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EXISTENCE OF TRAVELLING WAVES FOR A CLASS OF INTEGRO-DIFFERENTIAL EQUATIONS FROM POPULATION DYNAMICS

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Abstract: Existence of travelling waves solutions is studied for a class of integro-differential equations arising in population dynamics. The proof of wave existence is based on the Leray-Schauder method which uses topological degree and a priori estimates of the solution.

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1. Introduction

In this paper we study the existence of travelling wave solutions for integro-differential equations of the form

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + F\left(u, J\left(u\right)\right), \tag{1.1}$$

where $J\left(u\right)$ is the nonlocal term,

$$J\left(u\right) = \int_{-\infty}^{\infty} \phi\left(x-y\right) u\left(y,t\right) dy, \tag{1.2}$$

$\phi : \mathbb{R} \to \mathbb{R}$ is a bounded nonnegative function with a compact support, $\int_{-\infty}^{\infty} \phi\left(y\right) dy =$

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$$F(w, J(w)) = f(w) J(w) - g(w).$$

The properties of the functions $f, g : \mathbb{R} \to \mathbb{R}$ will be specified below. Equations of this type arise in population dynamics, where u denotes the density of the population and the integral $J(u)$ shows that the interaction of the individuals can be nonlocal [4], [6].

Denote $F_0(w) = F(w, w) = wf(w) - g(w)$. Suppose that $f, g \in C^1(\mathbb{R})$ and that equation $F_0(w) = 0$ has three solutions, $w^{\pm}$ and w^*, $w^+ < w^* < w^-$, with $F_0'(w^+) < 0$, $F_0'(w^-) < 0$, $F_0'(w^*) > 0$. This means that, regarded as stationary solutions of the ODE $dw/dt = wf(w) - g(w)$, w^+, w^- are stable, while w^* is unstable. Therefore,

$$\begin{cases} F_0'(w^{\pm}) = f(w^{\pm}) + w^{\pm} f'(w^{\pm}) - g'(w^{\pm}) < 0, \\ F_0'(w^*) = f(w^*) + w^* f'(w^*) - g'(w^*) > 0. \end{cases} \tag{1.3}$$

According to the stability of the points w^+ and w^- this case is called bistable. It is known that the reaction-diffusion equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + F_0(u)$$

has a travelling wave solution, in this case, with the limits $w(\pm\infty) = w^{\pm}$ at $\pm\infty$. Moreover, this solution is unique up to a translation and it is globally stable [7]. In the monostable case, where one of the points w^+ and w^- is unstable, the wave is not unique.

The same classification is used for the integro-differential equations. Equation (1.1) in the bistable case with $f(w) = w(1 - w)$ and $g(w) = w$ is studied in [3]. Another bistable case was analyzed in [1], for functions ϕ with narrow support.

A travelling wave solution of the nonlocal reaction-diffusion equation (1.1) has the form $u(x, t) = w(x - ct)$, where the constant $c \in \mathbb{R}$ is the wave speed and should be found together with the function w. They satisfy the equation

$$w'' + cw' + F(w, J(w)) = 0. \tag{1.4}$$

We are interested in solutions of equation (1.4) with the limits

$$w(\pm\infty) = w^{\pm}, \tag{1.5}$$

where the values $w^{\pm}$ are such that $F(w^{\pm}, w^{\pm}) = 0$.

We will use conventional Hölder spaces $E = C^{2+\alpha}(\mathbb{R})$, $E^0 = C^{\alpha}(\mathbb{R})$, with $0 < \alpha < 1$, and weighted Hölder spaces. Let $\mu(x) = 1 + x^2$ and E_μ, E_μ^0 be the corresponding weighted spaces endowed with the norms $||u||_\mu = ||\mu u||_E$ and $||u||_{0\mu} = ||\mu u||_{E^0}$.

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We need weighted spaces in order to define a topological degree in unbounded domains where the Leray-Schauder degree is not applicable [5]. Since any function $u \in E_\mu$ has the zero limit at $\pm\infty$, we are looking for the solutions w of (1.4) under the form $w = u + \psi$, where $\psi \in C^\infty(\mathbb{R})$ is chosen such that $\psi(x) = w^+$ for $x \geq 1$ and $\psi(x) = w^-$ for $x \leq -1$. Thus equation (1.4) can be written as

$$(u+\psi)'' + c(u+\psi)' + F(u+\psi, J(u+\psi)) = 0. \tag{1.6}$$

Denote by A the operator in the left-hand side of (1.6), that is $A : E_\mu \to E_\mu^0$,

$$Au = (u+\psi)'' + c(u+\psi)' + F(u+\psi, J(u+\psi)). \tag{1.7}$$

In order to prove the existence of travelling waves solutions of problem (1.1) we will employ the Leray-Schauder method. To do this, one combines the topological degree constructed in [2] with a priori estimates in weighted spaces which we deduce in the sequel, both for the function w and the wave velocity $c(\tau)$.

Let us briefly recall the Leray-Schauder method. Let $\mathcal{E}$, $\mathcal{E}_0$ be two Banach spaces, $A : \mathcal{E} \to \mathcal{E}_0$ be a given operator, and let Ω be a bounded domain from the functions space $\mathcal{E}$. To study the existence of solutions of the equation $Au = 0$ in Ω, the following property of the degree is used:

If there exists a topological degree $\gamma(A, \Omega)$ *associated to* A *and* Ω*, and it is different from zero,* $\gamma(A, \Omega) \neq 0$*, then the equation* $Au = 0$ *admits at least one solution in the domain* Ω.

We construct a continuous deformation of the operator, A_τ, $\tau \in [0,1]$, such that for $\tau = 1$, $A_1 = A$ (our operator); for $\tau = 0$, A_0 is a simple operator, whose topological degree $\gamma(A_0, \Omega)$ can be determined and $\gamma(A_0, \Omega) \neq 0$. By the definition of topological degree, if the equation $A_\tau u = 0$ does not admit solutions at the boundary $\partial\Omega$ of Ω, then the topological degree does not depend on τ, i.e. $\gamma(A_1, \Omega) = \gamma(A_0, \Omega)$. So $\gamma(A, \Omega) \neq 0$. From the above property it follows that the equation $Au = 0$ has at least one solution in Ω.

To verify that equation $A_\tau u = 0$ does not admit solutions on $\partial\Omega$, it suffices to prove a priori estimates: we show that for any solution u of this equation, we have $||u||_E < R$ for some $R > 0$. This means that the equation $A_\tau u = 0$ has not solutions on the boundary of the ball $B(0, R) \subset E$, neither outside it. If $B(0, R) \subseteq \Omega$, then equation $A_\tau u = 0$ does not admit solutions on the boundary $\partial\Omega$ of Ω.

The structure of the paper is the following. In the next section we introduce a class of homotopies A_τ, $\tau \in [0,1]$ and obtain some auxiliary results. The boundedness and the sign of the wave speed are studied in Section 3. A priori estimates of the solutions of (1.4) are obtained in Section 4. The main result on the existence of travelling waves is also proved here.

2. Estimates of the Solutions

In [2], the Fredholm property and the properness of the integro-differential operator A depending on a parameter $\tau \in [0,1]$, have been studied in weighted Hölder spaces. Denote by A_τ the operator A for c and J depending on $\tau \in [0,1]$, that is $A_\tau : E_\mu \to E_\mu^0$,

$$A_\tau u = (u+\psi)'' + c(\tau)(u+\psi)' + F(u+\psi, J_\tau(u+\psi)). \tag{2.1}$$

The properness of A_τ and its Fredholm property in the weighted spaces E_μ and E_μ^0 have allowed one to define the topological degree in [2]. These results will be applied in the present paper to the study of travelling wave solutions of equation (1.1) in the bistable case, for $F(w, J(w)) = f(w) J(w) - g(w)$. In (2.1) we take $F(w, J_\tau(w)) = f(w) J_\tau(w) - g(w)$, where

$$J_\tau(w) = \int_{-\infty}^{\infty} \phi_\tau(x-y) u(y) dy, \ \phi_\tau(x) = \frac{(\varepsilon_0 - 1)\tau + 1}{\varepsilon_0} \phi\left(\frac{(\varepsilon_0 - 1)\tau + 1}{\varepsilon_0} x\right), \tag{2.2}$$

for $\varepsilon_0 > 0$ small enough. Remark that for $\tau = 1$, $A_1 u \equiv Au$. We study here the problem

$$w'' + c(\tau) w' + f(w) J_\tau(w) - g(w) = 0, \tag{2.3}$$

$$\lim_{x\to\pm\infty} w(x) = w^\pm. \tag{2.4}$$

Suppose that $f, g \in C^1(\mathbb{R})$ and that (1.3) holds. The linearization of A_τ about a function $u_1 \in E_\mu$ is the operator $L_\tau : E_\mu \to E_\mu^0$,

$$L_\tau u \equiv A'_\tau(u_1) u = u'' + c(\tau) u' + \frac{\partial F}{\partial u}(u_1+\psi, J_\tau(u_1+\psi)) u + \frac{\partial F}{\partial U}(u_1+\psi, J_\tau(u_1+\psi)) J_\tau(u), \tag{2.5}$$

where $\partial F/\partial u$ and $\partial F/\partial U$ are the derivatives of $F(u,U)$ with respect to the first and to the second variable, respectively.

For the linearized operator L_τ, we introduce the limiting operators. For $w_1 = u_1 + \psi$ there exist the limits $\lim_{x\to\pm\infty} w_1(x) = w^\pm$, hence $J_\tau(w_1) = J_\tau(u_1+\psi) \to w^\pm$ as $x \to \pm\infty$ and the limiting operators associated to L_τ are given by

$$L_\tau^\pm u = u'' + c(\tau) u' + a^\pm u + b^\pm J_\tau(u), \tag{2.6}$$

where

$$a^\pm = \frac{\partial F}{\partial u}(w^\pm, w^\pm) = f'(w^\pm) w^\pm - g'(w^\pm), \ b^\pm = \frac{\partial F}{\partial U}(w^\pm, w^\pm) = f(w^\pm).$$

In this section, we give some auxiliary results on the solutions of problem (2.3) − (2.4). First we recall a result for the linear parabolic problem

$$\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2} + a(x,t) J_\tau(v) + b(x,t) v, \ v(x,0) = v_0(x), \ x \in \mathbb{R}. \tag{2.7}$$

Lemma 1. (see [3]) *Assume that* a *and* b *are continuous functions such that* $a(x,t) \geq 0$, *for all* $x, t \in \mathbb{R}$ *and* $a(x,t) + b(x,t) < 0$ *for* $|x|$ *large enough and* $t \in [0,T]$, *for some* $T > 0$. *If* $v_0(x) \geq 0$ *on* $\mathbb{R}$ *and* $v_0 \neq 0$ *(not identically* 0*), then* $v(x,t) > 0$ *for any* $x \in \mathbb{R}$ *and* $t \in [0,T]$.

One now proves that every nonincreasing solution of (2.3)−(2.4) is in fact strictly decreasing on $\mathbb{R}$.

Lemma 2. *One supposes that* $f(w) \geq 0$, $(\forall)\, w \in [w^+, w^-]$ *and* (1.3) *holds. Let* w_0 *be a non-constant and nonincreasing on* $\mathbb{R}$ *solution of problem* (2.3) − (2.4). *Then* w_0 *is strictly decreasing on* $\mathbb{R}$.

Proof. One differentiates (2.3), uses the change of function $w' = -v$ and employs the equality $(J_\tau(w))' = J_\tau(w')$ to deduce that

$$\begin{aligned} v'' + c(\tau) v' + a(x) J_\tau(v) + b(x) v = 0, \\ a(x) = f(w), \\ b(x) = J_\tau(w) f'(w) - g'(w). \end{aligned} \tag{2.8}$$

Here $a = f(w) \geq 0$, $(\forall)\, w \in [w^+, w^-]$ and by (1.3), $a(x) + b(x) = f(w) + J_\tau(w) f'(w) - g'(w) < 0$ for $|x|$ large enough.

Let v_0 be a solution of (2.8), $v_0 \geq 0$, $v_0 \neq 0$. We show that $v_0' > 0$ on $\mathbb{R}$. Applying Lemma 1 for problem (2.7) with the initial condition v_0 above and observing that any travelling wave of (2.7) is a solution of (2.8), it follows that $v_0 > 0$ on $\mathbb{R}$, i. e. $w_0' < 0$ on $\mathbb{R}$, as claimed.

Lemma 3. *Assume that* $f(w) \geq 0$ *for* $w \in [w^+, w^-]$. *Let* w_j *be a sequence of solutions of problem* (2.3) − (2.4) *such that* $w_j \to w_0$ *in* $C^1(\mathbb{R})$, *with* $w^+ < w_0(x) < w^-$ *and* $w_0'(x) \leq 0$, $(\forall)\, x \in \mathbb{R}$. *Then for* j *large enough, we have* $w^+ < w_j(x) < w^-$ *and* $w_j'(x) < 0$, $(\forall)\, x \in \mathbb{R}$.

Proof. Let us first show that $w^+ < w_j(x)$, $(\forall)\, x \in \mathbb{R}$, j large enough. Supposing by contradiction that this does not happen, it follows that $W_j(x) = w_j(x) - w^+$ has a negative minimum in some point x_j. Thus

$$W_j(x_j) < 0, \ W_j'(x_j) = 0, \ W_j''(x_j) \geq 0.$$

Moreover, by the uniform convergence $W_j \to w_0 - w^+$ (as $j \to \infty$) and $w_0 - w^+ > 0$, $w_0(x) \to w^+$ as $x \to \infty$, we derive that for each given interval $[a, b]$ of x, $W_j(x) > 0$ for j sufficiently large. Hence if the functions $W_j(x)$ have negative values, then the corresponding x should go to $+\infty$. Thus $x_j \to +\infty$ and $J_\tau(W_j)(x_j) \to J_\tau(w_0 - w^+)(+\infty) = 0$. Then

$$W_j'' + c(\tau) W_j' + J_\tau(w_j) f(w_j) - g(w_j) - w^+ f\left(w^+\right) + g\left(w^+\right) = 0.$$

At the point $x = x_j$ we have

$$W_j''(x_j) + c(\tau) W_j'(x_j) + J_\tau(W_j)(x_j) f(w_j(x_j)) + w^+\left[f(w_j(x_j)) - f\left(w^+\right)\right] - \left[g(w_j(x_j)) - g\left(w^+\right)\right] = 0. \quad (2.9)$$

We can write

$$\begin{cases} f(w_j(x_j)) - f(w^+) = f'(w^+)[w_j(x_j) - w^+] + \alpha_j [w_j(x_j) - w^+] \\ g(w_j(x_j)) - g(w^+) = g'(w^+)[w_j(x_j) - w^+] + \beta_j [w_j(x_j) - w^+] \end{cases},$$

with α_j, $\beta_j \to 0$ as $x_j \to +\infty$. Then (2.9) can be written as

$$W_j''(x_j) + c(\tau) W_j'(x_j) + J_\tau(W_j)(x_j) f(w_j(x_j)) - W_j(x_j) f\left(w^+\right) + \left[f\left(w^+\right) + w^+ f'\left(w^+\right) - g'\left(w^+\right)\right] W_j(x_j) + \left(w^+\alpha_j - \beta_j\right) W_j(x_j) = 0. \quad (2.10)$$

By

$$J_\tau(W_j)(x_j) = \int_{-\infty}^{\infty} \phi(x_j - y) W_j(y)\, dy \geq \int_{-\infty}^{\infty} \phi(x_j - y) W_j(x_j)\, dy = W_j(x_j), \quad (2.11)$$

and

$$J_\tau(W_j)(x_j) f(w_j(x_j)) - W_j(x_j) f\left(w^+\right) = \left[J_\tau(W_j)(x_j) - W_j(x_j)\right] f(w_j(x_j)) + W_j^2(x_j)\left(f'\left(w^+\right) + \alpha_j\right),$$

equation (2.10) can be written as

$$W_j''(x_j) + c(\tau) W_j'(x_j) + \left[J_\tau(W_j)(x_j) - W_j(x_j)\right] f(w_j(x_j)) + \left[f\left(w^+\right) + w^+ f'\left(w^+\right) - g'\left(w^+\right)\right] W_j(x_j) + W_j^2(x_j)\left(f'\left(w^+\right) + \alpha_j\right) \left(w^+\alpha_j - \beta_j\right) W_j(x_j) = 0.$$

Since $W_j^2(x_j) \to 0$, $(w^+\alpha_j - \beta_j) W_j(x_j) \to 0$ faster than $W_j(x_j) \to 0$, then with the aid of (1.3), (2.11) and $W_j(x_j) < 0$, $W_j'(x_j) = 0$, $W_j''(x_j) \geq 0$, we obtain a contradiction in signs. Therefore, $w^+ < w_j(x)$ for all $x \in \mathbb{R}$ and all j large enough. Similarly $w_j(x) < w^-$ for all $x \in \mathbb{R}$ and all j large enough.

We now prove that $w_j'(x) < 0$, $(\forall)\, x \in \mathbb{R}$. Assume that this is not true, i. e. there exists $x_j \in \mathbb{R}$ such that $w_j'(x_j) = 0$.

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If x_j is bounded, then $x_j \to x_0$ on a subsequence and since $w_j \to w_0$ in $C^1(\mathbb{R})$, we derive that $w_0'(x_0) = 0$. This contradicts Lemma 2.

If $x_j \to \pm\infty$, then differentiating (2.3) and denoting $v = -w'$, one obtains like in the proof of Lemma 2 that

$$v'' + c(\tau)\, v' + a(x)\, J_\tau(v) + b(x)\, v = 0,\ a(x) = f(w(x)),\ b(x) = J_\tau(w)\, f'(w) - g'(w).$$

If $v_j(x) \geq 0$ for all $x \in \mathbb{R}$, from $v_j(x_j) = 0$ we arrive again at a contradiction with Lemma 2. This implies that each v_j has also negative values. On the other hand, by $w_j' \to w_0'$ uniformly on $\mathbb{R}$ and $w_0' < 0$ on $\mathbb{R}$, we obtain that v_j is positive in every bounded interval $[a,b]$ of x, for j large enough. Since $w_j(x)$ has finite limits as $x \to \pm\infty$, we get $v_j(x) \to 0$ as $x \to \pm\infty$. Therefore, the minimum value of $v_j(x)$, which is obviously negative, is reached at some $\widehat{x}_j \to \infty$. So $v_j(\widehat{x}_j) < 0$ and by the uniform convergence $w_j'(x) \to w_0'(x)$ and $w_0 \to w^{\pm}$, we find that $v_j(\widehat{x}_j) \to 0$ as $j \to \infty$.

Set $\delta = v_j(\widehat{x}_j)$ and $V_j(x) = v_j(x) - \delta$. Observe that $V_j(x) \geq 0$ on $\mathbb{R}$, $V_j(\widehat{x}_j) = 0$ and

$$V_j'' + c(\tau)\, V_j' + a(x)\, J_\tau(V_j) + b(x)\, V_j + [a(x) + b(x)]\,\delta = 0. \tag{2.12}$$

Since $\widehat{x}_j$ is a point of minimum for V_j, it follows that $V_j''(\widehat{x}_j) \geq 0$, $V_j(\widehat{x}_j) = 0$, $J_\tau(V_j)(\widehat{x}_j) \geq 0$. In addition, if $\widehat{x}_j \to \pm\infty$, then

$$\begin{aligned} a(\widehat{x}_j) + b(\widehat{x}_j) &= f(w(\widehat{x}_j)) + J_\tau(w_j)(\widehat{x}_j)\, f'(w_j(\widehat{x}_j)) - g'(w_j(\widehat{x}_j)) \\ &\to f\left(w^{\pm}\right) + w^{\pm} f'\left(w^{\pm}\right) - g'\left(w^{\pm}\right) < 0 \end{aligned}$$

via (1.3). Hence $a(\widehat{x}_j) + b(\widehat{x}_j) < 0$ for j large enough and $a(\widehat{x}_j) > 0$ for all j.

Thus we arrive at a contradiction in signs in equation (2.12). The lemma is proved.

Since f, g are bounded functions on $[w^+, w^-]$, as in Lemma 6 from [3] we can easily obtain the boundedness of the derivatives w', w''.

Lemma 4. *If the solution w of equation* (2.3) *satisfies the estimate* $|w(x)| \leq R$, $(\forall)\, x \in \mathbb{R}$, *then* $|w'(x)| \leq C$, $|w''(x)| \leq C$, $(\forall)\, x \in \mathbb{R}$, *where* $C > 0$ *is a constant depending on* R *and the bounds for* f *and* g.

3. Estimates of the Wave Speed

In the following we find estimates independent of τ for the wave speed $c(\tau)$.

Theorem 5. *If* $w(x)$ *is a monotonically decreasing solution of problem* $(2.3) - (2.4)$ *and* $c(\tau)$ *is the corresponding wave speed, then there exist some constants* c_1, c_2 *independent of* τ, *such that* $c_1 < c(\tau) < c_2$, $(\forall)\, \tau \in [0,1]$.

Proof. Recall that $w(x)$ is a traveling wave solution of the integro-differential equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + J_\tau(u) f(u) - g(u).$$

Denote by $u(x,t)$ another solution of this equation, monotone with respect to x, having the limits $w^\pm$ at $\pm\infty$. We will estimate first this solution from above. Denote by $w^* \in (w^+, w^-)$ the unstable stationary solution of the equation $dw/dt = wf(w) - g(w)$, i. e. $w^* f(w^*) - g(w^*) = 0$. There exist two constants $a > 0$, $b \in \mathbb{R}$ such that

$$G(u) = J_\tau(u) f(u) - g(u) \le aJ_\eta(u - w^*) + b(u - w^*), \ w^* \le u \le w^-, \qquad (3.1)$$

where $\eta(x)$ is a piece-wise constant function, equal $M = \sup_x \phi(x)$ in the support $[-N, N]$ of ϕ and zero otherwise and

$$J_\eta(z) = \int_{-\infty}^{\infty} \eta(x-y) z(y,t)\, dy.$$

Indeed,

$$\begin{aligned} G(u) &= f(u) J_\tau(u - w^*) + w^*[f(u) - f(w^*)] - [g(u) - g(w^*)] \\ &= f(u) J_\tau(u - w^*) + (u - w^*)\left(w^* f'(\xi_1) - g'(\xi_2)\right), \end{aligned}$$

for $\xi_1, \xi_2 \in (w^*, w^-)$ intermediate points. Since $f \ge 0$, f, f', g' are bounded on $[w^*, w^-]$ and $J_\tau(u - w^*) \le J_\eta(u - w^*)$, we easily arrive at (3.1).

Consider now the auxiliary equation

$$\frac{\partial \widehat{u}}{\partial t} = \frac{\partial^2 \widehat{u}}{\partial x^2} + aJ_\eta(\widehat{u} - w^*) + b(\widehat{u} - w^*) \qquad (3.2)$$

with the initial condition

$$\widehat{u}(x,0) \ge \max(w^*, u(x,0)).$$

We look for a solution of equation (3.2) in the form $\widehat{u}(x,t) = w^* + e^{-\lambda(x - c_2 t)}$. Then λ verifies the equation

$$\lambda^2 - c_2\lambda + Ma\frac{e^{\lambda N} - e^{-\lambda N}}{\lambda} + b = 0.$$

For c_2 large enough, there exists $\lambda > 0$ solution of this equation and, consequently $\widehat{u}(x,t) = w^* + e^{-\lambda(x - c_2 t)}$ is a solution of (3.2).

By a comparison theorem, we obtain that $\widehat{u}(x,t) \ge w^*$, $\widehat{u}(x,t) \ge u(x,t)$, for all $x \in \mathbb{R}$ and $t \ge 0$, and by (3.1) we have also $G(\widehat{u}(x,t)) \le aJ_\eta(\widehat{u} - w^*) + b(\widehat{u} - w^*)$. If we chose $u(x,0) = w(x)$, we obtain $u(x,t) \equiv w(x - c(\tau)t) \le \widehat{u}(x,t)$. Similarly one can find an estimate from below for $w(x)$. If $c(\tau)$ is the speed of propagation of $w(x)$, this implies that $c_1 \leq c(\tau) \leq c_2$, where c_1, c_2 are some fixed values. The proof is complete.

We are going to establish the sign of the speed $c(\tau)$. We show that the waves connecting a stable point with an unstable one can exist only for a specific sign of the speed. Recall first an auxiliary result from [3].

Lemma 6. (see [3]) *Let ϕ be non-negative and even. If v is a decreasing positive function, then for each N*

$$\int_N^\infty [J_\tau(v)(x) - v(x)]\,dx \geq 0.$$

If v is an increasing positive function, then for each N

$$\int_{-\infty}^N [J_\tau(v)(x) - v(x)]\,dx \geq 0.$$

Lemma 7. *Assume that ϕ is an even and non-negative function and w is a decreasing solution of equation (2.3).*

If w has the limits $w(-\infty) = w^-$, $w(+\infty) = w^*$, then $c(\tau) > 0$ on $[0,1]$.

If w has the limits $w(-\infty) = w^*$, $w(+\infty) = w^+$, then $c(\tau) < 0$ on $[0,1]$.

Proof. Let w be a decreasing solution of (2.3) with the limits $w(-\infty) = w^-$, $w(+\infty) = w^*$. Set $v(x) = w(x) - w^*$. Then (2.3) becomes

$$v'' + c(\tau)v' + J_\tau(v) f(v+w^*) + w^*[f(v+w^*) - f(w^*)] - [g(v+w^*) - g(w^*)] = 0.$$

Writing the square brackets in the form

$$\begin{cases} f(v+w^*) - f(w^*) = vf'(w^*) + \alpha v \\ g(v+w^*) - g(w^*) = vg'(w^*) + \beta v \end{cases}, \tag{3.3}$$

with $\alpha(x) \to 0$, $\beta(x) \to 0$ as $x \to \infty$, the above equality can be written as

$$v'' + c(\tau)v' + J_\tau(v) f(v+w^*) + v\left[w^* f'(w^*) - g'(w^*)\right] + v(\alpha w^* - \beta) = 0.$$

Integrating with respect to x from N to $+\infty$, with N large enough, with the aid of (3.3) one arrives at

$$-v'(N) - c(\tau)v(N) + \int_N^\infty J_\tau(v)v\left[f'(w^*) + \alpha\right]dx + f(w^*)\int_N^\infty J_\tau(v)\,dx \\ + \left[w^* f'(w^*) - g'(w^*)\right]\int_N^\infty v(x)\,dx + \int_N^\infty v(\alpha w^* - \beta)\,dx = 0. \tag{3.4}$$

In view of (1.3) and of Lemma 6, we have

$$f(w^*)\int_N^\infty J_\tau(v)\,dx + \left[w^* f'(w^*) - g'(w^*)\right]\int_N^\infty v(x)\,dx \geq$$

$$f\left(w^*\right)\int_N^\infty \left[J_\tau\left(v\right)\left(x\right)-v\left(x\right)\right]dx\geq 0.$$

Next, $v'\left(N\right)<0$, $v\left(N\right)>0$ and all the other integrals from (3.4) are small for N sufficiently large. This is not possible except for the situation when $c\left(\tau\right)>0$, $\left(\forall\right)\tau\in\left[0,1\right]$. The second assertion follows in a similar way.

4. A Priori Estimates of Monotone Solutions of Problem (2.3)-(2.4)

First we show that monotone solutions of problem (2.3) – (2.4) have an exponential behavior at infinity.

Lemma 8. *Let w_τ be a monotonically decreasing solution of* (2.3) – (2.4) *. There exists a constant $\varepsilon>0$ such that for all x that satisfy $\left|w_\tau\left(x\right)-w^+\right|\leq\varepsilon$, we have in fact*

$$\left|w_\tau\left(x\right)-w^+\right|\leq Ke^{-ax}.$$

Analogously, for such x that $\left|w_\tau\left(x\right)-w^-\right|\leq\varepsilon$, the estimate

$$\left|w_\tau\left(x\right)-w^-\right|\leq Ke^{-bx}$$

holds. Here, the constants $K,a,b>0$ are independent of τ and w_τ.

Proof. Let $W\left(x\right)=w_\tau\left(x\right)-w^+$. Then $W\left(x\right)\geq 0$, W is decreasing and

$$W''+c\left(\tau\right)W'+J_\tau\left(W\right)f\left(w_\tau\right)+w^+\left[f\left(w_\tau\right)-f\left(w^+\right)\right]-\left[g\left(w_\tau\right)-g\left(w^+\right)\right]=0.$$

Using estimates similar to (3.3) this equality becomes

$$\begin{aligned}W''+c\left(\tau\right)W'+\left[J_\tau\left(W\right)f\left(w_\tau\right)-Wf\left(w^+\right)\right]&\\+\left[f\left(w^+\right)+w^+f'\left(w^+\right)-g'\left(w^+\right)\right]W+\left(w^+\alpha-\beta\right)W&=0.\end{aligned}\qquad(4.1)$$

In order to estimate $J_\tau\left(W\right)f\left(w_\tau\right)-Wf\left(w^+\right)$, we write

$$\begin{aligned}&J_\tau(W)f(w_\tau)-Wf(w^+)=J_\tau(W)f(w^++W)-Wf(w^+)\\&=J_\tau(W)(f(w^+)+f'(\tilde{w})W)-Wf(w^+)=(J_\tau(W)-W)f(w^+)+J_\tau(W)f'(\tilde{w})W,\end{aligned}$$

where $\tilde{w}$ is some intermediate point. Then equation (4.1) becomes

$$\begin{aligned}W''+c\left(\tau\right)W'+(J_\tau(W)-W)f(w^+)+J_\tau(W)f'(\tilde{w})W&\\+[f(w^+)+w^+f'(w^+)-g'(w^+)]W+(\alpha w^+-\beta)W&=0\end{aligned}$$

or

$$W''+c\left(\tau\right)W'+J_\tau(W)f(w^+)+\left(w^+f'(w^+)-g'(w^+)+\sigma(x)\right)W=0,\qquad(4.2)$$

where

$$\sigma(x) = J_\tau(W) f'(\tilde{w}) + (\alpha w^+ - \beta) \to 0, \quad x \to \infty.$$

Hence, the corresponding limiting equation is

$$W'' + c(\tau) W' + J_\tau(W) f(w^+) + \left(w^+ f'(w^+) - g'(w^+)\right) W = 0.$$

By virtue of conditions (1.3), this limiting equation does not have nonzero bounded solutions. Hence the operator from the left-hand side of (4.2) is a Fredholm operator and therefore $W(x)$ decays exponentially at infinity. The estimate can be obtained independent of τ since τ changes on a compact interval. The lemma is proved.

Lemma 9. *Let $\varepsilon > 0$ be the constant from Lemma 8. Then there exists a constant $\chi > 0$ such that, for any monotone solution w_τ of problem (2.3) – (2.4) and for any x with $|w_\tau(x) - w^+| > \varepsilon$ and $|w_\tau(x) - w^-| > \varepsilon$, we have $|w'_\tau(x)| \geq \chi$. Constant χ is independent of τ and of w_τ.*

The proof of the lemma is similar to the proof in [3], so we omit it.

Lemma 10. *For any given $r \geq 0$, denote by $\mathcal{M}_r$ the set of all monotone solutions w of problem (2.3) – (2.4), $\tau \in [0,1]$, such that $|x_w| \leq r$, where x_w is the solution of equation $w_\tau(x_w) = w_0$. In this case, there is $C_r > 0$ such that, for every $w \in \mathcal{M}_r$, we have*

$$||w - \psi||_{E_\mu} \leq C_r.$$

Proof. Let $0 < \varepsilon < w_0$, $\chi > 0$ be the constants arising in Lemmas 8 and 9 respectively, $r \geq 0$ be arbitrarily fixed, and $w \in \mathcal{M}_r$. Consider x_1, x_2 such that

$$w^- - w(x_1) = \varepsilon, \;\; w(x_2) - w^+ = \varepsilon. \tag{4.3}$$

Then $x_1 < x_w < x_2$ and $|x_w| \leq r$. By Lemma 8 we know that outside $[x_1, x_2]$, $w - \psi$ has an exponential behavior, i. e.

$$||w - \psi||_{E_\mu} = ||w - \psi||_{E_\mu([x_1,x_2])},$$

where $E_\mu([x_1,x_2])$ is $C^{2+\alpha}([x_1,x_2])$ with the weight μ. It remains to prove that $||w - \psi||_{E_\mu([x_1,x_2])} \leq C_r$ for some $C_r > 0$. Observe that

$$|w(x_2) - w(x_1)| = \left|w'(\xi)\right| (x_2 - x_1) \geq \chi (x_2 - x_1),$$

where $\xi \in (x_1, x_2)$ is an intermediate point. Hence

$$\chi(x_2 - x_1) \leq w(x_1) - w(x_2) = w^- - w^+ - 2\varepsilon < w^- - w^+,$$

$$\begin{cases} |x_1| \leq |x_1 - x_w| + |x_w| \leq |x_1 - x_2| + |x_w| \leq N + r \\ |x_2| \leq |x_2 - x_w| + |x_w| \leq |x_2 - x_1| + |x_w| \leq N + r \end{cases},$$

where $N = (w^- - w^+)/\chi$. Then,

$$||w-\psi||_{E_\mu([x_1,x_2])} = ||(w-\psi)\mu||_{C^{2+\alpha}([x_1,x_2])} = ||(w-\psi)\mu||_{C^\alpha([x_1,x_2])} + ||[(w-\psi)\mu]'||_{C^\alpha([x_1,x_2])} + ||[(w-\psi)\mu]''||_{C^\alpha([x_1,x_2])}.$$

We estimate now each norm:

$$||(w-\psi)\mu||_{C^\alpha([x_1,x_2])} \leq ||w-\psi||_{C^\alpha([x_1,x_2])}||\mu||_{C([x_1,x_2])} + ||\mu||_{C^\alpha([x_1,x_2])}||w-\psi||_{C([x_1,x_2])}.$$

Analogously for the first and the second derivative. Therefore, there exists $C_r > 0$ in order that $||w-\psi||_{E_\mu} \leq C_r$, as claimed. The lemma is proved.

Proposition 11. *Denote by w_M and w_N any monotone and any nonmonotone solution of problem* (2.3) – (2.4) *, respectively.*

a) There exists $R > 0$ such that $||w_M - \psi||_{E_\mu} \leq R$;

b) There exists $r > 0$ such that $||w_M - w_N||_{E_\mu} \geq r$. In other words, we can separate monotone solutions of (2.3) – (2.4) *from nonmonotone solutions.*

Proof. Consider the wave speed c depending on u, where $u(x) = w(x) - \psi(x)$. Denote the solution of the equation $w(x) = w_0$ by x_w.

a) We show the existence of some $r > 0$ such that all monotone solutions w of problem (2.3) – (2.4) with $c = c(u)$ belong to $\mathcal{M}_r$ from Lemma 10, that is $|x_w| \leq r$. Assume by contradiction that there exist two sequences x_k and $w_k(x)$, such that $w_k(x_k) = w_0$ and $|x_k| \to \infty$. Let $u_k(x) = w_k(x) - \psi(x)$, $c = c(u_k)$ the corresponding wave speed, and

$$v_k(x) = w_k(x + x_k) - \psi(x).$$

Remark that $v_k + \psi \in \mathcal{M}_0$ which is $\mathcal{M}_r$ for $r = 0$ and $x_w = 0$. Then, by Lemma 10, one derives that

$$||v_k||_{E_\mu} = ||w_k - \psi||_{E_\mu} \leq C_0, \tag{4.4}$$

for some $C_0 > 0$. Let γ be an increasing function satisfying the conditions $\gamma(-\infty) = 0$, $\gamma(\infty) = 1$, $\int_{-\infty}^0 \gamma(x)\,dx < \infty$, and let $c(u) = \ln\rho(u)$, where $\rho(u)$ is the integral

$$\rho(u) = \left(\int_{-\infty}^{\infty} \left(u(x) + \psi(x) - w^+\right)^2 \gamma(x)\,dx\right)^{1/2}.$$

Making the change of variable $x + x_k = y$, we get

$$\rho(u_k) = \left(\int_{-\infty}^{\infty} \left(v_k(x) + \psi(x) - w^+\right)^2 \gamma(x + x_k)\,dx\right)^{1/2}.$$

We now find the limits of $\rho(u_k)$ as $x_k \to \infty$ and as $x_k \to -\infty$. If $x_k \to \infty$, then $\gamma(x + x_k) \to 1$, $|v_k| \to 0$ and $\psi(x) - w^+$ does not approach 0 when x is in a neighbourhood of $-\infty$. So, $\rho(u_k) \to \infty$. If $x_k \to -\infty$, then $\gamma(x + x_k) \to 0$ and ψ, v_k are bounded by functions from $L^2(\mathbb{R})$. For v_k this follows from (4.4). So $\rho(u_k) \to 0$. Now we can conclude that $c(u_k) = \ln \rho(u_k)$ tends to ∞ as $x_k \to \infty$ and tends to $-\infty$ as $x_k \to -\infty$. This contradicts the boundedness of $c(u)$ (see Theorem 5). Thus we have proved that $w \in \mathcal{M}_r$ for some $r > 0$. The claim follows by Lemma 10.

b) If w_M and w_N are arbitrary monotone and nonmonotone solutions of problem (2.3) – (2.4), respectively, denote

$$u_M = w_M - \psi, \ u_N = w_N - \psi.$$

Assume that conclusion of part b) is not true. Then at least on some sequences u_M^k, u_N^k we have

$$||u_M^k - u_N^k||_{E_\mu} \to 0 \text{ as } k \to \infty. \tag{4.5}$$

By part a) of the proposition, at least on a subsequence denoted again k, u_M^k is strongly convergent in $C_\mu^2(\mathbb{R})$ to some limit $u_M^0 \in E_\mu$: $||u_M^k - u_M^0||_{C_\mu^2(\mathbb{R})} \to 0$ as $k \to \infty$. Then, $w^0 = u_M^0 + \psi$ is a solution of (2.3) – (2.4) for some $\tau \in [0,1]$ and c and

$$||w_M^k - w^0||_{C_\mu^2(\mathbb{R})} = ||u_M^k - u_M^0||_{C_\mu^2(\mathbb{R})} \to 0 \text{ as } k \to \infty. \tag{4.6}$$

Now from (4.5) and (4.6), one deduces that

$$||w_N^k - w^0||_{C_\mu^2(\mathbb{R})} = ||u_N^k + \psi - w^0||_{C_\mu^2(\mathbb{R})} \leq ||u_N^k - u_M^k||_{C_\mu^2(\mathbb{R})} + ||w_M^k - w^0||_{C_\mu^2(\mathbb{R})} \to 0 \text{ as } k \to \infty.$$

Since $w_N^k \to w^0$ in C^1 and w^0 is nonincreasing (as the limit of the sequence of nonincreasing solutions $w_M^k = u_M^k + \psi$), Lemma 3 leads to $\left(w_N^k\right)'(x) < 0$ on $\mathbb{R}$ for N large enough. But w_N^k is a nonmonotone solution. This contradiction ends the proof.

Theorem 12. *There exists a monotone travelling wave solution of problem* (1.4) – (1.5) *where the function* $F(w, J(w)) = f(w) J(w) - g(w)$*satisfies (1.7), that is a constant* c *and a monotone function* $w \in C^2(\mathbb{R})$, *satisfying equation* (1.4) *and the limits* (1.5).

Proof. First we separate monotone solutions of problem (2.3) – (2.4) from nonmonotone solutions. To this end, consider a ball in E_μ, say $||u||_{E_\mu} \leq R$, which contains all solutions of equation (2.3), both monotone and nonmonotone. Proposition 11 assures the existence of some $r > 0$ in order that for all solutions of (2.3) from the ball $||u||_{E_\mu} \leq R + 1$, we have

$$||w_M - w_N||_{E_\mu} \geq r.$$

In [2] we have shown that operator $A_\tau(u)$ is proper with respect to τ and u. In other words, the set of all solutions of equation $A_\tau(u) = 0$ is compact in the ball $||u||_{E_\mu} \leq R + 1$. Therefore, from any covering of the set of monotone solutions u_M by open balls $B(u_M, r)$ (with the center at u_M and the radius r), we can select a finite subcovering. Denote by G_k, $k = 1, ..., N$, the set of domains formed by the union of the balls of this subcovering and by ∂G_k their boundaries. Obviously, any monotone solution $u_M \in \bigcup_{k=1}^{N} G_k$, for every $\tau \in [0,1]$, while any nonmonotone solution $u_N \notin \bigcup_{k=1}^{N} (G_k \cup \partial G_k)$. Thus we have separated monotone from nonmonotone solutions of equation (2.3).

In paper [2] a topological degree is constructed for the general $\tau-$dependent operators A_τ in (2.1). This is available in particular for operators A_τ with $F(w, J_\tau(w)) = f(w) J_\tau(w) - g(w)$, where ϕ_τ, J_τ are defined in (2.2). In [1] it is proved that the topological degree for the particular case of the operator A_0 (where the support of ϕ is narrow) equals 1. In the present paper, we have got a priori estimates for the solutions of $(2.3)-(2.4)$ (Proposition 11). Consequently, one can use Leray- Schauder method to study the existence of travelling wave solutions for $(1.4)-(1.5)$. Since the topological degree does not change under a continuous deformation of the operator which does not vanish at the boundary of the domain, it follows that for $A_1 u = Au$, it is also 1. Hence we deduce the existence of a monotone travelling wave solution for problem $(1.4)-(1.5)$. This completes the proof.

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References

[1] N. Apreutesei, A. Ducrot, V. Volpert, Travelling waves for integro-differential equations in population dynamics, *Discrete Contin. Dyn. Syst., Ser. B*, **11**, No. 3 (2009), 541-561.

[2] N. Apreutesei, V. Volpert, Properness and topological degree for nonlocal reaction-diffusion operators, *Abstr. Appl. Anal.*, In Press.

[3] I. Demin, V. Volpert, Existence of waves for a nonlocal reaction-diffusion equation. *Math. Model. Nat. Phenom.*, **5**, No. 5 (2010), 80-101.

[4] S. Genieys, V. Volpert, P. Auger, Pattern and waves for a model in population dynamics with nonlocal consumption of resources, *Math. Model. Nat. Phenom.*, **1**, No. 1 (2006), 63-80.

[5] V. Volpert, *Elliptic Partial Differential Equations. Volume 1. Fredholm Theory of Elliptic Problems in Unbounded Domains*, Birkhauser (2011).

[6] V. Volpert, S. Petrovskii, Reaction-diffusion waves in biology, *Physics of Life Reviews*, **6** (2009), 267-310.

[7] A. Volpert, Vit. Volpert, Vl. Volpert, *Traveling Wave Solutions of Parabolic Systems*, Translation of Mathematical Monographs, Vol. 140, Amer. Math. Society, Providence (1994).