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ON THE MAIN SCALARS OF A FIVE-DIMENSIONAL FINSLER SPACE

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Abstract: In this paper we have find the main scalars of the five-dimensional Finsler space.The h-connection vectors and v-connection vectors have been proposed also.Further, the v-scalar curvature S has been also obtained in terms of main scalars of five-dimensional Finsler space.

AMS Subject Classification: 53B40, 53C60
Key Words: Finsler space, main scalars, h-and v-connection vectors, scalar curvature

1. Introduction

A theory of intrinsic n-dimensional Finsler space has been studied by Matsumoto and Miron [6] and is called " Miron frame" by Matsumoto.The theory of two dimensional Finsler space wsa studied in[7] In three-dimensional Finsler space there are three main scalars H, I, J in which the sum of H and I is LC ([1], [2] , [3]) ,called unified main scalar whereas in four-dimensional Finsler spce there are eight main scalars $H, I, J, K, H', I', J', K'$ [8] in which the sum of H, I,and K is LC, called unified main scalar. There is one h-connection vector and one v-connection vector in three-dimensional Finsler space whereas in four-dimensional Finsler space there are three h-connection vectors and three v-connection vectors. The orthonormal frame field $\left(l^i, m^i, n^i, p^i, q^i\right)$, called the Miron frame plays an important role in five-dimensional Finsler space.

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2. Scalar Components in the Miron Frame

Let F^5 be a five-dimensional Finsler space with fundamental function $L(x,y)$. The metric tensor g_{ij} and Cartan C-tensor C_{ijk} of F^5 are defined by

$$g_{ij} = \frac{1}{2}\dot\partial_i\dot\partial_j L^2, \qquad C_{ijk} = \frac{1}{2}\dot\partial_k g_{ij} = \frac{1}{4}\dot\partial_i\dot\partial_j\dot\partial_k L^2.$$

Throughout the paper, the symbols $\dot\partial_i = \frac{\partial}{\partial y^i}$ and $\partial_i = \frac{\partial}{\partial x^i}$ have been used. The frame $\left\{e^i_{(\alpha)}\right\}, \alpha = 1,2,3,4,5$ is called the Miron frame of F^5, where $e^i_{(1)} = l^i = \frac{y^i}{L}$ is called the normalized supporting element, $e^i_{(2)} = m^i = \frac{C^i}{C}$ is called the normalized torsion vector, $e^i_{(3)} = n^i, e^i_{(4)} = p^i, e^i_{(5)} = q^i$ are constructed by $g_{ij}e^i_{(\alpha)}e^j_{(\beta)} = \delta_{\alpha\beta}$.Here, C is the length of torsion vector $C_i = C_{ijk}g^{jk}$.The Greek letters $\alpha,\beta,\gamma,\delta$ vary from 1 to 5 throughout the paper. Summation convention is applied for both the Greek and Latin indices. In the Miron frame an arbitrary tensor can be expressed by scalar components along the unit vectors l^i, m^i, n^i, p^i, q^i.For instance; let $T = T^i_j$ be a tensor field of $(1,1)$ type, then the scalar components $T_{\alpha\beta}$ of T are defined by $T_{\alpha\beta} = T^i_j e_{(\alpha)i}e^j_{(\beta)}$and the components T^i_j of the tensor T are expressed as [5] $T^i_j = T_{\alpha\beta}e^i_{(\alpha)}e_{(\beta)j}$.From the equation $g_{ij}e^i_{(\alpha)}e^j_{(\beta)} = \delta_{\alpha\beta}$, we have

$$g_{ij} = l_il_j + m_im_j + n_in_j + p_ip_j + q_iq_j \tag{2.1}$$

Next, the C-tensor $C_{ijk} = \frac{1}{2}\dot\partial_k g_{ij}$ satisfies $C_{ijk}l^i = 0$ and is symmetric in i, j, k.Therefore, if $C_{.\alpha\beta\gamma}$ *are* scalar components of LC_{ijk} i.e., if

$$LC_{ijk} = C_{\alpha\beta\gamma}e_{(\alpha)i}e_{(\beta)j}e_{(\gamma)k}, \tag{2.2}$$

then, we have

$$\begin{aligned} LC_{ijk} = {} & C_{222}m_im_jm_k + C_{223}\Pi_{(ijk)}(m_im_jn_k) + C_{233}\Pi_{(ijk)}(m_in_jn_k) \\ & +C_{333}\ (n_in_jn_k) + C_{224}\Pi_{(ijk)}(m_im_jp_k) + C_{444}\ (p_ip_jp_k) \\ & +C_{244}\Pi_{(ijk)}(m_ip_jp_k) + C_{225}\Pi_{(ijk)}(m_im_jq_k) \\ & +C_{255}\Pi_{(ijk)}(m_iq_jq_k) + C_{555}\ (q_iq_jq_k) + C_{334}\Pi_{(ijk)}(n_in_jp_k) \\ & +C_{344}\Pi_{(ijk)}(n_ip_jp_k) + C_{335}\Pi_{(ijk)}(n_in_jq_k) + C_{355}\Pi_{(ijk)}(n_iq_jq_k) \\ & +C_{445}\Pi_{(ijk)}(p_ip_jq_k) + C_{455}\Pi_{(ijk)}(p_iq_jq_k) \\ & +C_{234}\Pi_{(ijk)}\{m_i(n_jp_k + n_kp_j)\} + C_{235}\Pi_{(ijk)}\{m_i(n_jq_k + n_kq_j)\} \\ & +C_{245}\Pi_{(ijk)}\{m_i(p_jq_k + p_kq_j)\}+C_{345}\Pi_{(ijk)}\{n_i(p_jq_k + p_kq_j)\}, \end{aligned} \tag{2.3}$$

where $\Pi_{(ijk)}\{...\}$ denote the cyclic interchange of i, j, k and summation.

For instance;$\Pi_{(ijk)}\{A_iB_jC_k\} = A_iB_jC_k + A_jB_kC_i + A_kB_iC_j.$

Contracting (2.2) with g^{jk}, we get $LCm_i = C_{\alpha\beta\beta}e_{(\alpha)i}$.Thus, if we put

$$C_{222}=H, C_{233}=I, C_{244}=K, C_{333}=J, C_{344}=J', \tag{2.4}$$
$$C_{444}=H', C_{334}=I', C_{234}=K', C_{255}=M, C_{355}=J'',$$
$$C_{455}=M', C_{555}=H'', C_{335}=I'', C_{445}=K'', C_{235}=N,$$
$$C_{245}=N', C_{345}=M'',$$

then, we have

$$H+I+K+M=LC, C_{223}=-\left(J+J'+J''\right), \tag{2.5}$$
$$C_{224}=-\left(H'+I'+M'\right), C_{225}=-\left(\ H''+I''+K''\right).$$

Hence, we have the following:

3. Proposition

In a five-dimensional Finsler space there are seventeen main scalars $H, I, J, K, H', I', J', K', H'', I'', J'', K'', M, M', M'', N, N'$ in which the sum of H, I, K and M is LC which is called unified main scalar.

Using (2.4) and (2.5), the equation (2.3) can be rewritten as

$$\begin{aligned} LC_{ijk} = {} & Hm_im_jm_k \ -\left(J+J'+J''\right)\Pi_{(ijk)}\left(m_im_jn_k\right)+I\Pi_{(ijk)}\left(m_in_jn_k\right) \\ & +J\left(n_in_jn_k\right)-\left(H'+I'+M'\right)\ \Pi_{(ijk)}\left(m_im_jp_k\right)+H'\left(p_ip_jp_k\right) \\ & +K\Pi_{(ijk)}\left(m_ip_jp_k\right)-\left(H''+I''+K''\right)\Pi_{(ijk)}\left(m_im_jq_k\right) \\ & +M\Pi_{(ijk)}\left(m_iq_jq_k\right)+H''\left(q_iq_jq_k\right)+I'\Pi_{(ijk)}\left(n_in_jp_k\right) \\ & +J'\Pi_{(ijk)}\left(n_ip_jp_k\right)+I''\Pi_{(ijk)}\left(n_in_jq_k\right)+J''\Pi_{(ijk)}\left(n_iq_jq_k\right) \\ & +K''\Pi_{(ijk)}\left(p_ip_jq_k\right)+M'\Pi_{(ijk)}\left(p_iq_jq_k\right)+K'\Pi_{(ijk)}\left\{m_i\left(n_jp_k+n_kp_j\right)\right\} \\ & +\ N\Pi_{(ijk)}\left\{m_i\left(n_jq_k+n_kq_j\right)\right\}+N'\Pi_{(ijk)}\left\{m_i\left(p_jq_k+p_kq_j\right)\right\} \\ & +M''\Pi_{(ijk)}\left\{n_i\left(p_jq_k+p_kq_j\right)\right\}. \end{aligned} \tag{2.6}$$

The Cartan's connection $C\Gamma=\left(\Gamma^i_{jk}, G^i_j, C^i_{jk}\right)$ will be used in the following section of this paper. The h- and v- covariant derivatives of the frame field $e_{(\alpha)i}$ are given by [4]

$$e_{(\alpha)i|j=H_{(\alpha)\beta\gamma}}e_{(\beta)i}e_{(\gamma)j}, \qquad Le_{(\alpha)i|j}=V_{(\alpha)\beta\gamma}e_{(\beta)i}e_{(\gamma)j}, \tag{2.7}$$

where $H_{(\alpha)\beta\gamma}$ and $V_{(\alpha)\beta\gamma}$, γ being fixed, are given by

$$H_{\alpha)\beta\gamma}=\begin{matrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & h_\gamma & J_\gamma & k_\gamma \\ 0 & -h_\gamma & 0 & h'_\gamma & J'_\gamma \\ 0 & -J_\gamma & -h'_\gamma & 0 & k'_\gamma \\ 0 & -k_\gamma & -J'_\gamma & -k'_\gamma & 0 \end{matrix}, V_{\alpha)\beta\gamma}=\begin{matrix} 0 & \delta_{2\gamma} & \delta_{3\gamma} & \delta_{4\gamma} & \delta_{5\gamma} \\ -\delta_{2\gamma} & 0 & u_\gamma & v_\gamma & w_\gamma \\ -\delta_{3\gamma} & -u_\gamma & 0 & u'_\gamma & v'_\gamma \\ -\delta_{4\gamma} & -v_\gamma & -u'_\gamma & 0 & w'_\gamma \\ -\delta_{5\gamma} & -w_\gamma & -v'_\gamma & -w'_\gamma & 0 \end{matrix} \tag{2.8}$$

In (2.8), we have put

$$H_{2)3\gamma} = -H_{3)2\gamma} = h_\gamma, H_{2)4\gamma} = -H_{4)2\gamma} = J_\gamma, H_{2)5\gamma} = -H_{5)2\gamma} = k_\gamma,$$
$$H_{3)4\gamma} = -H_{4)3\gamma} = h'_\gamma, H_{3)5\gamma} = -H_{5)3\gamma} = J'_\gamma, H_{4)5\gamma} = -H_{5)4\gamma} = k'_\gamma,$$
$$V_{2)3\gamma} = -V_{3)2\gamma} = u_\gamma, V_{2)4\gamma} = -V_{4)2\gamma} = v_\gamma, V_{2)5\gamma} = -V_{5)2\gamma} = w_\gamma,$$
$$V_{3)4\gamma} = -V_{4)3\gamma} = u'_\gamma, V_{3)5\gamma} = -V_{5)3\gamma} = v'_\gamma, V_{4)5\gamma} = -V_{5)4\gamma} = w'_\gamma. \tag{2.9}$$

Hence, we have the following:

4. Proposition

In a five-dimensional Finsler space there exist six h-connection vectors $h_i, J_i, k_i, h'_i, J'_i, k'_i$ *whose scalar components with respect to the frame* $\left\{e^i_{(\alpha)}\right\}$ *are* $h_\gamma, J_\gamma, k_\gamma, h'_\gamma, J'_\gamma, k'_\gamma$ *i.e.,* $h_i = h_\gamma e_{(\gamma)i}, J_i = J_\gamma e_{(\gamma)i},\ k_i = k_\gamma e_{(\gamma)i},\ h'_i = h'_\gamma e_{(\gamma)i}, J'_i = J'_\gamma e_{(\gamma)i},\ k'_i = k'_\gamma e_{(\gamma)i}$.

5. Proposition

In a five-dimensional Finsler space there exist six v-connection vectors $u_i, v_i, w_i, u'_i, v'_i, w'_i$ *whose scalar components with respect to the frame* $\left\{e^i_{(\alpha)}\right\}$ *are* $u_\gamma, v_\gamma, w_\gamma, u'_\gamma, v'_\gamma, w'_\gamma$ *i.e.,*$u_i = u_\gamma e_{(\gamma)i}, v_i = v_\gamma e_{(\gamma)i}, w_i = w_\gamma e_{(\gamma)i},\ u'_i = u'_\gamma e_{(\gamma)i}, v'_i = v'_\gamma e_{(\gamma)i},\ w'_i = w'_\gamma e_{(\gamma)i}$.

In view of equations(2.8), (2.9) and using the propositions (2.2) and (2.3), the equations (2.7) may be explicitly written as

(2.10) a
$$l_{i|j} = 0, m_{i|j} = n_i h_j + p_i J_j + q_i k_j, n_{i|j} = -m_i h_j + p_i h'_j + q_i J'_j,$$
$$p_{imidj} = -m_i J_j - n_i h'_j + q_i k'_j, q_{i|j} = -m_i k_j - n_i J'_j - p_i k'_j, \qquad \text{and}$$

(2.10) b
$$Ll_{i|j} = m_i m_j + n_i n_j + p_i p_j + q_i q_j = g_{ij} - l_i l_j = h_{ij},$$
$$Lm_{i|j} = -l_i m_j + n_i u_j + p_i v_j + q_i w_j,$$
$$Ln_{i|j} = -l_i n_j - m_i u_j + p_i u'_j + q_i v'_j,$$
$$Lp_{i|j} = -l_i p_j - m_i v_j - n_i u'_j + q_i w'_j,$$
$$Lq_{i|j} = -l_i q_j - m_i w_j - n_i v'_j - p_i w'_j.$$

Since m_i, n_i, p_i, q_i are homogeneous functions of degree zero in y^i, we have
$$Lm_{i|j}l^j = Ln_{i|j}l^j = Lp_{i|j}l^j = Lq_{i|j}l^j = 0$$
which in view of equation (2.10)b and proposition (2.3) gives

$$u_1 = v_1 = w_1 = u'_1 = v'_1 = w'_1 = 0. \tag{2.11}$$

The h-scalar derivative of the adopted components $T_{\alpha\beta}$ of the tensor T^i_j of $(1,1)$ type is defined as[4]

$$T_{\alpha\beta,\gamma} = (\delta_k T_{\alpha\beta})\, e^k_{\gamma)} + T_{\mu\beta} H_{\mu)\alpha\gamma} + T_{\alpha\mu} H_{\mu)\beta\gamma}, \tag{2.12}$$

where $\delta_k = \partial_k - G^r_k \dot{\partial}_r$.

Similarly, the v-scalar derivative of the adopted components $T_{\alpha\beta}$ of the tensor T^i_j of $(1,1)$ type is defined as [4]

$$T_{\alpha\beta;\gamma} = L\,(\dot{\partial}_k T_{\alpha\beta})\, e^k_{\gamma)} + T_{\mu\beta} V_{\mu)\alpha\gamma} + T_{\alpha\mu} V_{\mu)\beta\gamma}. \tag{2.13}$$

Thus, $T_{\alpha\beta,\gamma}$ and $T_{\alpha\beta;\gamma}$ are the adopted components of $T^i_{j|k}$ and $T^i_j\,|_k$ respectively i.e.,

$$T^i_{j|k} = T_{\alpha\beta,\gamma} e^i_{(\alpha)} e_{(\beta)j} e_{(\gamma)k}, \; L T^i_j\,|_k = T_{\alpha\beta;\gamma} e^i_{(\alpha)} e_{(\beta)j} e_{(\gamma)k}. \tag{2.14}$$

From (2.2) it follows that

$$L^2 C_{hij}\,|_k = (C_{\alpha\beta\gamma;\delta} - C_{\alpha\beta\gamma}\delta_{1\delta})\, e_{\alpha)h} e_{\beta)i} e_{\gamma)j} e_{\delta)k}. \tag{2.15}$$

The explicit form of $C_{\alpha\beta\gamma;\delta}$ is obtained as

(2.16)

(a) $C_{1\beta\gamma;\delta} = -C_{\beta\gamma\delta}$,

(b) $C_{222;\delta} = H;\delta + 3\,(J + J' + J'')\, u_\delta + 3\,(H' + I' + M')\, v_\delta + 3\,(H'' + I'' + K'')\, w_\delta$,

(c) $C_{223;\delta} = -\,(J + J' + J'')\,;\delta + (H - 2I)\, u_\delta - 2K' v_\delta - 2N w_\delta + (H' + I' + M')\, u'_\delta + (H'' + I'' + K'')\, v'_\delta$,

(d) $C_{224;\delta} = -\,(H' + I' + M')\,;\delta - 2K' u_\delta + (H - 2K)\, v_\delta - 2N w_\delta - (J + J' + J'')\, u'_\delta + (H'' + I'' + K'')\, w'_\delta$,

(e) $C_{225;\delta} = -\,(H'' + I'' + K'')\,;\delta - 2N u_\delta - 2N' v_\delta + (H - 2M)\, w_\delta - (J + J' + J'')\, v'_\delta - (H' + I' + M')\, w'_\delta$,

(f) $C_{233;\delta} = I;\delta - (3J + 2J' + 2J'')\, u_\delta - I' v_\delta - I'' w_\delta - 2K' u'_\delta - 2N v'_\delta$,

(g) $C_{234;\delta} = K';\delta - (I' - K')\, u_\delta - (J' - K')\, v_\delta - M'' w_\delta - (K - I)\, u'_\delta - N' v'_\delta - N w'_\delta$,

(h) $C_{235;\delta} = N;\delta - (2I'' + K'' + H'')\, u_\delta - M'' v_\delta - (J + J' + 2J'')\, w_\delta - N' u'_\delta - (M - I)\, v'_\delta + K' w'_\delta$,

(i) $C_{244;\delta} = K;\delta - J' u_\delta - (3H' + 2I' + 2M') v_\delta + 2K' v'_\delta - (K'' + 2N')\, w'_\delta$,

(j) $C_{245;\delta} = N';\delta - M'' u_\delta - (H'' + I'' + 2K'') v_\delta + N u'_\delta - (H' + I' + 2M')\, w_\delta + K' v'_\delta - M w'_\delta$,

$(k)\ C_{255;\delta} = M;\delta - J''u_\delta - M'v_\delta - (3H'' + 2I'' + 2K'')\,w_\delta + 2Nv'_\delta\ + 2N'w'_\delta,$

$(l)\ C_{333;\delta} = J;\delta + 3\,[Iu_\delta - I'u'_\delta - I''v'_\delta]\,,$

$(m)\ C_{334;\delta} = I';\delta + 2K'u_\delta + Iv_\delta + (J - 2J')\,u'_\delta - 2M''v'_\delta - I''w'_\delta,$

$(n)\ C_{335;\delta} = I'';\delta + 2Nu_\delta - 2M''u'_\delta + (J - 2J')\,v'_\delta + Iw_\delta + I'w'_\delta,$

$(o)\ C_{344;\delta} = J';\delta + Ku_\delta + 2K'v_\delta -\ (H' - 2I')\,u'_\delta - K''v'_\delta\ - 2M''w'_\delta,$

$(p)\ C_{345;\delta}\ =\ M'';\delta + N'u_\delta + Nv_\delta +\ (I'' - K'')\,u'_\delta + K'w_\delta + (I' - M')\,v'_\delta + (J' - J'')\,w'_\delta,$

$(q)\ C_{355;\delta} = J'';\delta + Mu_\delta\ -\ M'u'_\delta + 2Nw_\delta - (H'' - 2I'')\,v'_\delta + 2M''w'_\delta,$

$(r)\ C_{444;\delta} = H';\delta + 3\,[Kv_\delta + J'u'_\delta - K''w'_\delta]\ ,$

$(s)\ C_{445;\delta} = K'';\delta + 2N'v_\delta + 2M''u'_\delta + Kw_\delta\ + J'v'_\delta + (H' - 2M')\,w'_\delta,$

$(t)\ C_{455;\delta} = M';\delta + Mv_\delta + J''u'_\delta + 2N'w_\delta\ + 2M''v'_\delta - (H' - 2K'')\,w'_\delta,$

$(u)\ C_{555;\delta} = H'';\delta + 3\,[Mw_\delta + J''v'_\delta + M'w'_\delta]\ ,$

where $H;\delta$ for instant; is the v-scalar derivative of the single scalar H., namely $H;\delta = L\,(\dot\partial_i H)\,e^i_{(\delta)}$.The tensor $C_{hij}\ |_k$is completely symmetric.Accordingly (2.13) yields

$$C_{\alpha\beta\gamma;\delta} + C_{\alpha\beta\delta;\gamma}\ = C_{\alpha\beta\gamma}\delta_{1\delta} - C_{\alpha\beta\delta}\delta_{1\gamma}.$$

This equation is explicitly written as

(2.17)

$(a)\quad -(J + J' + J'')\,;2{+}(H - 2I)\,u_2{-}2K'v_2{-}2Nw_2{+}(H' + I' + M')\,u'_2{+}(H'' + I'' + K'')\,v'_2 = H;3 + 3\,(J + J' + J'')\,u_3 + 3\,(H' + I' + M')\,v_3 +$

$3\,(H'' + I'' + K'')\,w_3$

$(b)\quad I;2 - (3J + 2J' + 2J'')\,u_2 - I'v_2 - I''u'_2 - 2K'w_2 - 2Nv'_2 = -(J + J' + J'')\,;3 + (H - 2I)\,u_3 - 2K'v_3 - 2Nu'_3 + (H' + I' + M')\,w_3 +$

$(H'' + I'' + K'')\,v'_3,$

$(c)\ K';2{-}(I' - K')\,u_2{-}(J' - K')\,v_2{-}M''w_2{-}(K - I)\,u'_2{-}N'v'_2{-}Nw'_2 = -(H' + I' + M')\,;3{-} 2K'u_3 + (H - 2K)\,v_3 - 2Nw_3 - (J + J' + J'')\,u'_3$

$+ (H'' + I'' + K'')\,w'_3\ =\ -(J + J' + J'')\,;4 + (H - 2I)\,u_4 - 2K'v_4 - 2Nw_4 + (H' + I' + M')\,u'_4 + (H'' + I'' + K'')\,v'_4,$

$(d)\quad N;2 - (2I'' + K'' + H'')\,u_2 - M''v_2 - (J + J' + 2J'')\,w_2 - N'u'_2 - (M - I)\,v'_2 + K'w'_2 = -(H'' + I'' + K'')\,;3 - 2Nu_3 - 2N'v_3 + (H - 2M)\,w_3$

$-(J + J' + J'')\,v'_3{-}(H' + I' + M')\,w'_3 = -(J + J' + J'')\,;5{+}(H - 2I)\,u_5{-} 2K'v_5 - 2Nw_5 + (H' + I' + M')\,u'_5 + (H'' + I'' + K'')\,v'_5,$

(e) $J;2+3\left[Iu_2-I'u'_2-I''v'_2\right]=I;3-\left(3J+2J'+2J''\right)u_3-I'v_3-I''w_3-2K'u'_3-2Nv'_3,$

(f) $I';2+2K'u_2+Iv_2+\left(J-2J'\right)u'_2-2M''v'_2-I''w'_2=K';3-\left(I'-K'\right)u_3-\left(J'-K'\right)v_3-M''w_3-\left(K-I\right)u'_3-N'v'_3-Nw'_3=I;4-$

$$\left(3J+2J'+2J''\right)u_4-I'v_4-I''w_4-2K'u'_4-2Nv'_4,$$

(g) $I'';2+2Nu_2-2M''u'_2+\left(J-2J'\right)v'_2+Iw_2+I'w'_2=N;3-\left(2I''+K''+H''\right)u_3-M''v_3-\left(J+J'+2J''\right)w_3-N'u'_3-\left(M-I\right)v'_3+K'w'_3$

$$=I;5-\left(3J+2J'+2J''\right)u_5-I'v_5-I''w_5-2K'u'_5-2Nv'_5,$$

(h) $J';2+Ku_2+2K'v_2-\left(H'-2I'\right)u'_2-K''v'_2-2M''w'_2=K;3-J'u_3-(3H'+2I'+2M')v_3+2K'v'_3-\left(K''+2N'\right)w'_3=K';4-$

$$\left(I'-K'\right)u_4-\left(J'-K'\right)v_4-M''w_4-\left(K-I\right)u'_4-N'v'_4-Nw'_4,$$

(i) $M'';2+N'u_2+Nv_2+\left(I''-K''\right)u'_2+K'w_2+\left(I'-M'\right)v'_2+\left(J'-J''\right)w'_2=N';3-M''u_3-(H''+I''+2K'')v_3+Nu'_3-\left(H'+I'+2M'\right)w_3$

$+K'v'_3-Mw'_3=N;4-\left(2I''+K''+H''\right)u_4-M''v_4-\left(J+J'+2J''\right)w_4-N'u'_4-\left(M-I\right)v'_4+K'w'_4=K';5-\left(I'-K'\right)u_5-\left(J'-K'\right)v_5$
$-M''w_5-\left(K-I\right)u'_5-N'v'_5-Nw'_5,$

(j) $J'';2+Mu_2-M'u'_2+2Nw_2-\left(H''-2I''\right)v'_2+2M''w'_2=M;3-J''u_3-M'v_3-\left(3H''+2I''+2K''\right)w_3+2Nv'_3+2N'w'_3=N;5-$

$$\left(2I''+K''+H''\right)u_5-M''v_5-\left(J+J'+2J''\right)w_5-N'u'_5-\left(M-I\right)v'_5+K'w'_5,$$

(k) $-\left(H'+I'+M'\right);2-2K'u_2+\left(H-2K\right)v_2-2N'w_2-\left(J+J'+J''\right)u'_2+\left(H''+I''+K''\right)w'_2=H;4+3\left(J+J'+J''\right)u_4+3\left(H'+I'+M'\right)v_4$

$$+3\left(H''+I''+K''\right)w_4,$$

(l) $K;2-J'u_2-(3H'+2I'+2M')v_2+2K'v'_2-\left(K''+2N'\right)w'_2=-\left(H'+I'+M'\right);4-2K'u_4+\left(H-2K\right)v_4-2Nw_4-\left(J+J'+J''\right)u'_4+$

$$\left(H''+I''+K''\right)w'_4,$$

(m) $H';2+3\left[Kv_2+J'u'_2-K''w'_2\right]=K;4-J'u_4-(3H'+2I'+2M')v_4+2K'v'_4-\left(K''+2N'\right)w'_4,$

(n) $K'';2+2N'v_2+2M''u'_2+Kw_2+J'v'_2+\left(H'-2M'\right)w'_2=N';4-M''u_4-(H''+I''+2K'')v_4+Nu'_4-\left(H'+I'+2M'\right)w_4+K'v'_4-$

$Mw_4' = K;5 - J'u_5 - (3H' + 2I' + 2M')v_5 + 2K'v_5' - (K'' + 2N')\,w_5',$

(o) $M';2 + Mv_2 + J''u_2' + 2N'w_2 + 2M''v_2' - (H' - 2K'')\,w_2' = M;4 - J''u_4 - M'v_4 - (3H'' + 2I'' + 2K'')\,w_4 + 2Nv_4' + 2N'w_4' = N';5-$

$M''u_5 - (H'' + I'' + 2K'')v_5 + Nu_5' - (H' + I' + 2M')\,w_5 + K'v_5' - Mw_5',$

(p) $I';3 + 2K'u_3 + Iv_3 + (J - 2J')\,u_3' - 2M''v_3' - I''w_3' = J;4 + 3\,[Iu_4 - I'u_4' - I''v_4'],$

(q) $J';3 + Ku_3 + 2K'v_3 - (H' - 2I')\,u_3' - K''v_3' - 2M''w_3' = I';4 + 2K'u_4 + Iv_4 + (J - 2J')\,u_4' - 2M''v_4' - I''w_4',$

(r) $M'';3 + N'u_3 + Nv_3 + (I'' - K'')\,u_3' + K'w_3 + (I' - M')\,v_3' + (J' - J'')\,w_3' = I'';4 + 2Nu_4 - 2M''u_4' + (J - 2J'')\,v_4' + Iw_4 + I'w_4'$

$= I';5 + 2K'u_5 + Iv_5 + (J - 2J')\,u_5' - 2M''v_5' - I''w_5',$

(s) $H';3 + 3\,[Kv_3 + J'u_3' - K''w_3'] = J';4 + Ku_4 + 2K'v_4 - (H' - 2I')\,u_4' - K''v_4' - 2M''w_4',$

(t) $K'';3 + 2N'v_3 + 2M''u_3' + Kw_3 + J'v_3' + (H' - 2M')\,w_3' = M'';4 + N'u_4 + Nv_4 + (I'' - K'')\,u_4' + K'w_4 + (I' - M')\,v_4' + (J' - J'')\,w_4'$

$= J';5 + Ku_5 + 2K'v_5 - (H' - 2I')\,u_5' - K''v_5' - 2M''w_5',$

(u) $M';3 + Mv_3 + J''u_3' + 2N'w_3 + 2M''v_3' - (H'' - 2K'')\,w_3' = J'';4 + Mu_4 - M'u_4' + 2Nw_4 - (H'' - 2I'')\,v_4' + 2M''w_4' = M'';5 + N'u_5 +$

$Nv_5 + (I'' - K'')\,u_5' + K'w_5 + (I' - M')\,v_5' + (J' - J'')\,w_5',$

(v) $-(H'' + I'' + K'');2 - 2Nu_2 - 2N'v_2 + (H - 2M)\,w_2 - (J + J' + J'')\,v_2' - (H' + I' + M')\,w_2' = H;5 + 3\,(J + J' + J'')\,u_5 + 3\,(H' + I' + M')\,v_5$

$+3\,(H'' + I'' + K'')\,w_5,$

(w) $N';2 - M''u_2 - (H'' + I'' + 2K'')v_2 + Nu_2' - (H' + I' + 2M')\,w_2 + K'v_2' - Mw_2' = -(H'' + I'' + K'');4 - 2Nu_4 - 2N'v_4 + (H - 2M)\,w_4 -$

$(J + J' + J'')\,v_4' - (H' + I' + M')\,w_4' = -(H' + I' + M');5 - 2K'u_5 + (H - 2K)\,v_5 - 2Nw_5 - (J + J' + J'')\,u_5' + (H'' + I'' + K'')\,w_5',$

(x) $M;2 - J''u_2 - M'v_2 - (3H'' + 2I'' + 2K'')\,w_2 + 2Nv_2' + 2N'w_2' = -(H'' + I'' + K'');5 - 2Nu_5 - 2N'v_5 + (H - 2M)\,w_5 - (J + J' + J'')\,v_5' -$

$(H' + I' + M')\,w_5',$

(y) $H'';2+3\left[Mw_2+J''v'_2+M'w'_2\right]=M;5-J''u_5-M'v_5-(3H''+2I''+2K'')\,w_5+2Nv'_5\ +2N'w'_5,$

(z) $I'';3+2Nu_3-2M''u'_3+(J-2J')\,v'_3+Iw_3+I'w'_3=J;5+3\left[Iu_5-I'u'_5-I''v'_5\right],$

(A) $J'';3+Mu_3\ -\ M'u'_3+2Nw_3-(H''-2I'')\,v'_3+2M''w'_3\ =\ I'';5+2Nu_5-2M''u'_5+(J-2J'')\,v'_5+Iw_5+I'w'_5,$

(B) $H'';3+3\left[Mw_3+J''v'_3+M'w'_3\right]=J'';5+Mu_5-M'u'_5+2Nw_5-(H''-2I'')\,v'_5+2M''w'_5,$

(C) $K'';4+2N'v_4+2M''u'_4+Kw_4+J'v'_4+(H'-2M')\,w'_4=H';5+3\left[Kv_5+J'u'_5-K''w'_5\right],$

(D) $M';4+Mv_4+J''u'_4+2N'w_4+2M''v'_4-(H'-2K'')\,w'_4\ =\ K'';5+2N'v_5+2M''u'_5+Kw_5+J'v'_5+(H'-2M')\,w'_5,$

(E) $H'';4+3\left[Mw_4+J''v'_4+M'w'_4\right]\ =\ M';5+Mv_5+J''u'_5+2N'w_5+2M''v'_5-(H'-2K'')\,w'_5.$

The v-curvature tensor S_{hijk} of $C\Gamma$ is defined by $S_{hijk}=C_{hkr}C^r_{ij}-C_{hjr}C^r_{ik}$.If $S_{\alpha\beta\gamma\delta}$ are scalar components of L^2S_{hijk} i.e.,if

$L^2S_{hijk}=L^2S_{\alpha\beta\gamma\delta}e_{(\alpha)h}e_{(\beta)i}e_{(\gamma)j}e_{(\delta)k},$

then from (2.2) it follows that

$$S_{\alpha\beta\gamma\delta}=C_{\alpha\beta\theta}C_{\beta\gamma\theta}-C_{\alpha\gamma\theta}C_{\beta\delta\theta} \tag{2.18}$$

The v-Ricci tensor $S._{hj}$ is defined as $S_{hj}=S_{hijk}g^{ik}$. Therefore, the scalar components of L^2S_{hj} are given by $S_{\alpha\beta\gamma\delta}$.

Hence, in view of (2.18) it follows that the v-scalar curvature

$S=\frac{1}{2}L^2S_{hj}g^{hj}$ are given by

$L^2S=C_{\alpha\beta\theta}C_{\alpha\beta\theta}-C_{\alpha\alpha\theta}C_{\beta\beta\theta}.$

In terms of main scalars, this equation gives

$$S=\left(K^2+I^2+M^2-HI-HK-KI-HM-MI-MK\right)+2\left(J^2+H'^2+H''^2\right)$$
$$+3\begin{pmatrix} I'^2+J'^2+M'^2+K'^2+N^2+N'^2+J''^2 \\ +I''^2+K''^2+M''^2+JJ'+JJ''+J'J''+H'I'+M'I'+H'M' \\ I''K''+K''H''+I''K'' \end{pmatrix} \tag{2.19}$$

6. Theorem

The v-scalar curvature S in terms of main scalars is given by (2.19) .

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