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AN ESTIMATE OF VELOCITY PROFILE IN LAMINAR BOUNDARY LAYER

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Abstract: This paper examines the boundary layer thickness developed for an incompressible viscous fluid flowing over a flat plate. The solution of the Blasius problem is considered. In this paper, the authors propose a polynomial variation for velocity u in the boundary layer for the governing partial differential equations(pdes) that are solved by the integral method. A known solution that is close to the Blasius solution given by the sine function together with Balsius solution are compared with the proposed profile. The boundary layer thickness (δ),the skin friction coefficients (C_f) and the coefficients of drag (C_D) obtained are compared in all cases. The proposed velocity profile agrees fairly well to within 3.6% error with the known solutions.

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1. Introduction

From the engineering point of view, the flow of viscous fluids over flat or curved surfaces is important in the determination of, for example, friction or viscous forces on bodies such as ship, airplane or turbine plate. The friction force is obtained from an application of boundary layer theory. Boundary layer is important in the design of for example, diffusers and nozzles. Physiologically, blood and the breath

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flowing through our bodies have boundary layers. We briefly describe boundary layer theory below. Consider an external flow that is incompressible and viscous that is flowing past a stationary semi-infinite flat plate. If the plate is infinite, the boundary layer may become turbulent. Close to this body, the fluid particles get retarded. The magnitude of the retardation depends on the viscosity of the fluid. The thin layer close to the body in which fluid particles start getting retarded is called the boundary layer. The thickness of this layer is often symbolized by δ. An estimate of this layer has been investigated by a number of authors, for example Blasius in[1].

The flow described above is governed by the Navier-Stoke's equations with negligible body forces that may be written as

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}.\nabla)\mathbf{v} = -\frac{1}{\rho}\nabla P + \frac{\mu}{\rho}\nabla^2\mathbf{v} \tag{1}$$

in which $\mathbf{v}$ is the fluid velocity, P is the pressure acting, μ represents viscosity and ρ is the density of the fluid. The term $(\mathbf{v}.\nabla)\mathbf{v}$ in equation (1) is called the inertial term while the term $\frac{\mu}{\rho}\nabla^2\mathbf{v}$ is called the viscous term. The inertial term gives rise to inertial forces that occur in the fluid while the viscous term generate viscous forces. Let u represent the velocity in the boundary layer and U be the velocity of the fluid flowing in the mainstream outside the boundary layer. Due to the rapid increase in velocity from zero on the body to u in boundary layer, the velocity gradient $\frac{\partial u}{\partial y}$ is large. Hence the viscous stress $\mu\frac{\partial u}{\partial y}$ is not negligible in the boundary layer. Thus, outside the boundary layer viscous forces are negligible in comparison. On the other hand, inertial forces are much lager than viscous forces outside the boundary layer. Hence, within the boundary layer, inertial and viscous forces are in equilibrium. Let us define a dimensionless parameter here called the Reynolds number often denoted Re. This is the ratio of inertial forces to viscous forces in the fluid.

2. Mathematical Formulation

For this flow, Continuity and the Navier-Stoke's equations(Equation (1)), may be written as

$$\nabla.\mathbf{v} = 0 \tag{2}$$

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}.\nabla)\mathbf{v} = -\frac{1}{\rho}\nabla P + \nu\nabla^2\mathbf{v} \tag{3}$$

where $\nu = \frac{\mu}{\rho}$ is the kinematic viscosity. Here we assume that the flow is steady, $2-D$ and isothermal. The equations in this case then become

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{4}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \tag{5}$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial P}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) \tag{6}$$

where u and v are the velocity components in $x-$ and $y-$ directions respectively. Simplification based on order of magnitude analysis for parameters, i.e. determining which terms in the equations are 'very small' relative to the other terms as in[2], reduce the equations under consideration to

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{7}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = U\frac{dU}{dx} + \nu\frac{\partial^2 u}{\partial y^2} \tag{8}$$

often called Prandtl's equations. If the following boundary conditions are prescribed, i.e. $u = v = 0$ at $y = 0$ on the flat plate(called the no slip condition) and $u \to U$ as $y \to \infty$, then the pdes in (7) and (8) together with these boundary conditions to be solved constitute the Blasius problem. The solution of these equations describe the velocity profile of the fluid in the boundary layer. This solution was obtained by H. Blasius in 1908. Blasius transformed the partial differential equations into a third order nonlinear ordinary differential equation of the form

$$\frac{1}{2}f\frac{d^2 f}{d\eta^2} + \frac{d^3 f}{d\eta^3} = 0 \tag{9}$$

He achieved this by introducing a non-dimensional variable η called similarity variable by letting it to be $\eta = y\sqrt{\frac{U}{\nu x}}$ (where x is the distance from the leading edge of the flat plate) and defining a function $f(\eta)$ such that $\psi(x, y) = f(\eta)\sqrt{\nu x U}$ where ψ is the stream function given by $u = \frac{\partial \psi}{\partial y}$ and $v = -\frac{\partial \psi}{\partial x}$. The boundary conditions changed accordingly to $f(0) = f'(0) = 0$ and $f'(\eta) = 1$ as $\eta \to \infty$. He solved this equation by series expansion [3] and obtained the result

$$f = \sum_{n=0}^{\infty}\left(-\frac{1}{2}\right)^n \frac{\alpha^{n+1}}{(3n+2)!} C_n \eta^{(3n+2)}$$

where $\alpha = 0.3320$ and $C_0 = 1$, $C_1 = 1$, $C_2 = 11$, $C_3 = 375\ldots$.

C.Toepfer[5] solved numerically the same equation by applying a Runge-Kutta method. L. Howarth[6] solved the same equation with increased accuracy. A table of values for η, f, f' and f'' as generated by[6] for η between 0 and 1 are as shown below.

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η	f	f'	f''
0	0	0	0.33206
0.2	0.00664	0.06641	0.33199
0.4	0.02656	0.13277	0.33147
0.6	0.05974	0.19804	0.33008
0.8	0.10611	0.26471	0.32739
1.0	0.16557	0.32979	0.32301

From the complete table of Howarth (η up to 8.8), the boundary layer thickness is defined as the distance where $f' = \frac{u}{U} = 0.99$ i.e. $u = 99\%$ of U. The value of η corresponding to this 5 and by definition, $\delta\sqrt{\frac{U}{\nu x}} = 5$ from which Blasius found the following result

$$\frac{\delta}{x} = \frac{5}{\sqrt{Re_x}} \tag{10}$$

where Re_x is Reynolds number dependent on x. Let $f(\eta) = \frac{u}{U}$ where $\eta = \frac{y}{\delta}$. Values for $f(\eta)$ and $\frac{y}{\delta}$ can be plotted to yield velocity profile in the boundary layer.

3. Integral Method

The boundary layer equations in the form of equations (7) and (8) may be integrated as follows: Integrate the second equation with respect to y from $y = 0$ to $y = h > \delta$. This activity results in

$$\int_0^h \left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - U\frac{dU}{dx} \right) dy = \nu \int_0^h \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) dy$$

or

$$\int_0^h \left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - U\frac{dU}{dx} \right) dy = \nu \left[\frac{\partial u}{\partial y}\right]_0^h = -\nu\left(\frac{\partial u}{\partial y}\right)_{y=0} = -\frac{\tau_w}{\rho}$$

where

$$\tau_w = \mu\left(\frac{\partial u}{\partial y}\right)_{y=0} \tag{11}$$

is called the wall shear stress. Therefore we have that

$$\int_0^h \left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - U\frac{dU}{dx} \right) dy = -\frac{\tau_w}{\rho} \tag{12}$$

Now v may be written as

$$v = \int_0^y \left(\frac{\partial v}{\partial y}\right) dy$$

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and on using the continuity equation, we have that

$$v = \int_0^y \left(\frac{\partial v}{\partial y}\right) dy = -\int_0^y \left(\frac{\partial u}{\partial x}\right) dy$$

Therefore the term

$$\int_0^h v\frac{\partial u}{\partial y}dy$$

may be written as

$$\int_0^h v\frac{\partial u}{\partial y}dy = -\int_0^h \left(\frac{\partial u}{\partial y}\right)\left[\int_0^y \frac{\partial u}{\partial x}dy\right] dy$$

which may further be written as

$$\int_0^h v\frac{\partial u}{\partial y}dy = -\int_0^h u'v$$

where

$$u' = \left(\frac{\partial u}{\partial y}\right)$$

and

$$v = \int_0^y \frac{\partial u}{\partial x}dy$$

Integrating by parts, we have

$$\int_0^h u'v = [uv]_0^h - \int_0^h uv'$$

so that

$$\int_0^h v\frac{\partial u}{\partial y}dy = -\left[u\int_0^y \frac{\partial u}{\partial x}dy\right]_0^h + \int_0^h u\frac{\partial u}{\partial x}dy$$

or

$$\int_0^h v\frac{\partial u}{\partial y}dy = -U\int_0^h \frac{\partial u}{\partial x}dy + \int_0^h u\frac{\partial u}{\partial x}dy$$

Substituting this result in equation (11) we find

$$\int_0^h \left(2u\frac{\partial u}{\partial x} - U\frac{dU}{dx} - U\frac{\partial u}{\partial x}\right) dy = -\frac{\tau_w}{\rho}$$

Now add and subtract the quantity

$$\frac{dU}{dx}\int_0^h udy$$

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to the left hand side of the above expression and re-arrange the terms to obtain

$$\int_0^h 2u\frac{\partial u}{\partial x}dy - \frac{dU}{dx}\int_0^h (U-u)dy - \frac{d}{dx}\int_0^h Uudy = -\frac{\tau_w}{\rho}$$

or

$$\int_0^h \frac{\partial}{\partial x}(u^2)dy - \frac{d}{dx}\int_0^h Uudy - \frac{dU}{dx}\int 0^h(U-u)dy = -\frac{\tau_w}{\rho}$$

or

$$\int_0^h \frac{\partial}{\partial x}(u^2 - Uu)dy - \frac{dU}{dx}\int_0^h (U-u)dy = -\frac{\tau_w}{\rho}$$

or

$$\int_0^h \frac{\partial}{\partial x}\left[u(U-u)\right]dy + \frac{dU}{dx}\int_0^h (U-u)dy = \frac{\tau_w}{\rho}$$

which implies that

$$\frac{d}{dx}\int_0^\delta u(U-u)dy + \frac{dU}{dx}\int_0^\delta (U-u)dy = \frac{\tau_w}{\rho} \quad (13)$$

in which h is taken as δ. This is the momentum integral equation.

4. Velocity Profile in Boundary Layer

It is possible to assume a velocity profile in equation (12) and obtain an estimate of the boundary layer thickness. The following velocity profiles have been done[1, 4, 5].

4.1. Sine Function

Assume that the velocity profile in the boundary layer is approximated by the sin function in the form $u = a\sin by$ in which a and b are constants. The velocity profile must satisfy the following boundary conditions on the flat plate

(i) $u = 0$ at $y = 0$. This is the no-slip condtion

(ii) $u = U$ at $y = \delta$. This is the condition of continuity when passing from boundary layer profile to potential velocity U

(iii) $\frac{\partial u}{\partial y} = 0$ at $y = \delta$. This condition represents continuity of tangent at the point where the boundary layer and potential solutions are joined

(iv) $\mu\frac{\partial^2 u}{\partial y^2} = \frac{dP}{dx} = 0$ at $y = 0$ i.e. zero pressure gradient along the plate

Using these conditions we obtain the velocity profile

$$u = U \sin(\frac{\pi y}{2\delta}) \tag{14}$$

This is the sinusoidal approximation to velocity profile[1]. It satisfies all the above boundary conditions. Note that since we are considering uniform flow over a flat plate, the derivative of the mainstream velocity U with respect to x in (13) is zero. This yields the equation

$$\frac{d}{dx}\int_0^\delta u(U-u)dy = \frac{\tau_w}{\rho} \tag{15}$$

Consider the integral $\int_0^\delta u(U-u)dy$ in (15). This integral may be written as $U^2\int_0^\delta \frac{u}{U}(1-\frac{u}{U})dy$ or as $U^2\delta\int_0^1 f(\eta)(1-f(\eta))d\eta$. Using this expression in (15) we obtain

$$U^2\int_0^1 f(\eta)\left(1-f(\eta)\right)d\eta\frac{d\delta}{dx} = \frac{\tau_w}{\rho} \tag{16}$$

or

$$U^2C\frac{d\delta}{dx} = \frac{\tau_w}{\rho} \tag{17}$$

where C is given by

$$C = \int_0^1 f(\eta)\left(1-f(\eta)\right)d\eta \tag{18}$$

The wall shear stress in (11) and the velocity profile in (14) yields

$$\tau_w = \frac{\mu U\pi}{2\delta} \tag{19}$$

Also the same velocity profile used in (18) gives C as

$$C = \frac{4-\pi}{2\pi} \tag{20}$$

Using (19) and (20) in (17) and observing that at the leading edge of the flat plate, $x = 0, \delta = 0$, we obtain

$$\int_0^\delta d(\delta^2) = \frac{2\pi^2\nu}{(4-\pi)U}\int_0^x dx$$

from which we find

$$\frac{\delta}{x} = \pi\sqrt{\frac{2}{(4-\pi)}}\sqrt{\frac{\nu}{Ux}}$$

or

$$\frac{\delta}{x} = \frac{4.795}{\sqrt{Re_x}} \tag{21}$$

[5] records a fourth order profile of the form

$$f(\eta) = 2\eta - 2\eta^2 + \eta^4$$

with result

$$\frac{\delta}{x} = \frac{5.84}{\sqrt{Re_x}} \tag{22}$$

We compare thi with the Blasius result.

4.2. Proposed Velocity Profile

Let the velocity distribution be given by $u = a + by + cy^2 + dy^3 + ey^4 + fy^5$. In addition to the boundary conditions listed in subsection 4.1, we have that

$$\frac{\partial^2 u}{\partial y^2} = 0 \quad \text{at} \quad y = \delta \tag{23}$$

i.e continuty of curvature of profile at boundary between boundary layer and potential solutions. Also from condition (iv) we have that

$$\frac{\partial^3 u}{\partial y^3} = 0 \quad \text{on} \quad y = 0 \tag{24}$$

Application of the no slip condition yields $a = 0$. The velocity profile is then

$$u = by + cy^2 + dy^3 + ey^4 + fy^5 \tag{25}$$

Applying condition (ii) gives

$$b\delta + c\delta^2 + d\delta^3 + e\delta^4 + f\delta^5 = U \tag{26}$$

From equation (25) we have

$$\frac{\partial u}{\partial y} = b + 2cy + 3dy^2 + 4ey^3 + 5fy^4 \tag{27}$$

and using condition (iii) we have

$$b + 2c\delta + 3d\delta^2 + 4e\delta^3 + 5f\delta^4 = 0 \tag{28}$$

From equation (27) we have

$$\frac{\partial^2 u}{\partial y^2} = 2c + 6dy + 12ey^2 + 20fy^3 \tag{29}$$

from which we find on using condition (iv)

$$c = 0 \tag{30}$$

Using condition in equation (23) in (29) we obtain

$$6d\delta + 12e\delta^2 + 20f\delta^3 = 0 \tag{31}$$

and finally from (29) we obtain

$$\frac{\partial^3 u}{\partial y^3} = 6d + 24ey + 60fy^2 \tag{32}$$

and on using condition in (24) we obtain

$$d = 0 \tag{33}$$

This yields the following system of equations in b, e and f as unknowns

$$b\delta + e\delta^4 + f\delta^5 = U$$

$$b + 4e\delta^3 + 5f\delta^4 = 0$$

$$3e\delta^2 + 5f\delta^3 = 0 \tag{34}$$

Solution of (34) yields the profile

$$\frac{u}{U} = \frac{5}{3}\frac{y}{\delta} - \frac{5}{3}(\frac{y}{\delta})^4 + (\frac{y}{\delta})^5 \tag{35}$$

or

$$f(\eta) = \frac{5}{3}\eta - \frac{5}{3}\eta^4 + \eta^5 \tag{36}$$

Test for whether the boundary conditions are satisfied or not reveals that the proposed velocity profile also satisfies all the boundary conditions. Using (11) and (18), $\tau_w = \frac{5\mu U}{3\delta}$ and $C = \frac{4650}{37422}$ for this profile. Solving using (17), the result for this profile yields

$$\frac{\delta}{x} = \frac{5.18}{\sqrt{Re_x}} \tag{37}$$

Again we compare this with the Blasius solution.The quantity

$$C_f = \frac{\tau_w}{\frac{1}{2}\rho U^2}$$

is called the coefficient of skin friction. C_f for the three methods are given below

$$\text{Blasius} = \frac{0.664}{\sqrt{Re_x}} \quad \text{Sine function} = \frac{0.655}{\sqrt{Re_x}} \quad \text{Proposed} = \frac{0.644}{\sqrt{Re_x}}$$

The quantity C_D given by

$$2C_f\sqrt{Re_x} = C_D\sqrt{Re_L}$$

where L is length of plate is called the coefficient of drag. C_D for the three methods are given below

$$\text{Blasius} = \frac{1.328}{\sqrt{Re_L}} \quad \text{Sine function} = \frac{1.310}{\sqrt{Re_L}} \quad \text{Proposed} = \frac{1.288}{\sqrt{Re_L}}$$

To obtain a sketch of the velocity profiles for comparison purposes, a few values of data that running from 0 to 1 for the dimensionless distance $\frac{y}{\delta}$ were calculated

0.00 0.2 0.4 0.5 0.6 0.8 1.0

to generate the dimensionless velocity $\frac{u}{U}$ values for the sine function and Proposed methods. The resulting calculated dimensionless velocity values are as follows:

Sine function : 0.00 0.331 0.6342 0.7599 0.8618 0.9783 1.0000

Proposed profile : 0.00 0.3090 0.5878 0.7071 0.8090 0.9511 1.0000

Graphical sketches of $\frac{u}{U}$ against $\frac{y}{\delta}$ were plotted and are depicted in the figure 1 below.

5. Discussion and Conclusion

The laminar boundary layer characteristics of boundary layer thickness, skin friction and drag coefficients for the sine function and proposed methods compare fairly well with the results of Blasius. In particular, the proposed method compares to within 3.6% error.

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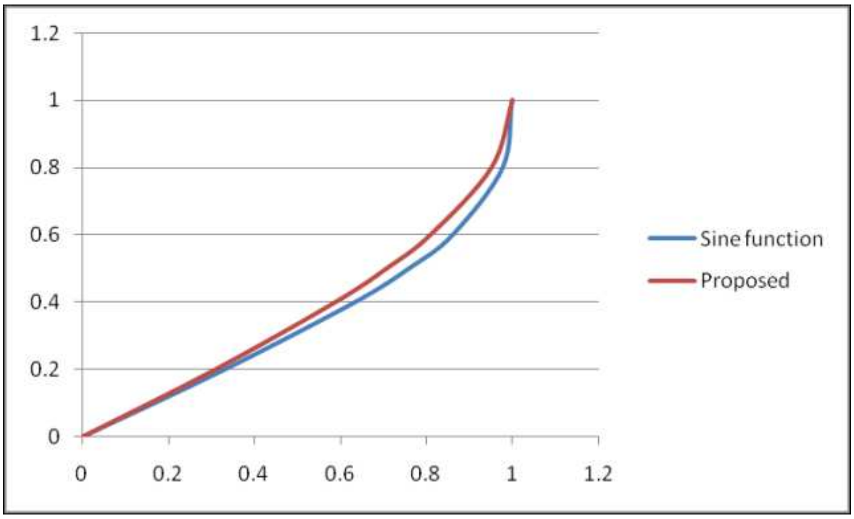

Figure 1: Velocity profiles for sine function and proposed method.

References

[1] P.D. MCormack, L. Crane, *Physical Fluid Dynamics*, Academic Press (1973).

[2] N. Curle, H.J. Davies, *Modern Fluid Dynamics*, Volume 1, Lindon, Van Nostrand (1968).

[3] W.F. Hughes, J.A. Brighton, *Fluid Dynamics*, McGraw-Hill Professional (1999).

[4] C.P. Kothandaraman, R. Rudramoorthy, *Fluid Mechanics and Machinery*, Second Edition, New Age International Publishers, Kolkatal (2007).

[5] H. Schlichting, *Boundary Layer Theory*, McGraw-Hill Series in Mechanical Engineering, Seventh Edition, New York (1979), 126-149.

[6] L. Howarth, On the solution of the laminar boundary layer equations, *Pro. Roy. Soc.*, London, (1938), 547-579.