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FINITELY GENERATED s_1 IDEALS IN COMMUTATIVE RINGS

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Abstract: Let R be a commutative ring with identity, and let a be a nonzero element of R.

The principal ideal I = <a>is called s_1 ideal of R if, and only if the following holds:

If ab = a, b∈R. There exists a/ ∈R, a/ ≠ 0 such that a/b = 0 [1].

Now, let a_1, a_2, ..., a_n be any n nonzero elements in R. This paper deals with a new definition for the finitely generated ideal I = <a_1, a_2, ..., a_n>of the ring R that we call finitely generated s_1 ideal. We prove the following result among others, if A^- = B^-, where A=(a_1, a_2,..., a_n) ∈$R^n$and B=(b_1, b_2,...,b_n) ∈R^n then <a_1, a_2, ..., a_n>is $s_1$ideal of R if, and only if <b_1,b_2, ..., b_n>is $s_1$ideal of R.

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1. Introduction

Throughout this work we use the following notations:

R: Commutative ring with identity

R^n: Set of all n-tuples with components in a ring R and operations defined componentwise.

$M_n(R)$: Set of all $n \times n$ matrices whose entries belong to R

<a_1, a_2,..., a_n>: The finitely generated ideal in the ring R with generators (a_1, a_2, ..., a_n) ∈R^n.

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Let $A=(a_1,a_2,\ldots,a_n)$ be an orderd n-tuples in R^n we denote by:

$A^- = \{x=(x_1, x_2,\ldots,x_n) \in R^n : AX^T = \sum_{i=1}^{n} a_i x_i = 0\}$.

This paper deals with a new definition for a finitely generated ideal I=<a_1, a_2, ..., a_n>,

$a_i \in R$ and $a_i \neq 0$ i=1,2,...,n that we call s_1 ideal of R if, and only if the following holds:

If AM=A where $A=(a_1,a_2,\ldots,a_n) \in R^n$ and $M=(m_{ij}) \in M_n(R)$, (1)

there exists $A' = (a'_1,a'_2,\ldots,a'_n) \in R^n, a'_i \neq 0$, i=1,2,...,n such that:

$A'M = 0$. (2)

This definition will serve as our main tool throughout this work first we give two examples the first one is for a principal ideal which is $s_1$ideal and the second one is for a principal ideal which is not $s_1$ideal

1. I=<2>is $s_1$ideal of z_6. [1]

2. I=<4>is not $s_1$ideal of z_{12}

2. Main Results

In this section we state down the following results:

Theorem 1. *If $A^- = B^-$ where $A=(a_1,a_2, \ldots, a_n) \in R^n$ and $B=(b_1,b_2,\ldots, b_n) \in R$, then $<a_1, a_2,\ldots,a_n>$ is s_1 ideal of a ring R if, and only if $<b_1,b_2,\ldots,b_n>$ is s_1 ideal of R.*

Theorem 2. *If $I=<a_1,a_2,\ldots,a_n>$, $J=<b>$ are two ideals of a ring R, b is not a zero divisor of R, then $IJ= <a_1b, a_2b,\ldots,a_nb>$ is s_1 ideal of R if, and only if $I=<a_1,a_2,\ldots,a_n>$ is s_1 ideal of R.*

Theorem 3. *If $<a_1,a_2,\ldots,a_n>$ is s_1 ideal of a ring R, then $<a_i>$ is s_1 ideal of R for each i=1,2,...,n.*

Theorem 4. *Let R and S be any two commutative rings with identity, let φ be an isomorphism from R into S then $<a_1,a_2,\ldots,a_n>$ is s_1 ideal of R if, and only if $<\varphi(a_1),\varphi(a_2),\ldots\varphi(a_n)>$ is s_1 ideal of S.*

Theorem 5. *Let R be a commutative ring with identity if $<a_1,a_2,\ldots,a_n>$ is s_1 ideal of R[x],* then $<a_1,a_2,\ldots,a_n>$ is $s_1$ideal of R.

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3. The Proofs

In this section we prove our main results:

Proof of theorem 1. Assume that $\langle a_1,a_2,\ldots,a_n\rangle$ is s_1 ideal of a ring R with A- = B-, to prove that $\langle b_1,b_2,\ldots,b_n\rangle$ is s_1 ideal of R, let

BM=B, where $B=(b_1,b_2,\ldots,b_n) \in R^n$ and $M=(m_{ij}) \in M_n(R)$ i,j=1,2,...,n

It follows that B(M-I)=0, where I is the identity matrix. This means that each column of (M-I) belong to B-, but B-=A- this implies that each column of (M-I) belong to A-.

Thus A (M-I) =0, which means that:

AM=A. Since $\langle a_1,a_2,\ldots,a_n\rangle$ is s_1 ideal of the ring R, using (2) there exists

$A\prime = (a_1\prime,a_2\prime,\ldots,a_n\prime) \in R^n$ such that:

$A'M=0$ (3)

It follows that $\langle b_1, b_2,\ldots,b_2\rangle$ is s_1 of a ring R. Using exactly the same way, we can prove that if $\langle b_1,b_2,\ldots,b_n\rangle$ is s_1 ideal of R then $\langle a_1, a_2, \ldots, a_n\rangle$ is s_1 ideal of R.

Proof of theorem 2. To prove that the ideal $IJ=\langle a_1b,a_2b,\ldots,a_nb\rangle$[2]

is $s_1$ideal of a ring R if, and only if $I=\langle a, a, \ldots,a\rangle$ is s_1 ideal of R, it is enough to prove that E-=A- where $E=(a_1 b,a_2b,\ldots,a_n b)\in R^n$ and $A=(a_1,a_2,\ldots,a_n) \in R^n$

Now let $x=(x_1, x_2\ldots,x_n) \in E+$, this means that

$\sum_{i=1}^{n} ba_ix_i =o$ which implies that

$b\sum_{i=1}^{n} a_ix_i =0$, since b is not a zero divisor of R we get [3]

$\sum_{i=1}^{n} a_ix_i =0$, hence $x\in A+$

In the same way we prove that if $x\in A+$ then $x\in E+$, using theorem (1) we get the result.

Proof of theorem 3. Assume that $\langle a_1,a_2,\ldots,a_n\rangle$ is $s_1$ideal of a ring R, to prove that $\langle a_i\rangle$ is s_1 ideal of R, i=1,2,...,n.

Let $a_ib_i = a_i$, i=1,2,...,n

We get $(a_1,a_2,\ldots,a_n) \begin{bmatrix} b1\,0\,0\ldots0 \\ 0\ b2\,0..0 \\ \ldots\ldots\ldots\ldots \\ 0\ldots\ \ 0\,bn \end{bmatrix} = (a_1,a_2,\ldots,a_n)$

Since $\langle a_1,a_2,\ldots,a_n\rangle$ is s_1 ideal of R, using (2) we get there exists $(a_1\prime, a_2\prime,\ldots,a_n\prime) \in R^n$, $a_i\prime \neq 0$

i=1,2,...,n such that:

$$(a_1', a_2',\ldots,a_n')\begin{bmatrix} b1\,0\ldots 0 \\ 0\; b2\,0..0 \\ \ldots\ldots\ldots\ldots \\ 0\ldots\ldots 0\,bn \end{bmatrix} = 0$$

Thus $a_i' b_i = 0$, i = 1,2,...n and therefore $<a_i>$is s_1 ideal of R for each i=1,2,...,n.

Proof of theorem 4. To prove that $<\varphi(a_1),\varphi(a_2),\ldots,\varphi(a_n)>$is s_1 ideal of S, let

$(\varphi(a_1),\varphi(a_2),\ldots,\varphi(a_n)(m_{ij}) = (\varphi(a_1),\varphi(a_2),\ldots,\varphi(a_n))$, where $(m_{ij}) \in M_n(S)$.This implies that

$(a_1,a_2,\ldots,a_n)(\varphi^{-1}m_{ij}) =(a_1,a_2,\ldots,a_n)$, $(\varphi^{-1}m_{ij}) \in M_n(R)$. [4]

Since $<a_1,a_2,\ldots,a_n>$is $s_1$ideal of R, using (2) we get that there exists $(a_1',a_2',\ldots,a_n') \in R^n$, $a_i' \neq 0$ i=1,2,...,n

Such that:

$(a_1',a_2',\ldots,a_n')(\varphi^{-1}m_{ij}) =0$, hence

$(\varphi(a_1'),\varphi(a_2'),\ldots,\varphi(a_{n'})(m_{ij}) =0$, which means that $<\varphi(a_1),\varphi(a_2),\ldots,\varphi(a_n)>$is s_1 ideal of S.

In the same way we can prove that if $<\varphi(a_1),\varphi(a_2),\ldots,\varphi(a_n)>$is s_1 ideal of S then

$<a_1,a_2,\ldots,a_n>$is s_1 ideal of R.

Proof of theorem 5. To prove that$<a_1,a_2,\ldots,a_n>$is s_1 ideal of R, let

$(a_1,a_2,\ldots,a_n)(m_{ij}) =(a_1,a_2,\ldots,a_n)$, where $(m_{ij}) \in M_n(R)$, i,j=1,2,...,n.

Since $<a_1,a_2,\ldots,a_n>$is s_1 ideal of R[x], using (2) there exists $(f_1(x),f_2(x),\ldots,f_n(x)) \in (R[x])^n$

Such that:

$(f_1(x),f_2(x),\ldots,f_n(x))(m_{ij}) =0$,where $f_i(x)=a_0^i+a_1^i x+\bullet\bullet\bullet+a_n^i x$, i=1,2,...,n

This means that we there exists $(a_0^1,a_0^2,\ldots,a_0^n) \in R^n$ with

$(a_0^1,a_0^2,\ldots,a_0^n)(m_{ij}) =0$ [5]

Hence we get that $<a_1,a_2,\ldots,a_n>$is s_1 ideal of R.

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