International Electronic Journal of Pure and Applied Mathematics
Volume 5 No. 1 2012, 7-13

A UNIQUE COMMON FIXED POINT THEOREM FOR SIX MAPS IN CONE METRIC SPACES

M. Rangamma[1], K. Prudhvi[2 §]
[1,2]Department of Mathematics
Osmania University
Hyderabad, Andhra Pradesh, INDIA

Abstract: In this paper, we prove a unique common fixed point theorem for six maps in cone metric spaces. Our result generalizes and extends some recent results.

AMS Subject Classification: 47H10, 54H25
Key Words: cone metric space, common fixed point, coincidence point, asymptotically regular

1. Introduction and Preliminaries

Recently Huang and Zhang [4] have introduced the concept of cone metric spaces and established some fixed point theorems for contractive mappings in these spaces. Subsequently Abbas and Jungck [1] and Abbas and Rhoades [2] have studied common fixed point theorems in cone metric spaces (see also, [4], [6] and the references mentioned therein). In [3] Di Bari and Vetro have introduced the concept of φ-map and proved some fixed point theorems generalizing some known results. In this paper we extend the fixed point Theorem 3.1 of R.P. Pant et al [5] to six maps.

In all that follows, E is a real Banach space and θ is the zero element of E. For the mappings $f, g : X \longrightarrow X$, let $C(f, g)$ denotes set of coincidence points of f, g that is $C(f, g) := \{z \in X : fz = gz\}$.

We recall some definitions of cone metric spaces and some of their properties [4].

Definition 1.1. Let E be a real Banach space and P be a subset of E. The set P is called a cone if and only if:

(a) P is closed, nonempty and $P \neq \{\theta\}$;

(b) $a, b \in R$,$a, b \geq 0$,$x, y \in P \Longrightarrow ax + by \in P$;

(c) $x \in P$ and $-x \in P \Longrightarrow x = \theta$.

Received: October 14, 2011

§Correspondence author

International Electronic Journal of Pure and Applied Mathematics – IEJPAM, Volume 5, No. 1 (2012)

Definition 1.2. Let P be a cone in a Banach space E define partial ordering $\leq$ with respect to p by $x \leq y$ if and only if $y - x \in P$. We shall write $x < y$ to indicate $x \leq y$ but $x \neq y$ while $x \ll y$ will stand for $y - x \in intP$, where $intP$ denotes the interior of the set P. This Cone P is called an order cone.

Definition 1.3. Let E be a Banach Space and $P \subset E$ be an order cone. The order cone P is called normal if there exists $K > 0$ such that for all $x, y \in E$, $\theta \leq x \leq y$ implies $\parallel x \parallel \leq K \parallel y \parallel$. The least positive number K satisfying the above inequality is called the normal constant of P.

Definition 1.4. Let X be a nonempty set of E. Suppose that the map $d : X \times X \longrightarrow E$ satisfies:

$(d1)$ $\theta \leq d(x, y)$ for all $x, y \in X$ and $d(x, y) = \theta$ if and only if $x = y$;

$(d2)$ $d(x, y) = d(y, x)$ for all $x, y \in X$;

$(d3)$ $d(x, y) \leq d(x, z) + d(y, z)$ for all $x, y, z \in X$. Then d is called a cone metric on X and (X, d) is called a cone metric space. It is obvious that the cone metric spaces generalize metric spaces.

Definition 1.5. Let (X, d) be a cone metric space. We say that $\{x_n\}$ is:

(i) a Cauchy sequence if for every c in E with $\theta \ll c$, there is N such that for all $n, m > N$, $d(x_n, x_m) \ll c$;

(ii) a convergent sequence if for any $\theta \ll c$, there is an N such that for all $n > N$, $d(x_n, x) \ll c$, for some fixed x in X. We denote this $x_n \longrightarrow x$ $(n \longrightarrow \infty)$. A cone metric space X is said to be complete if every Cauchy sequence in X is convergent in X.

Definition 1.6. Let$f, g : X \longrightarrow X$. Then the pair $(f, g)$is said to be(IT)-Commuting at $z \in X$ if $f(g(z))$=$g(f(z))$ with $f(z)$=$g(z)$.

Definition 1.7. Let P be an order cone. A non-decreasing function $\varphi : P \longrightarrow P$ is called a φ-map if:

(i) $\varphi(\theta) = \theta$ and $\theta < \varphi(\omega)$ for, $\omega \in P \setminus \theta$,

(ii) $\omega \in IntP$ implies $\omega - \varphi(\omega) \in IntP$,

(iii) $\lim_{n \longrightarrow \infty} \varphi^n(\omega) = \theta$ for every $\omega \in P \setminus \theta$.

Definition 1.8. We define common asymptotic regularity of two functions in the following way. Let f, g, h and r, s, t be self-maps on a cone metric space (X, d). The pairs (f, g) and (r, s) are said to be common asymptotically regular with respect to h and t respectively at $x_0 \in X$ if there exists a sequence $x_n inX$ Such that $hx_{2n+1} = fx_{2n} = rx_{2n+2} = tx_{2n+3}$,

$$hx_{2n+2} = gx_{2n+1} = sx_{2n+3} = tx_{2n+4}, \quad n = 0, 1, 2, 3, \ldots$$

and

$$lim_{n \longrightarrow \infty} d(hx_n, hx_{n+1}) = \theta = lim_{n \longrightarrow \infty} d(tx_n, tx_{n+1}).$$

International Electronic Journal of Pure and Applied Mathematics – IEJPAM, Volume 5, No. 1 (2012)

2. Common fixed point theorem

The following theorem is an extends and improve the Theorem 3.1, see [5].

Theorem 2.1. *Let(X, d) be a cone metric space, P be an order cone and f, g, h and r, s, t be self-maps. Let the pairs (f, g) and (r, s) be common asymptotically regular with respect to h and r respectively at $x_0 \in X$ and the following conditions are satisfied:*

$(E1)$ $f(X) = g(X) = r(X) = s(X)$;

$(E2)$ $d(fx, gy) \leq \varphi(m_1(x, y))$ for all $x, y \in X$, where $m_1(x, y) = d(hx, hy) + \gamma[d(fx, hx) + d(gy, hy)]$, for some $\gamma(0 \leq \gamma \leq 1)$

$(E3)$ $d(rx, sy) \leq \varphi(m_2(x, y))$ for all $x, y \in X$, where $m_2(x, y) = d(tx, ty) + \gamma[d(rx, tx) + d(sy, ty)]$.

If $f(X)$ or $g(X)$ or $r(X)$ or $s(X)$ or $h(X)(= t(X))$ is a complete subspace of X, then:

(i) $C(f, h)$ is non-empty.

(ii) $C(r, t)$ is non-empty.

(iii) $C(g, h)$ is non-empty.

(iv) $C(s, t)$ is non-empty. Further,

(v) f and h have a common fixed point provided that f and h are (IT)-commuting at a point $u \in C(f, h)$.

(vi) g and h have a common fixed point provided that g and h are (IT)-commuting at a point $v \in C(g, h)$.

(vii) r and t have a common fixed point provided that r and t are (IT)-commuting at a point $u_1 \in C(r, t)$.

(viii) s and t have a common fixed point provided that s and t are (IT)-commuting at a point $v_1 \in C(s, t)$.

(ix) f, g, r, s and h, t have a unique common fixed point provided that (v), (vi), (vii) and (viii) all are true.

Proof. Suppose x_0 is an arbitrary point of X. Since (f, g) and (r, s) are common asymptotically regular with respect to h and t respectively at $x_0 \in X$. Then there exists $\{x_n\}$ in X such that
$hx_{2n+1} = fx_{2n} = rx_{2n+2} = tx_{2n+3}$ and $hx_{2n+2} = gx_{2n+1} = sx_{2n+3} = tx_{2n+4}$, for all n= 0,1,2,..., and $lim_{n\longrightarrow\infty} d(hx_n, hx_{n+1}) = \theta = lim_{n\longrightarrow\infty} d(tx_n, tx_{n+1})$. First we shall show that $\{hx_{2n}\}$ is a cauchy sequence. Suppose $\{hx_{2n}\}$ is not cauchy sequence. Then there exists $\mu > 0$ and increasing sequences m_k and n_k of positive integers, such that $m_k$even and n_k odd and for all k, $m_k > n_k$,

$$d(hx_{m_k}, hx_{n_k}) \geq \mu \ \ and \ \ d(hx_{m_k-1}, hx_{n_k}) < \mu. \tag{1}$$

By the triangle inequality,
$d(hx_{m_k}, hx_{n_k}) \leq d(hx_{m_k}, hx_{n_k-1}) + d(hx_{n_k-1}, hx_{n_k})$. Letting $k \longrightarrow \infty$, we get that

International Electronic Journal of Pure and Applied Mathematics – IEJPAM, Volume 5, No. 1 (2012)

$lim_{k \longrightarrow \infty} d(hx_{m_k}, hx_{n_k}) < \mu$. Therefore there exists k_0 such that

$$d(hx_{m_k}, hx_{n_k}) < \mu \quad for\ all\ k \geq k_0. \tag{2}$$

By (1) and (2), we get that

$$\mu \leq d(hx_{m_k}, hx_{n_k}) < \mu \quad for\ all\ k \geq k_0$$

implies $lim_{k \longrightarrow \infty} d(hx_{m_k}, hx_{n_k}) = \mu$. By (E2), we have $d(hx_{m_k+1}, hx_{n_k+1}) = d(fx_{m_k}, gx_{n_k})$. That is $d(hx_{m_k+1}, hx_{n_k+1}) \leq \varphi(d(hx_{m_k}, hx_{n_k}) + \gamma[d(hx_{m_k+1}, hx_{m_k}) + d(hx_{n_k+1}, hx_{n_k})]$. Letting $k \longrightarrow \infty$, we get that $d(hx_{m_k+1}, hx_{n_k+1}) \leq \varphi(\mu)$ and as per the definition (1.7)of φ-map $\varphi(\mu) < \mu$. Thus $\{hx_n\}$ is a cauchy sequence. Similarly we can prove that $\{tx_n\}$ is a cauchy sequence. Suppose $h(X)(= t(X))$is a complete subspace of X. Then $\{hx_n\}$ being contained in $h(X)$ call it z. Let $u = h^{-1}$. Thus $hu = z$ for some $u \in X$. Note that the sub spaces$\{hx_{2n+1}\}$ and $\{hx_{2n+2}\}$ also converges to z. By $(E2)$, we obtain

$$\begin{aligned} d(fu, gx_{2n+1} &\leq \varphi(m_1(u, x_{2n+1})) \\ &= \varphi(d(hu, hx_{2n+1}) + \gamma[d(fu, hu) + d(gx_{2n+1}, hx_{2n+2})]). \end{aligned}$$

Letting $k \longrightarrow \infty$, we get

$$\begin{aligned} d(fu, z) &\leq \varphi(d(hu, z) + \gamma[d(fu, hu) + \theta]) \\ d(fu, hu) &\leq \varphi(\gamma(d(fu, hu)) < d(fu, hu), 0 \leq \gamma \leq 1, \end{aligned}$$

a contradiction. Therefore

$$fu = hu = z. \tag{3}$$

Thus C(f,h) is non empty. This proves (i). Since we get $\{tx_n\}$ is a cauchy sequence in $h(X)$ which is complete. Hence $\{tx_n\}$ has a limit in $h(X)$. Note that $lim(hx_n) = lim(tx_n)$. We get $lim_{n \longrightarrow \infty} tx_n = z \in t(X)$. Therefore there exists $u^1 \in X$ such that $tu^1 = z$ implies $u^1 = t^{-1}z$. Note that the subsequences $\{tx_{2n+3}\}$and $\{tx_{2n+4}\}$are also converges to z. By (E3), we obtain

$$\begin{aligned} d(ru^1, sx_{2n+3}) &\leq \varphi(m_2(u^1, x_{2n+3})) \\ &= \varphi((tu^1, tx_{2n+3}) + \gamma[d(ru^1, tu^1) + d(sx_{2n+3}, tx_{2n+3})]) \\ &= \varphi(d(tu^1, tx_{2n+3}) + \gamma[d(ru^1, tu^1) + d(tx_{2n+4}, tx_{2n+3})]). \end{aligned}$$

Letting $n \longrightarrow \infty$, we get

$$\begin{aligned} d(ru^1, tu^1) &\leq \varphi(d(z, z) + \gamma[d(ru^1, tu^1) + d(z, z)]) \\ &\leq \varphi(\gamma[d(ru^1, tu^1)]) < d(ru^1, tu^1), \end{aligned}$$

International Electronic Journal of Pure and Applied Mathematics – IEJPAM, Volume 5, No. 1 (2012)

a contradiction. Therefore

$$ru^1 = tu^1 = z. \tag{4}$$

Thus C(r,t) is non empty. This proves(ii). In view of (3) and (4) it follows that

$$fu = hu = ru^1 = tu^1 = z. \tag{5}$$

Since $f(X) = g(X) = r(X) = s(X) = h(X)(= t(X))$. Therefore there exists $v, v^1 \in X$ such that,

$$fu = hv \; and \; ru^1 = tv^1. \tag{6}$$

We claim that $hv = gv$ and $tv^1 = sv^1$. Using (E2) and (E3),we obtain

$$\begin{aligned} d(hv, gv) = d(fu, gv) &\leq \varphi(m_1(u, v)) \\ &= \varphi(d(hu, hv) + \gamma[d(fu, hu) + d(gv, hv)]) \\ &\leq \varphi(d(fu, fu) + \gamma[d(fu, fu) + d(hv, gv)]) \\ or \; d(hv, gv) &\leq \varphi(d(z, z) + \gamma[d(z, z) + d(hv, gv)]) \\ d(hv, gv) &\leq \varphi(\gamma[d(hv, gv)]) < d(hv, gv)]), \end{aligned}$$

a contradiction. Therefore

$$hv = gv. \tag{7}$$

In view of (7), (6)and (3) it follows

$$gv = hv = fu = hu = z. \tag{8}$$

From (5) and (8) it follows

$$gv = hv = fu = hu = ru^1 = tv^1 = z. \tag{9}$$

Thus C(g,h) is non empty. This proves (iii). Now using (E3),we obtain

$$\begin{aligned} d(tv^1, sv^1) = d(ru^1, sv^1) &\leq \varphi(m_2(u^1, v^1)) \\ &= \varphi(d(tu^1, tv^1) + \gamma[d(ru^1, tu^1) + d(sv^1, tv^1)]) \\ &= \varphi(d(z, z) + \gamma[d(z, z) + d(tv^1, sv^1)]) \\ d(tv^1, sv^1) &\leq \varphi(\gamma[d(tv^1, sv^1)]) < d(tv^1, sv^1), \end{aligned}$$

a contradiction. Therefore

$$sv^1 = tv^1. \tag{10}$$

In view of (5), (6)and (10) it follows,

$$sv^1 = tv^1 = ru^1 = tu^1 = z. \tag{11}$$

Thus C(s,t) is non empty. This proves (iv). In view of (9),(11) it follows,

$$fu = hu = gv = hv = ru^1 = tu^1 = sv^1 = tv^1. \tag{12}$$

Since, $(f,h),(g,h)$ and $(r,t),(s,t)$ are (IT)-commuting, then $fhu = hfu$ implies $ffu = fhu = hfu = hhu$,

$$ghv = hgv \; implies \; ggv = ghv = hgv = hhv, \tag{13}$$

$rtu^1 = tru^1$ *implies* $rru^1 = rtu^1 = tru^1 = ttu^1$,
$stv^1 = tsv^1$ *implies* $ssv^1 = stv^1 = tsv^1 = ttv^1$. In view of (E2) it follows that

$$\begin{aligned} d(ffu, fu) = d(ffu, gv) &\leq \varphi(m_1(fu, v)) \\ &= \varphi(d(hfu, hv) + \gamma[d(ffu, hfu) + d(gv, hv)]) \\ d(ffu, fu) &\leq \varphi(d(ffu, fu)) < d(ffu, fu), \end{aligned}$$

a contradiction (since $\varphi(\omega) < \omega$). Therefore, $ffu = fu = hfu (= z)$,

$$fu \text{ is a common fixed point of } f \text{ and } h. \tag{14}$$

Similarly, we get $ggv = gv$.

$$\text{Therefore, } gv \text{ is a common fixed point of } g \text{ and } h. \tag{15}$$

Followed by (13)$ggv = gv = hgv(= z)$. Since $fv = gv$. From (14) and (15), we conclude that

$$fu(= z) \text{ is a common fixed point of } f,\ g \text{ and } h. \tag{16}$$

Now in view of $(E3)$, *we obtain*

$$\begin{aligned} d(rru^1, ru^1) = d(rru^1, ru^1) &\leq \varphi(m_2 d(ru^1, v^1)) \\ &= \varphi(d(tru^1, tv^1) + \gamma[d(rru^1, tru^1) + d(sv^1, tv^1)]) \\ d(rru^1, ru^1) &\leq \varphi(d(rru^1, ru^1)) < d(rru^1, ru^1), \end{aligned}$$

a contradiction. Therefore, $rru^1 = ru^1, rru^1 = tru^1 = ru^1 = z$,

$$ru^1 \text{ is a common fixed point of } r, \text{ and } t. \tag{17}$$

Similarly, we get $ssv^1 = tsv^1 = sv^1 = z$,

$$sv^1(= z) \text{ is a common fixed point of } s \text{ and } t. \tag{18}$$

International Electronic Journal of Pure and Applied Mathematics – IEJPAM, Volume 5, No. 1 (2012)

Since $ru^1 = sv^1 = z$. From (17) and (18), we conclude that

$$ru^1(= z) \text{ is a common fixed point of } r,\ s \text{ and } t. \tag{19}$$

Since $fu = ru^1(= z)$. Therefore from (16) and (19), we conclude that f, g, r, s and h, t are having a common fixed point. Finally in order to to prove uniqueness, let w be another common fixed point of f, g, r, s and h, t. Consider,

$$\begin{aligned} d(z,w) = d(fz, gw) &\leq \varphi(m_1(z,w)) \\ &= \varphi(d(hz, hw) + \gamma[d(hz, fz) + d(h, w)]) \\ &\leq \varphi(d(z,w)) < d(z,w), \end{aligned}$$

a contradiction. Therefore $z = w$.

Hence f, g, r, s and h, t have a unique common fixed point. □

References

[1] M. Abbas, G. Jungck, Common fixed point results for non commuting mappings without continuity in cone metric spaces, *J. Math. Anal. Appl.*, **341** (2008), 416-420.

[2] M. Abbas, B.E. Rhoades, Fixed and periodic point results in cone metric spaces, *Appl. Math. Lett.*, **22** (2009), 511-515.

[3] C. Di Bari, P. Vetro, φ-pairs and common fixed points in cone metric spaces, *Rendiconti del Circolo Mathematico di Palermo*, **57**, No. 2 (2008), 279-285.

[4] L.G. Huang, X. Zhang, Cone metric spaces and fixed point theorems of contractive mappings, *J. Math. Anal. Appl.*, **332**, No. 2 (2007), 1468-1476.

[5] R.P. Pant, Rakesh Mohan, Pankaj Kumar Mishrma, Some common fixed point theorems in cone metric spaces, *IJSTM*, **2**, No. 2 (2011), 48-56.

[6] S. Rezapour, Halbarani, Some notes on the paper "Cone metric spaces and fixed point theorem of contractive mappings", *J. Math. Anal. Appl.*, **345** (2008), 719-724.