A NOTE ON $\pi$-REGULAR AND $\pi S$-UNITALITY
OVER NOETHERIAN REGULAR $\delta$-NEAR RINGS
($\pi$-R&$\pi S$-U-NR-$\delta$-NR)

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Abstract: In this paper, we begin with to show that the characterization of $\pi$-regularity and $\pi S$-Unitality over Noetherian regular $\delta$-near rings, also consider their application in near rings as well.
Next we introduce more general concepts of $\pi$-Regularity and $\pi S$-Unitality characterization over Noetherian Regular $\delta$-near rings and then given some examples in near-rings also investigated their properties and characterization.

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1. Introduction

The concept of Von-Nuemann regularity of Noetherian Regular $\delta$-Near Rings of Near Rings studied by many authors Biedleman, Choudari, Goel, Heathorly, Hongan, Ligh, Maron and Murthy. Their main results are suggested in the book G Pilz [12].
The Von-Neumann regularity of rings and its generalization were studied by Fischer, Snider, Hirano Tominga, Savage, Ligh, Schein and Ohori. In 1985, Ohori investigated the characterization of $\pi$-Regularity and strong $\pi$-regularity and $\pi$-Unitality of Rings.
A Near Ring $N$ is an algebraic system $(N,+,.)$ with two binary operations ‘+’ and ‘.’ Such that $(N,+)$ is a group where $N$ is not necessarily abelian with neutral element say 0, $(N,.)$ is a semi group and $(a + b ) c = ac + bc$ for all $a, b, c \in N$.

If $N$ has a unity 1, then $N$ is called Unitary.

A Near Ring $N$ is with the extra $N0 = 0 = 0N$ and $\forall a \in N$, $a0 = 0 = 0a$ is said to be zero symmetric.

An element $d \in N$ is distributive if $d ( a + b ) = da + db$, $\forall a, b \in N$.

We will use the notations as follows:

Given a Near Ring $N$, $N_0 = \{a \in N / a0 = 0\}$ which is called the zero symmetric part of $N$ and $N_c = \{a \in N / a0 = a\}$ which is called the constant part of $N$. The set of all distributive elements in $N$ is denoted by $N_d$.

Obviously, we see that $N_0$ and $N_c$ are sub Noetherian $\delta$-delta near rings of $N$, but $N_d$ is a semi group under multiplication of $N$.

Clearly, Noetherian Regular $\delta$-near ring $N$ is zero symmetric Near Ring in case $N = N_0$, in case of $N=N_c$, $N$ is called a constant Near Ring and in case $N = N_d$ then $N$ is called distributive Noetherian Regular $\delta$-near ring. For all basic results we shall refer to G Pilz [12] some preliminaries along with examples provided herewith in section 2.

**2. Preliminaries**

**Definition 2.1.** A Near-Ring is a set $N$ together with two binary operations “+” and “.” Such that

(i) $(N, +)$ is a Group not necessarily abelian, (ii) $(N,.)$ is a semi Group and (iii) for all $n_1, n_2, n_3 \in N$, $(n_1 + n_2) . n_3 = (n_1 . n_3 + n_2 . n_3)$ i.e. right distributive law.

**Examples 2.2** Let $M_{2\times2} = \{(aij) / Z ; Z is treated as a near-ring\}$. $M_{2\times2}$ under the operation of matrix addition ‘+’ and matrix multiplication ‘.’.

**Example 2.3.** $Z$ be the set of positive and negative integers with 0. $(Z, +)$ is a group. Define ‘.’ on $Z$ by $a \cdot b = a$ for all $a, b \in Z$. Clearly $(Z, +,.)$ is a near-ring.

**Example 2.4.** Let $Z_{12} = \{0, 1, 2, \ldots , 11\}$. $(Z_{12}, +)$ is a group under ‘+’ modulo 12. Define ‘.’ on $Z_{12}$ by $a \cdot b = a$ for all $a \in Z_{12}$. Clearly $(Z_{12}, +,.)$ is a near-ring.

**Definition 2.5.** A near-ring $N$ is Regular Near-Ring if each element $a \in N$ then there exists an element $x$ in $N$ such that $a = axa$.

**Definition 2.6.** A Commutative ring $N$ with identity is a Noetherian Regular $\delta$-Near Ring if it is Semi Prime in which every non-unit is a zero divisor and the Zero ideal is Product of a finite
number of principle ideals generated by semi prime elements and $N$ is left simple which has $N_0 = N$, $N_e = N$.

**Definition 2.7.** A near-ring $N$ is called a $\delta$-Near – Ring if it is left simple and $N_0$ is the smallest non-zero ideal of $N$ and a $\delta$-Near – Ring is a non-constant near ring.

**Definition 2.8.** A $\delta$-Near-Ring $N$ is isomorphic to $\delta$-Near-Ring and is called a Regular $\delta$-Near-Ring if every $\delta$-Near-Ring $N$ can be expressed as sub-direct product of near-rings $\{N_i\}$, $N_i$ is a non-constant near-ring or a $\delta$-Near-Ring $N$ is sub-directly irreducible $\delta$-Near-Rings $N_i$.

**Definition 2.9.** Let $N$ be a Commutative Ring. Let $N$ be a Noetherian Regular $\delta$-Near-Ring if each $P \in \mathfrak{A}(N_N)$ is strongly prime i.e., $P$ is a $\delta$-Near – Ring of $N$.

**Example 2.10.** Let $N = \begin{bmatrix} F & F \\ 0 & F \end{bmatrix}$ where $F$ is a field. Then $P(N) = \begin{bmatrix} 0 & F \\ 0 & 0 \end{bmatrix}$

Let, $\sigma : N \rightarrow N$ be defined by, $\sigma((a \ b = a \ 0 \ 0 \ c)) = 0 \ c$

It can be seen that a $\sigma$ endomorphism of $N$ and $N$ is a $\sigma(\ast)$-Ring or Noetherian Regular $\delta$-Near – Ring.

**Definition 3.11.** Let $(N, +, \bullet)$ be a near-ring. A subset $L$ of $N$ is called a ideal of $N$ provided that 1. $(N, +)$ is a normal subgroup of $(N, +)$, and 2. $m.(n + i) = m.n \in L$ for all $i \in L$ and $m, n \in N$.

### 3. General Concepts on $\pi$-Regularity Characterization Over Noetherian Regular $\delta$-Near Rings of Near Rings

In this section we discussed some definitions, properties and general concepts on $\pi$-Regularity characterization over Noetherian Regular $\delta$-Near Rings of Near Rings.

**Definition 3.1.** A Noetherian regular $\delta$-Near– Ring $N$, an element $a \in N$ is called nilpotent if $\exists$ a positive integer $n \ni a^n = 0$.

**Definition 3.2.** A subset $S \subset N$ is called Nilpotent if $\exists$ a positive integer $n \ni S^n = 0$ and $S \subset N$ is called Nil if every element in $S$ is Nilpotent which is introduced by G Pilz Near Rings, North Holans P.C., New York in 1983.

**Note 3.3.** Every Nilpotent subset of $N$ is NIL.

**Definition 3.4.** A (two sided ) $N$ – subgroup of $N$ is a subset $K$ of $N$ such that (i) $(K, +)$ is a subgroup of $(N, +)$, (ii) $NK \subset K$ and (iii) $KN \subset K$. If $K$ satisfies the properties (i),(ii) then it is called a right $N$ – subgroup of $N$.

**Definition 3.5.** A (two sided ) $N$ – subgroup of $N$ is a subset $M$ of $N$ such that (i) $(M, +)$ is a subgroup of $(N, +)$, (ii) $NM \subset M$ and (iii) $MN \subset M$. If $M$
satisfies the properties (i),(ii) then it is called a left N – subgroup of N, normal left N- subgroup of N and right N- subgroup of N respectively.

**Note 3.6.** Hence Normal right N – subgroup of N are the same of right ideals of Noetherian regular δ-Near Ring N.

**Definition 3.7.** A subset L of a Noetherian regular δ-Near Ring N together with (i) NL ⊂ L and (ii) LN ⊂ L is called N – subset of N.

If this L satisfies (i) then it is called a left N – subset of Noetherian regular δ-Near Ring N (NR-δ-NR) and If this L satisfies (ii) then it is called a right N – subset of Noetherian regular δ-Near Ring N (NR-δ-NR).

**Note 3.8.** A Noetherian regular δ-Near Ring N is said to be ‘Reduced’ if N has no non-zero nilpotent elements i.e., ∀ a ∈ N, a^n = 0 for some positive integer n ⇒ a =0 Mc Coy already given proof for N is Reduced ⇔ ∀ a ∈ N, a^2 = 0 ⇒ a =0.

**Definition 3.9.** A Noetherian δ-Near Ring is ‘regular’ (Von – Nuemann) if ∀ a ∈ N, ∃ an element x ∈ N such that a = axa. Since, ya and ay are idempotent elements in N, taking ya = e, Na = Neya = Nae ⊂ Ne = Nya ⊂ Na. Hence, Na = Ne. obviously, N is left S-Unital.

**Proposition 3.10.** Let N be a Noetherian Regular δ-Near Ring of a Near Ring. Then N is regular ⇔ N has the condition “∀ a ∈ N, ∃ e^2 = e ∈ N such that N a = N e and N is left S-Unital”.

**Proof.** Given N be a Noetherian Regular δ-Near Ring of a Near Ring.

⇒ To show that N is regular if and only if N has the condition “∀ a ∈ N, ∃ e^2 = e ∈ N such that N a = N e and N is left S-Unital”. For that, ∀ a ∈ N, ∃ y ∈ N such that a = aya. Since, ya and ay are idempotent elements in N, taking ya = e, Na = Neya = Nae ⊂ Ne = Nya ⊂ Na. Hence, Na = Ne. obviously, N is left S-Unital.

⇐ Let us assume that N has the given condition “∀ a ∈ N, ∃ e^2 = e ∈ N such that N a = N e and N is left S-Unital”.

Then S-Unitality ⇒ that a ∈ Na = Ne. so that ∃ y ∈ N such that a = ye.

⇒ e = ee ∈ Ne = Na, so that ∃ y ∈ N such that e = ya.

Thus we obtain a = xe = xee = xeya = aya ⇒ N is regular.

Hence the Proposition.

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**Proposition 3.11.** Every Noetherian regular δ-Near Ring N has no non-zero nil left N- subset.

**Proof.** Let N be a Noetherian regular δ-Near Ring and M be a nil left N- subset of N.

It is sufficient to show that M = {0 }.

Indeed, let a ∈ M since N is Noetherian regular δ-Near Ring(NR-δ-NR),

N has the condition that there exists e^2 = e ∈ N such that Na = Ne and N is left S-Unital.
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By known Proposition, since $M$ is a left $N$ – subset implies $a \in Na \in M$.
On the other hand, since $M$ is Nil, there exists a positive integer $n$ such that $a^n = 0$.
Given condition $e = ee \in Ne = Na \subset M$.
Also, $\exists$ a positive integer $m$ such that $e = e^m = 0$.
By the two conditions, we have $a \in N0$, so that $a = r.0 = (r.0)^n = a^n = 0$.
Therefore, $M = \{0\}$. Hence the Proposition.

**Corollary 3.12.** Every Noetherian regular $\delta$-Near Ring $N$ with identity has no non-zero nil left $N$-subgroup.

**Definition 3.13.** A Noetherian regular $\delta$-Near Ring $N$ is said to be $\pi$-Regular if $\forall a \in N$, $\exists$ a positive integer $n$ such that $a^n$ is a regular element i.e., $a^n = a^m x a^n$ for some $x \in N$. such an element ‘$a$’ is called $\pi$-Regular.

**Note 3.14.** Every Regular near ring and Noetherian regular $\delta$-Near Ring is $\pi$-Regular but not every $\pi$-Regular is a Noetherian regular $\delta$-Near Ring, Regular Near Ring.

**Example 3.15.** Let $N = \{0, a, b, c\}$ be an additive klein 4 – group this is a Noetherian regular $\delta$-Near Ring with the following multiplication table:

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>a</th>
<th>b</th>
<th>c</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>a</td>
<td>0</td>
<td>0</td>
<td>a</td>
<td>a</td>
</tr>
<tr>
<td>b</td>
<td>0</td>
<td>a</td>
<td>c</td>
<td>b</td>
</tr>
<tr>
<td>c</td>
<td>0</td>
<td>a</td>
<td>b</td>
<td>c</td>
</tr>
</tbody>
</table>

Thus Noetherian regular $\delta$-Near Ring $N$ is zero symmetric Noetherian regular $\delta$-Near Ring with identity say $e$. Moreover, $N$ is a $\pi$-Regular but not regular.

Indeed, $0 = 0a0$, $a^2 = a^2 b a^2$, $b^4 = b^4 a b^4$ and $c^2 = c^2 c c^2$ but $a$ is not a regular element.

**Example 3.16.** Let $N = Z_4 = \{0, 1, 2, 3\}$ be an additive group of integers mod. 4 and define multiplication as follows table:

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>3</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>2</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>3</td>
</tr>
</tbody>
</table>

Thus Noetherian regular $\delta$-Near Ring $N$ is zero symmetric Noetherian regular $\delta$-Near Ring with identity say $e$. Moreover, $N$ is a $\pi$-Regular but not regular.

Indeed, $0 = 0a0$, $a^2 = a^2 b a^2$, $b^4 = b^4 a b^4$ and $c^2 = c^2 c c^2$ but $a$ is not a regular element.
Proposition 3.17. Let $N$ be a Near Ring. Then $N$ is regular if and only if $N$ has the condition $\forall a \in N, \exists e^2 = e \in N$ such that $Na = Na^n$ and $N$ is left S-Unital.

Proof. Refer Proposition 1. [21].

Proposition 3.18. Every regular near ring $N$ has no non-zero nil left $N$-Subset.

Proof. Refer Proposition 3. [21].

4. General Concepts on $\pi S$-Unitality Characterization over Noetherian Regular $\delta$-Near Rings of Near Rings

In this section we discussed some definitions, properties and general concepts on $\pi S$-Unitality characterization over Noetherian Regular $\delta$-Near Rings of Near Rings.

Definition 4.1. A Noetherian regular $\delta$-Near Ring $N$ is said to be left S-Unital (respectively, right S-Unital) if $\forall a \in N, a \in Na$ (respectively, $a \in an$) such element ‘a’ is called left S-Unital(respectively, right S-Unital).

Definition 4.2. A Noetherian regular $\delta$-Near Ring $N$ is called S-Unital, if $N$ is both left S-Unital and right S-Unital.

Note 4.3. Every Noetherian Regular $\delta$-Near Ring $N$ with left identity or identity is clearly left S-Unital.

Note 4.4. Every Noetherian Regular Near Ring $N$ is S-Unital we studied and begin with to show that the characterization of regularity and S-Unitality in Noetherian Regular $\delta$-Near Rings $N$ of Near Ring.

Definition 4.5. A Near ring $N$ is called left $\pi S$-Unital (respectively right $\pi S$-Unital) if for each element $a \in N$, $\exists$ a positive integer $n$ such that $a^n$ is a S – Unital element, i.e., $a^n \in N a^n$ (respectively $a^n \in a^n N$) such an element ‘a’ is called left $\pi S$-Unital (respectively right $\pi S$-Unital).

$N$ is called $\pi S$-Unital, if $N$ is both left $\pi S$-Unital and right $\pi S$-Unital. Also, every left $\pi S$-Unital
(respectively right $\pi S$-Unital) Noetherian Regular $\delta$-Near Ring is left $\pi S$-Unital (respectively right $\pi S$-Unital), but not conversely we can give counter example for this as below :

In above example 3.15 clearly, $N$ is a left $\pi S$-Unital Noetherian Regular $\delta$-Near Ring. But in example 3.15, $N$ is left $\pi S$-Unital, indeed $0 = 1.0 = 2.0 = 3.0 \in N0$, and $1 = 3.1 \in N1, 2^2 = 0 = 0.2^2 \in N 2^2$ and $3 = 3.3 \in N3$. But this Noetherian Regular $\delta$-Near Ring $N$ is not S- Unital, because 2 is not a left S-Unital element.

These are some Properties, concepts on $\pi$-Regularity and $\pi S$-Unitality characterization Over Noetherian Regular $\delta$-Near Rings of Near Rings.
Theorem 4.6. Let $N$ be a Near Ring. Then $N$ is a $\pi$ - Regular if and only if $N$ has the condition $\forall a \in N, \exists e \in N$ such that $Na = Ne$ and $N$ is left $\pi S$- unital.

Proof. Refer Theorem 7. [21].

5. Main Results on $\pi$-Regularity and $\pi S$-Unitality Characterization over Noetherian Regular $\delta$-Near Rings of Near Ring

We derived some important results on $\pi$-Regularity and $\pi S$-Unitality characterization over Noetherian Regular $\delta$-Near Rings of Near Ring.

Theorem 5.1. Let $N$ be a Noetherian Regular $\delta$-Near Ring. Then $N$ is $\pi$-Regular if and only if $N$ has the condition that $\forall a \in N, \exists e \in N$ such that $N a = Ne$ and $N$ is left $\pi S$-Unital.

Proof. Suppose that $N$ is $\pi$-Regular. Then for any $a \in N$, there exists a positive integer $n$ and $x \in N$ such that $a^n \in N$. Hence $N$ is left $\pi S$-Unital.

Next, since $xa^n$ and $a^n x$ are idempotent elements in $N$, putting $xa^n = e$, $N a^n = Ne$, $xa^n \subset N$.

Hence $N a^n = Ne$.

Conversely, assume that $N$ has the given condition $\forall a \in N, \exists e \in N$ such that $N a = Ne$ and $N$ is left $\pi S$-Unital. Then the $\pi S$-Unitality $\Rightarrow a^n \in N a^n = Ne$, so that there exists $y \in N$ such that $a^n = ye$ . . . . . . (Equation 1).

On the otherhand, we see that $e = ee = Ne$, so that there exists $x \in N$ such that $e = xa^n$ . . . . . . (Equation 2). From these two conditions of Equ 1 and Equ 2, we obtain that $a^n = a^n x a^n$.

Therefore, $N$ is a $\pi S$-regular Noetherian Regular $\delta$-Near Ring.

Hence the theorem.

Example 5.2. $N = \{0, a\}$ is a non-zero $N$- subgroup which is Nil. Then for any Noetherian regular $\delta$-Near Ring $N$, the center of $N$ is denoted by the set $Z(N) = \{x \in N / ax = xa, \forall a \in N\}$.

When $N$ is distributive, i.e, $N = N_d$, $Z(N)$ is a sub near ring of Noetherian Regular $\delta$-Near Ring $N$. so that is a distributive $\pi$-Regular Near rings but which are not additive abelian.

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Theorem 5.3. The center of a distributive $\pi$-Regular Noetherian regular $\delta$-Near Ring $N$ is also $\pi$-regular.
Proof. Let \( N \) be a distributive \( \pi \)-Regular Noetherian regular \( \delta \)-Near Ring and let \( a \in Z(N) \).

Then \( \exists \ y \in N \) and \( \exists \ n \in Z^+ \ni x^n a x^n \in Z(N) \). then our claim is done. Indeed, let \( q \in N \). since, \( a \in Z(N) \). Thus we can deduce that \( q (a^n x) = (qa^n) x = a^n (qx) = a^n x a^n (qx) = a^n x (qx) a^n = a^n (xq) a^n \)

and \( (a^n x) = (xa^n) q = x(a^n q) = x(qa^n) = xq(a^n) x = xa^n a^n = a^n (xq x) a^n \).

Hence, \( a^n x \in Z(N) \). similarly we can obtain that \( xa^n \in Z(N) \),

thus \( q(xa^n) = q(a^n x) = (qa^n) x = (a^n x) q = x(a^n q) x \) &

\( (xa^n) = x(a^n x) q = xq(a^n x) = x(qa^n) x = x(a^n q) x \)

\( \Rightarrow q(xa^n) = (xa^n) q \) i.e., \( xa^n \in Z(N) \).

Hence \( Z(N) \) is a \( \pi \)-Regular.

Hence the theorem.

References


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