2 = \mu_1 \cap \mu_2$.

(2) $\Rightarrow$ (1) : Suppose (2) holds. Let R be a right ideal and L be a left ideal of S. Let $x \in R \cap L$. Then $x \in (R \cap L]$. This together with Prop.3.12(2) implies that $x \in (R]$ and $x \in (L]$. Hence $\chi_{(R]}(x) = \chi_{(L]}(x) = 1$(where $\chi_{(R]}$ and $\chi_{(L]}$ are the characteristic functions of $(R]$ and $(L]$ respectively). Then $(\chi_{(R]} \cap \chi_{(L]})(x) = 1$. Now by Corollary 3.4, $\chi_{(R]}$ is a fuzzy right ideal and $\chi_{(L]}$ is a fuzzy left ideal of S. Hence by hypothesis $\chi_{(R]} \circ \chi_{(L]} = \chi_{(R]} \cap \chi_{(L]}$. So $(\chi_{(R]} \circ \chi_{(L]})(x) = 1$. Hence by Proposition 3.16(3), $\chi_{(R\Gamma L]}(x) = 1$. So $x \in (R\Gamma L]$. Hence $R \cap L \subseteq (R\Gamma L]$. Also clearly $(R\Gamma L] \subseteq R \cap L$. Hence $(R\Gamma L] = R \cap L$. Consequently, S is regular. □

4. Corresponding Fuzzy Ideals

Many results of po-semigroups could be extended to po-Γ-semigroups directly and via operator po-semigroups [9](left, right) of a po-Γ-semigroup. In order to make operator po-semigroups of a po-Γ-semigroup work in the context of fuzzy sets as it worked in the study of po-Γ-semigroups [9], we obtain various relationships between fuzzy ideals of a po-Γ-semigroup and those of its operator po-semigroups. Here, among the other results, we obtain an inclusion preserving bijection between the set of all fuzzy ideals of a po-Γ-semigroup and that of its operator po-semigroup.

Definition 4.1. (see [9]) Let S be a Γ-semigroup. Let us define a relation ρ on $S \times \Gamma$ as follows :$(x, \alpha)\rho(y, \beta)$ if and only if $x\alpha s = y\beta s$ for all $s \in S$ and $\gamma x\alpha = \gamma y\beta$ for all $\gamma \in \Gamma$. Then ρ is an equivalence relation. Let $[x, \alpha]$ denote the equivalence class containing (x, α). Let $L = \{[x, \alpha] : x \in S, \alpha \in \Gamma\}$. Then L is a semigroup with respect to the multiplication defined by $[x, \alpha][y, \beta] = [x\alpha y, \beta]$. This semigroup L is called the left operator semigroup of the Γ-semigroup S. Dually the right operator semigroup R of Γ-semigroup S is defined where the multiplication is defined by $[\alpha, a][\beta, b] = [\alpha a\beta, b]$.

Let $((S, \Gamma), \leq)$ be a po-Γ-semigroup. We define a relation $\leq$ on L by $[a, \alpha] \leq [b, \beta]$ if and only if $a\alpha s \leq b\beta s$ for all $s \in S$ and $\gamma a\alpha \leq \gamma b\beta$ for all $\gamma \in \Gamma$. With respect to this relation L becomes a po-semigroup. In a similar way R can be made into a po-semigroup.

If there exists an element $[e, \delta] \in L([\gamma, f] \in R)$ such that $e\delta s = s$(resp. $s\gamma f = s$) for all $s \in S$ then $[e, \delta]$(resp. $[\gamma, f]$) is called the left(resp. right) unity of S.

Now let S be a Γ-semigroup with unities and L and R are po-semigroups. We define a relation $\leq$ in S by $a \leq b$ if and only if $[a, \alpha] \leq [b, \alpha]$ in L and $[\alpha, a] \leq [\alpha, b]$ in R for all $\alpha \in \Gamma$. We also define a relation $\leq$ in Γ by $\alpha \leq \beta$ if and only if $[a, \alpha] \leq [a, \beta]$

in L and $[\alpha, a] \leq [\beta, a]$ in R for all $a \in S$. With respect to these relations S becomes a po-Γ-semigroup.

Definition 4.2. (see [28]) For a fuzzy subset μ of R we define a fuzzy subset μ^* of S by $\mu^*(a) = \inf\limits_{\gamma \in \Gamma} \mu([\gamma, a])$, where $a \in S$. For a fuzzy subset η of S we define a fuzzy subset $\eta^{*'}$ of R by $\eta^{*'}([\alpha, a]) = \inf\limits_{s \in S} \eta(s\alpha a)$, where $[\alpha, a] \in R$. For a fuzzy subset δ of L, we define a fuzzy subset $\delta^+)$ of S by $\delta^+(a) = \inf\limits_{\gamma \in \Gamma} \delta([a, \gamma])$, where $a \in S$. For a fuzzy subset ν of S we define a fuzzy subset $\nu^{+'})$ of L by $\nu^{+'}([a, \alpha]) = \inf\limits_{s \in S} \nu(a\alpha s)$, where $[a, \alpha] \in L$.

Proposition 4.3. *Let S be a po-Γ-semigroup with unities and L be its left operator po-semigroup and μ is a fuzzy ideal L. Then μ^+ is a fuzzy ideal of S.*

Proof. Let $a, b \in S$ and $\gamma \in \Gamma$. Then

$$\begin{aligned} \mu^+(a\gamma b) &= \inf_{\alpha \in \Gamma} \mu([a\gamma b, \alpha]) = \inf_{\alpha \in \Gamma} \mu([a, \gamma][b, \alpha]) \\ &\geq \inf_{\alpha \in \Gamma} \mu([b, \alpha]) (\text{since } \mu \text{ is a fuzzy ideal of } L) \\ &= \mu^+(b). \end{aligned}$$

Similarly we can show that $\mu^+(a\gamma b) \geq \mu^+(a)$. Let $a, b \in S$ be such that $a \leq b$. Then $[a, \alpha] \leq [b, \alpha]$ in L for all $\alpha \in \Gamma$. Then $\mu([a, \alpha]) \geq \mu([b, \alpha])$ for all $\alpha \in \Gamma$. Now

$$\mu^+(a) = \inf_{\alpha \in \Gamma} \mu([a, \alpha]) \geq \inf_{\alpha \in \Gamma} \mu([b, \alpha]) (\text{since } \mu \text{ is a fuzzy ideal of } L) = \mu^+(b).$$

Hence μ^+ is a fuzzy ideal of S. □

Proposition 4.4. *Let S be a po-Γ-semigroup with unities and L be its left operator po-semigroup and σ is a fuzzy ideal of S. Then $\sigma^{+'}$ is a fuzzy ideal of L.*

Proof. Let $[a, \alpha], [b, \beta] \in L$. Then

$$\begin{aligned} \sigma^{+'}([a, \alpha][b, \beta]) &= \sigma^{+'}([a\alpha b, \beta]) = \inf_{s \in S} \sigma(a\alpha b\beta s) = \inf_{s \in S} \sigma(a\alpha(b\beta s)) \\ &\geq \inf_{s \in S} \sigma(b\beta s) (\text{since } \sigma \text{is a fuzzy ideal of } S) \\ &= \sigma^{+'}([b, \beta]). \end{aligned}$$

Similarly we can show that $\sigma^{+'}([a, \alpha][b, \beta]) \geq \sigma^{+'}([a, \alpha])$. Let $[a, \alpha], [b, \alpha] \in L$ be such that $[a, \alpha] \leq [b, \alpha]$. Then $a \leq b$ in S, implies $a\alpha s \leq b\alpha s$ $\forall \alpha \in \Gamma, \forall s \in S$. So

$\sigma(a\alpha s) \geq \sigma(b\alpha s)\ \forall \alpha \in \Gamma, \forall s \in S$. Then

$$\begin{aligned}\sigma^{+'}([a,\alpha]) &= \inf_{s\in S} \sigma(a\alpha s)\\ &\geq \inf_{s\in S} \sigma(b\alpha s)(\text{since } \sigma \text{is a fuzzy ideal of } S)\\ &= \sigma^{+'}([b,\alpha]).\end{aligned}$$

Hence $\sigma^{+'}$ is a fuzzy ideal of L. □

Theorem 4.5. *Let S be a po-Γ-semigroup with unities and L be its left operator po-semigroup. Then there exist an inclusion preserving bijection $\mu \mapsto \mu^{+'}$ between the set of all fuzzy ideals of S and set of all fuzzy ideals of L.*

Proof. Let μ be a fuzzy ideal of a po-Γ-semigroup S. Then by Proposition 4.7, $\mu^{+'}$ is a fuzzy ideal of the po-semigroup L. So by Proposition 4.6, $(\mu^{+'})^{+}$ is a fuzzy ideal of S. From Theorem 5.13[28], it is clear that $(\mu^{+'})^{+} = \mu$ and $(\mu^{+})^{+'} = \mu$. Also the inclusion preserving property follows from Theorem 5.13[28]. Hence the proof. □

Remark 3. The right operator analogues of Propositions 4.6, 4.7 and Theorem 4.8 are also true.

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