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COMMON FIXED POINT THEOREMS FOR MULTI-VALUED A^*-MAPPINGS ON A 2-METRIC SPACE

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Abstract: In this paper, we extended the idea of A^*-mappings from metric spaces to 2-metric space. Further, we proved some fixed point theorems for multi-valued A^*-mappings in complete 2-metric space setup. These results extend and improve several results for multi-valued mappings of complete metric spaces to multi-valued mappings of complete 2-metric space.

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Key Words: 2-metric space, fixed points, multi-valued mappings, A^*-mappings

1. Introduction

The concept of 2-metric spaces has been initiated by S. Gahler [9] and these spaces has subsequently been studied many authers like K. Iseki [5], B.E. Rhoades [7], Saha and Dey [10], investigating the existence of fixed point common fixed point for various contractive mappings in 2-metric space. In [1], M. Akram, A.A. Zafar and A.A. Siddiqui introduced a general class of contraction called A-Contraction. In [1], it is proved that the class of A-Contraction is a proper super class of Kannan's Contractions including several other famous contractions like Bianchini's [4] and Reich's [8] contractions. In [2] M. Akram and A.A. Siddiqui proved a fixed point theorem for single valued maps on generalized complete metric spaces. Saha and

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Dey in [11], proved some fixed point theorems for the class of A-contractions on 2-metric space. In [3], M. Akram, A. A. Siddiqui, and A. A. Zafar introduced general multi-valued A^*-maps and proved some fixed point theorems for these maps. In this paper, we extended the idea of A^*-mappings from metric space to 2-metric space and proved some fixed point theorems for multi-valued mappings on complete 2-metric spaces. Lets R_+ denote the set of all non-negative real numbers and A^* stands for the set of all functions $\alpha : R_+^3 \longrightarrow R_+$ satisfying

1. α is continuous on the R_+^3 of all triplets of non-negative reals (with respect to the Euclidean metric on R^3);

2. α is non-decreasing in each coordinate variables;

3. $a \leq kb$ for some $k \in [0, 1)$, whenever $a \leq \alpha(a, b, b)$ or $a \leq \alpha(b, a, b)$ or $a \leq \alpha(b, b, a)$ for all a, b.

Definition 1. Let X be a non-empty set. A real valued function d on $X \times X \times X$ is said to be a 2-metric on X if

1. given distinct element x, y of X, there exists an element z of X such that $d(x, y, z) \neq 0$

2. $d(x, y, z) = 0$ when at least two of x, y, z are equal.

3. $d(x, y, z) = d(x, z, y) = d(y, z, x)$ for all x, y, z in X and

4. $d(x, y, z) \leq d(x, y, w) + d(x, w, z) + d(w, y, z)$ for all x, y, z, w in X.

When d is a 2-metric on X. Then the ordered pair (X, d) is called a 2-metric space.

Definition 2. A sequence $\{x_n\}$ in X is said to be a Cauchy sequence if for each $a \in X$ $\lim d(x_n, x_m, a) = 0$ as $m, n \to \infty$.

Definition 3. A sequence $\{x_n\}$ in X is convergent to an element $x \in X$ if for each $a \in X \lim d(x_n, x, a) = 0$.

Definition 4. A complete 2-metric space is one in which every Cauchy sequence in X converges to an element of X.

Definition 5. A self map T on a 2-metric space X is said to be A-Contraction if for each $u \in X$ it satisfies the following condition

$$\delta(Tx, Ty, u) \leq \alpha(\delta(x, y, u), \delta(x, Tx, u), \delta(y, Ty, u)) \tag{1}$$

for all $x, y \in X$ and for some $\alpha \in A$.

Through out the sequel, $CB(X)$ would denote the set of all closed and bounded subsets of X, where X denote a complete 2-metric space. For the sets A and B in a 2-metric space X, we use the symbols,

$$\delta(A, B, u) = \sup\{d(a, b, u) : a \in A, b \in B, \forall\ u \in X\} \tag{2}$$

and

$$D(A, B, u) = \inf\{d(a, b, u) : a \in A, b \in B, \forall\ u \in X\} \tag{3}$$

A point $z \in X$ is said to be a fixed point of a multi-valued map $T : X \to 2^X$ if $z \in T(z)$.

2. Main Results

Theorem 6. *If $S, T : X \to CB(X)$ are mappings of complete 2-metric space (X,d) such that for every $u \in X$*

$$\delta(Sx, Ty, u) \leq \alpha(\delta(Sx, x, u), \delta(Ty, y, u), \delta(x, y, u)) \tag{4}$$

for all $x, y \in X$. Then S and T have a common fixed point.

Proof. Let $x_0 \in X$, choose a point $x_{2n-1} \in X_{2n-1} = Sx_{2n-2}$ and $x_{2n} \in X_{2n} = Tx_{2n-1}$ for $n = 1, 2, 3, ...$ Then for any $n \in N$

$$\begin{aligned} \delta(X_{2n+1}, X_{2n+2}, u) &= \delta(Sx_{2n}, Tx_{2n+1}, u) \\ &\leq \alpha(\delta(Sx_{2n}, x_{2n}, u), \delta(Tx_{2n+1}, x_{2n+1}, u), \delta(x_{2n}, x_{2n+1}, u)) \\ &\leq \alpha(\delta(X_{2n+1}, X_{2n}, u), \delta(X_{2n+2}, X_{2n+1}, u), \delta(X_{2n}, X_{2n+1}, u)) \end{aligned}$$

Now using the defination of α, there exists some $k \in [0, 1)$, such that, we get

$$\delta(X_{2n+1}, X_{2n+2}, u) \leq k\delta(X_{2n}, X_{2n+1}, u).$$

Continuing in a similar way, we get

$$\delta(X_{2n+1}, X_{2n+2}, u) \leq k^{2n+1}\delta(X_0, X_1, u), \text{for all } n \in N.$$

Which infact gives that

$$d(x_n, x_{n+1}, u) \leq \delta(X_n, X_{n+1}, u) \leq k^n\delta(X_0, X_1, u), \text{for all } n \in N.$$

Thus for any $m, n \in N$ with $m < n$, we see that

$$d(x_m, x_n, u) \leq \sum_{i=m}^{n-1} d(x_i, x_{i+1}, u) \leq \delta(X_0, X_1, u) \sum_{i=m}^{n-1} k^i \to 0$$

as $m, n \to \infty$. Therefore, $\{x_n\}$ be a Cauchy sequence in X. Since X is a complete 2-metric space. So, $x_n \to z \in X$. Consider,

$$\begin{aligned}
\delta(x_{2n+1}, Tz, u) &\leq \delta(X_{2n+1}, Tz, u) \\
&= \delta(Sx_{2n}, Tz, u) \\
&\leq \alpha(\delta(Sx_{2n}, x_{2n}, u), \delta(Tz, z, u), \delta(x_{2n}, z, u)) \\
&\leq \alpha(\delta(X_{2n+1}, X_{2n}, u), \delta(Tz, z, u), \delta(x_{2n}, z, u)) \\
&\leq \alpha(k^{2n}\delta(X_0, X_1, u), \delta(Tz, z, u), \delta(x_{2n}, z, u))
\end{aligned}$$

Taking limit as $n \to \infty$, we get

$$\begin{aligned}
\delta(z, Tz, u) &\leq \alpha(0, \delta(Tz, z, u), \delta(z, z, u)) \\
&= \alpha(0, \delta(Tz, z, u), 0) \\
&\leq \alpha(0, \delta(z, Tz, u), 0) \\
&\leq k(0) = 0.
\end{aligned}$$

with $k \in [0, 1)$. Which finally gives, $\delta(z, Tz, u) = 0$. But, we know that $D(z, Tz, u) \leq \delta(z, Tz, u) = 0$. It means that $D(z, Tz, u) = 0$. This implies $z \in \overline{Tz}$. But $\overline{Tz} = Tz$ as $Tz \in CB(X)$. Therefore, $z \in Tz$. Similarly, one can show that $z \in Sz$. Thus z is a common fixed point of S and T. □

Corollary 7. *If $S, T : X \to CB(X)$ are mappings such that for each $u \in X$,*

$$d(Sx, Ty, u) \leq h\left(\delta(x, Sx, u) + \delta(y, Ty, u), \right)$$

for all $x, y \in X$ and $0 < h < 1/2$. Then S and T have common fixed point.

Proof. By defining $\alpha : R_+^3 \longrightarrow R_+$ by $\alpha(u, v, w) = h(u + v)$, where $0 < h < 1/2$. One can easily see that $\alpha \in A^*$. So, by taking $u = \delta(Sx, x, u)$, $v = \delta(Tx, x, u)$ and $w = \delta(x, y, u)$, we have $\delta(Sx, Ty, u) \leq h\left(\delta(x, Sx, u) + \delta(y, Ty, u)\right) = h\left(u + v\right) = \alpha(u, v, w)$ because $\delta(Sx, Ty, u) \leq \alpha(\delta(x, Sx, u), \delta(y, Ty, u), \delta(x, y, u))$. Hence by Theorem 6, We have the conclusion. □

Corollary 8. *If $S, T : X \to CB(X)$ are mappings such that for each $u \in X$*

$$\delta(Sx, Ty, u) \leq h\left(\delta(x, Sx, u)\delta(y, Ty, u)\right)^{1/2},$$

for all $x, y \in X$ and $0 < h < 1$. Then S and T have common fixed point.

Proof. If we define $\alpha : R_+^3 \longrightarrow R_+$ by $\alpha(u, v, w) = h(uv)^{1/2}$ where where $0 < h < 1$. One can easily see that $\alpha \in A^*$. Now taking $u = \delta(x, Sx, u)$, $v = \delta(y, Ty, u)$ and $w = \delta(x, y, u)$, we have

$$\alpha(\delta(x, Sx, u), \delta(y, Ty, u), \delta(x, y, u)) = h\left(\delta(x, Sx, u)\ \delta(y, Ty, u)\right)^{1/2}.$$

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Using the equation given above, we have

$$\begin{aligned}\delta(Sx,Ty,u) &\leq h\,(\delta(x,Sx,u)\ \delta(y,Ty,u))^{1/2}\\ &= \alpha(\delta(x,Sx,u),\delta(y,Ty,u),\delta(x,y,u))\\ \delta(Sx,Ty,u) &\leq \alpha(\delta(x,Sx,u),\delta(y,Ty,u),\delta(x,y,u)).\end{aligned}$$

Hence by Theorem 6, we get the conclusion. □

Theorem 9. *If $T : X \to CB(X)$ be a mapping of complete 2-metric space (X,d) such that for every $u \in X$*

$$\delta(Tx,Ty,u) \leq \alpha(\delta(Tx,x,u),\delta(Ty,y,u),\delta(x,y,u)) \tag{5}$$

for all $x, y \in X$. Then T has a fixed point.

Proof. If we take $S = T$ in Theorem 6. We have required result. □

Theorem 10. *Let $f : X \to X$ and $S, T : X \to CB(X)$ be mappings such that*

- *f, S and T are continuous.*
- *$Sx \subseteq f(X)$ and $Tx \subseteq f(X)$.*
- *f weakly commutes with S and T.*
- *for every $u \in X$*

$$\delta(Sx,Ty,u) \leq \alpha(\delta(Sx,x,u),\delta(Ty,y,u),\delta(x,y,u))$$

for all $x, y \in X$. Then f, S and T have a common fixed point.

Proof. Let $x_0 \in X$, we may define a sequence $x_n \in X$ as $x_{2n-1} = fx_{2n-2} \in X_{2n-1} = Sx_{2n-2}$ and $x_{2n} = fx_{2n-1} \in X_{2n} = Tx_{2n-1}$, where $n = 1, 2, 3, ...$

$$d(fx_{2n}, fx_{2n+1}, u) \leq \delta(X_{2n+1}, X_{2n+2}, u) = \delta(Sx_{2n}, Tx_{2n+1}, u),$$

by using this

$$\begin{aligned}\delta(Sx_{2n},Tx_{2n+1},u) &\leq \alpha(\delta(Sx_{2n},x_{2n},u),\delta(Tx_{2n+1},x_{2n+1},u),\delta(x_{2n},x_{2n+1},u))\\ &\leq \alpha(\delta(X_{2n+1},X_{2n},u),\delta(X_{2n+2},X_{2n+1},u),\delta(X_{2n},X_{2n+1},u)).\end{aligned}$$

This implies that

$$\begin{aligned}\delta(X_{2n+1},X_{2n+2},u) &\leq \alpha(\delta(X_{2n},X_{2n+1},u),\delta(X_{2n+1},X_{2n+2},u),\delta(X_{2n},X_{2n+1},u))\\ &\leq k\delta(X_{2n},X_{2n+1},u), \text{by property (3) of } \alpha.\end{aligned}$$

Repeating this step, we have

$$\begin{aligned} d(fx_{2n}, fx_{2n+1}, u) &\leq \delta(X_{2n+1}, X_{2n+2}, u) \\ &\leq k^{2n+1}\ \delta(X_0, X_1, u). \end{aligned}$$

for all $n \in N$. Which in fact gives that

$$d(fx_{n-1}, fx_n, u) \leq \delta(X_n, X_{n+1}, u),$$

for all $n \in N$. Thus for any $m, n \in N$ with $m < n$, we see that

$$\begin{aligned} d(fx_m, fx_n, u) &\leq d(fx_m, fx_{m+1}, u) + d(fx_{m+1}, fx_{m+2}) + ... + d(fx_{n-1}, fx_n, u) \\ &\leq \delta(X_0, X_1, u)\ \{k^m + k^{m+1} + ... + k^{n-m-1}\} \\ &= \delta(X_0, X_1)\ k^m\{1 + k + k^2 + ...\} \\ &= \delta(X_0, X_1, u)\ k^m(1/1 - k). \end{aligned}$$

when $m \to \infty$, we get

$$d(fx_m, fx_n, u) = 0.$$

Thus $\{fx_n\}$ be a Cauchy sequence in X, and hence $fx_n \to x \in X$. Then

$$\begin{aligned} D(x_{2n}, Sx, u) &\leq \delta(X_{2n}, Sx, u) \\ &\leq \delta(Tx_{2n-1}, Sx, u) \\ &\leq \alpha(\delta(Tx_{2n-1}, x_{2n-1}, u), \delta(Sx, x, u), \delta(x_{2n-1}, x, u)) \\ &\leq \alpha(\delta(X_{2n}, X_{2n-1}, u), \delta(Sx, x, u), \delta(x_{2n-1}, x, u)) \\ &\leq \alpha(k^{2n-1}\ \delta(X_0, X_1, u), \delta(Sx, x, u), \delta(fx_{2n-2}, x, u)). \end{aligned}$$

Taking limit $n \to \infty$ and using continuty of α, we have

$$\begin{aligned} D(x, Sx, u) &\leq \alpha(0, \delta(Sx, x, u), 0) \\ &\leq k(0) = 0, \text{by the property(3) of } \alpha. \end{aligned}$$

Hence $D(x, Sx, u) \leq 0$. This implies that $D(x, Sx, u) = 0$. Which gives $x \in \overline{Sx}$. But $\overline{Sx} = Sx$ as $Sx \in CB(X)$. Suppose that f and S are continues. Since f is weakly commutes with S. So, we have

$$\begin{aligned} d(x, fx, u) &\leq d(x, fx, ffx_{2n}) + d(x, ffx_{2n}, u) + d(ffx_{2n}, fx, u) \\ &\leq d(x, fx, ffx_{2n}) + q\ H(Sx, fSx_{2n}, u) + d(ffx_{2n}, fx, u)\ , q > 1 \\ &\leq d(x, fx, ffx_{2n}) + q\ \{H(Sx, Sfx_{2n}, u) + H(Sfx_{2n}, fSx_{2n}, u)\} \\ &+ d(ffx_{2n}, fx, u) \\ &\leq d(x, fx, ffx_{2n}) + q\ \{H(Sx, Sfx_{2n}, u) + D(fx_{2n}, Sx_{2n}, u)\} \end{aligned}$$

$$+ \quad d(ffx_{2n}, fx, u).$$

As, we know from the construction of the sequence $fx_{2n} \in Sx_{2n} = X_{2n+1}$, therefore from the equation given above, we get

$$d(x, fx, u) \leq d(x, fx, ffx_{2n}) + q\ \{H(Sx, Sfx_{2n}, u) + 0\} + d(ffx_{2n}, fx, u).$$

Taking limit $n \to \infty$, we have

$$\begin{aligned} d(x, fx, u) &\leq d(x, fx, fx) + q\ H(Sx, Sx, u) + d(fx, fx, u) \\ &\leq 0. \end{aligned}$$

This gives $d(x, fx, u) \leq 0$. Which implies that $d(x, fx, u) = 0$ or $x = fx$. Thus $x = fx \in Sx \bigcap Tx$ as required. □

Corollary 11. *Let $f : X \to X$ and $S, T : X \to CB(X)$ be mapping satisfying conditions(1-3) of Theorem 10 and for each $u \in X$, such that*

$$\delta(Sx, Ty, u) \leq h\ (\delta(Sx, x, u) + \delta(Ty, y, u))$$

for all $x, y \in X$ and $h \in [0, 1/2)$. Then f, S and T have common fixed point.

Corollary 12. *Let $f : X \to X$ and $S, T : X \to CB(X)$ be mapping satisfying conditions(1-3) of Theorem 10 and for each $u \in X$,such that*

$$\delta(Sx, Ty, u) \leq h(\delta(Sx, x, u)\ \delta(Ty, y, u))^{1/2}$$

and for all $x, y \in X$ and $h \in [0, 1)$. Then f, S and T have common fixed point.

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