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A COMMON FIXED POINT THEOREM FOR THREE MAPS IN CONE METRIC SPACES

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Abstract: The existence of coincidence points and common fixed point theorems for three maps satisfying certain contractive conditions without exploiting the notation of continuity of any map involved therein, in cone metric space is proved. Our result extends and generalize the results of Abbas and Jungck [1].

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1. Introduction and Preliminaries

The study of common fixed points of mappings satisfying certain contractive conditions has been at the centre of vigorous research activity, being the applications of fixed point very important in several areas of mathematics. In 2007, Huang and Zhang [6] generalized the concept of a metric space, replacing the set of real numbers by an ordered Banach space and obtained some fixed point theorems for mapping satisfying different contractive conditions. Subsequently Abbas and Jungck [1] and Abbas and Rhoades [2] have studied common fixed point theorems in cone metric spaces (see also [6], [11] and the references mentioned there in) Jungck [8] defined a pair of self-mappings to be weakly compatible if they commute at their coincidence points. In recent years several authors have obtained coincidence point results for various class of mappings on a metric space, utilizing these concepts. For survey of coincidence point theory, its applications, comparision of different contractive conditions and related results,we refer to [4], [5] and the references therein.

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The aim of this paper is to present coincidence points and common fixed point results for three mappings which satisfy generalized contractive conditions.These theorems generalize and extends the results of Abbas and Jungck [1].

In all that follows, E is a real Banach space. For the mappings $f, g : X \longrightarrow X$,let $C(f, g)$ denotes set of coincidence points of f, g, i.e., $C(f, g) := \{z \in X : fz = gz\}$.

We recall some definitions of cone metric spaces and some of their properties [6].

Definition 1.1. Let E be a real Banach space and P be a subset of E.The set P is called a cone if and only if:

(a) P is closed, nonempty and $P \neq \{0\}$;

(b) $a, b \in R$,$a, b \geq 0$,$x, y \in P \Longrightarrow ax + by \in P$;

(c) $x \in P$ and $-x \in P \Longrightarrow x = 0$.

Definition 1.2. Let P be a cone in a Banach space E define partial ordering $\leq$ with respect to p by $x \leq y$ if and only if $y - x \in P$.We shall write $x < y$ to indicate $x \leq y$ but $x \neq y$ while $x \ll y$ will stand for $y - x \in intP$, where $intP$ denotes the interior of the set P. This Cone P is called an order cone.

Definition 1.3. Let E be a Banach Space and $P \subset E$ be an order cone.The order cone P is called normal if there exists $L > 0$ such that for all $x, y \in E$, $0 \leq x \leq y$ implies $\| x \| \leq L \| y \|$.

The least positive number L satisfying the above inequality is called the normal constant of P.

Definition 1.4. Let X be a nonempty set of E. Suppose that the map $d : X \times X \longrightarrow E$ satisfies:

$(d1)$ $0 \leq d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;

$(d2)$ $d(x, y) = d(y, x)$ for all $x, y \in X$;

$(d3)$ $d(x, y) \leq d(x, z) + d(y, z)$ for all $x, y, z \in X$.

Then d is called a cone metric on X and (X, d) is called a cone metric space.

It is obvious that the cone metric spaces generalize metric spaces.

Definition 1.5. Let (X, d) be a cone metric space.We say that $\{x_n\}$ is:

(i) a Cauchy sequence if for every c in E with $0 \ll c$, there is N such that for all $n, m > N$, $d(x_n, x_m) \ll c$;

(ii) a convergent sequence if for any $0 \ll c$, there is an N such that for all $n > N$, $d(x_n, x) \ll c$, for some fixed x in X. We denote this $x_n \longrightarrow x$ $(n \longrightarrow \infty)$. A cone metric space X is said to be complete if every Cauchy sequence in X is convergent in X.

Definition 1.6. Let$f, g : X \longrightarrow X$.Then the pair $(f, g)$is said to be(IT)-Commuting at $z \in X$ if $f(g(z))$=$g(f(z))$ with $f(z)$=$g(z)$.

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2. Common Fixed Point Theorem

In this section we obtain coincidence points and common fixed point theorems for three maps in cone metric spaces.

The following theorem extends and improves Theorem 2.3 of [1].

Theorem 2.1. *Let (X, d) be a cone metric space,and P a normal cone with normal constant L. Suppose mappings $f, g, h : X \longrightarrow X$ satisfy the contractive condition:*

$$d(fx, gy) \leq k(d(fx, hx) + d(gy, hy)), \quad for \ \ all \ \ x, y \in X. \tag{1}$$

where $k \in [0, 1/2)$ is a constant.If $f(X) \cup g(X) \subset h(X)$ and $h(X)$ is a complete subspace of X. Then the maps f,g and h have a coincidence point p in X. Moreover if (f, h) and (g, h) are (IT)-Commuting at p, then f,g and h have a unique common fixed point.

Proof. Suppose x_0 is an arbitrary point of X, and define the sequence $\{y_n\}$ in X such that

$$y_{2n} = fx_{2n} = hx_{2n+1} \quad \text{and} \quad y_{2n+1} = gx_{2n+1} = hx_{2n+2},$$

for all n= 0,1,2,. . . . By equation (1), we have

$$\begin{aligned} d(y_{2n}, y_{2n+1}) &= d(fx_{2n}, gx_{2n+1}) \\ &\leq k(d(fx_{2n}, hx_{2n}) + d(gx_{2n+1}, hx_{2n+1})) \\ &= k(d(y_{2n}, y_{2n-1}) + d(y_{2n+1}, y_{2n})) \\ (1-k)d(y_{2n}, y_{2n+1}) &\leq kd(y_{2n}, y_{2n-1}) \\ \Longrightarrow \quad d(y_{2n}, y_{2n+1}) &\leq \delta d(y_{2n-1}, y_{2n}), \quad where \ \ \delta = \frac{k}{1-k}. \end{aligned}$$

Similarly, it can be shown that $d(y_{2n+1}, y_{2n+2}) \leq d(y_{2n}, y_{2n+1})$.

Therefore, for all n,

$$d(y_{n+1}, y_{n+2}) \leq \delta d(y_n, y_{n+1}) \leq \cdots \leq \delta^{n+1} d(y_0, y_1).$$

Now, for any $m > n$,

$$\begin{aligned} d(y_n, y_m) &\leq d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2}) + \cdots + d(y_{m-1}, y_m). \\ &\leq [\delta^n + \delta^{n+1} + \cdots + \delta^{m-1}] d(y_1, y_0) \\ &\leq \frac{\delta^n}{1-\delta} d(y_1, y_0). \end{aligned}$$

From definition (1.3), we have

$$\| d(y_n, y_m) \| \leq \frac{\delta^n}{1-\delta} L \| d(y_1, y_0) \| .$$

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Which implies that $d(y_n, y_m) \to 0 \ as \ \ n, m \to \infty$,(since $\delta < 1$).

Hence $\{y_n\}$ is a Cauchy sequence, where $y_n = \{hx_n\}$.

Therefore $\{hx_n\}$ is a Cauchy sequence. Since $h(X)$ is complete, there exists q in $h(X)$ such that $hx_n \longrightarrow q$ as $n \longrightarrow \infty$. Consequently, we can find p in X such that $h(p) = q$. We shall show that $hp = fp = gp$. Consider,

$$\begin{aligned} d(fp, gx_{2n+1}) &\leq k(d(fp, hp) + d(gx_{2n+1}, hx2n+1)). \\ \text{Letting } \ n \longrightarrow \infty, \text{we get} & \\ d(fp, q) &\leq k(d(fp, hp) + d(q, q)) \\ &\leq kd(fp, q) \\ d(fp, q) &< 1/2(d(fp, q)), \ \text{ a contradiction.} \end{aligned}$$

Therefore

$$fp = q = hp. \tag{2}$$

similarly,

$$\begin{aligned} d(gp, fx_{2n}) &\leq k(d(gp, hp) + d(fx_{2n}, hx_{2n})). \\ Letting \ \ n \longrightarrow \infty, & \\ d(gp, q) &\leq k(d(gp, hp) + d(hp, hp)) \\ &\leq kd(gp, hp) \\ &< d(gp, q) \ \ \text{with } k < 1, \ \text{ a contradiction.} \end{aligned}$$

Therefore,

$$g(p) = q = h(p). \tag{3}$$

From the equations (2) and (3), it follows that

$$q = hp = fp = gp, \ \ p \ \ is \text{ a coincidence point of } f, g, h. \tag{4}$$

Since, (f, h), (g, h) are (IT)-commuting at p.

$$\begin{aligned} d(ffp, fp) = d(ffp, gp) &\leq k(d(ffp, hfp) + d(gp, hp)) \\ &\leq kd(ffp, fhp) < d(ffp, fp), \end{aligned}$$

a contradiction, since $k < 1$ and $fp = hp$, which implies $ffp = fp$.

$$fp \ = \ ffp = fhp = hfp,$$

$$\Rightarrow ffp = hfp = fp = q. \tag{5}$$

Therefore

$$fp(=q) \text{ is a common fixed point of } f \text{ and } h. \tag{6}$$

similarly, we get,

$$gp = ggp = ghp = hgp,$$

$$\Rightarrow ggp = hgp = gp = q. \tag{7}$$

Therefore

$$gp = fp(=q) \text{ is a common fixed point of } g \text{ and } h. \tag{8}$$

In view of (6) and (8), it follows that f, g and h have a common fixed point namely q. The uniqueness of the common fixed point of q follows equation (1). Indeed, let q_1 be another common fixed point of f, g and h. Consider,

$$\begin{aligned} 0 \le d(q, q_1) = d(fq, gq_1) &\le k(d(fq, hq) + d(gq1, hq_1)) \\ &= k(d(hq, hq) + d(hq_1, hq_1) \\ &= k(0 + 0). \end{aligned}$$

$$\Rightarrow \quad d(q, q_1) \le 0. \text{ Thus } q = q_1.$$

Therefore, f, g and h have a unique common fixed point. □

Remark 2.2. If we take $g = f$ and $h = g$ in Theorem 2.1, then we obtain Theorem 2.3 of [1].Also if we let $g = f$, $h = g$ and g is identity map on X in Theorem 2.1, then we obtain Theorem 3 of [6].

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