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ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF NONLINEAR FRACTIONAL ORDER RIEMANN-LIOUVILLE VOLTERRA-STIELTJES QUADRATIC INTEGRAL EQUATIONS

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Abstract: Our aim in this article is to study the existence and the asymptotic stability of solutions of a class of fractional order functional Riemann-Liouville Volterra-Stieltjes integral equations by using the Schauder fixed point theorem.

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1. Introduction

Fractional calculus is a generalization of ordinary differentiation and integration to arbitrary non-integer order. The subject is as old as the differential calculus, starting from some speculations of G.W. Leibniz (1697) and L. Euler (1730), and since then, it has continued to be developed up to nowadays. Integral equations are one of the most useful mathematical tools in both pure and applied analysis. This is particularly true of problems in mechanical vibrations and the related fields of engineering and mathematical physics. We can find numerous applications of

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differential and integral equations of fractional order in viscoelasticity, electrochemistry, control, porous media, electromagnetism, etc., [8, 19, 29]. There has been a significant development in ordinary and partial fractional differential and integral equations in recent years; see the monographs of Abbas et al. [5], Kilbas et al. [22], Miller and Ross [23], Podlubny [26], Samko et al. [27], and the papers by Abbas *et al.* [1, 2, 3, 4, 6], Banaś et al. [9, 10, 11, 12, 13], Darwish et al. [14], Dhage [15, 16, 17, 18] and the references therein.

In [9, 13], Banaś et al. used the technique associated with a certain measure of noncompactness related to monotonicity for the study of the existence of solutions for the following nonlinear Volterra-Stieltjes quadratic integral equation,

$$x(t) = a(t) + f(t, x(t)) \int_0^t u(t, \tau, x(\tau)) d_\tau g(t, \tau); \ t \geq 0, \tag{1}$$

where $a : [0, \infty) \to [0, \infty)$ and $f : [0, \infty) \times \mathbb{R} \to \mathbb{R}$ are given continuous functions.

Recently, and by using the techniques of some fixed point theorems, Abbas *et al.* in [4], proved some existence results concerning the existence and attractivity of the solutions for the following nonlinear quadratic Volterra integral equation of fractional order,

$$\begin{aligned} u(t,x) &= f(t,x,u(t,x),u(\alpha(t),x)) + \tfrac{1}{\Gamma(r)} \int_0^{\beta(t)} (\beta(t)-s)^{r-1} \\ &\quad \times g(t,x,s,u(s,x),u(\gamma(s),x))ds; \ (t,x) \in \mathbb{R}_+ \times [0,b], \end{aligned} \tag{2}$$

where $b > 0$, $r \in (0, \infty)$, $\alpha, \ \beta, \ \gamma : \mathbb{R}_+ \to \mathbb{R}_+$, $f : \mathbb{R}_+ \times [0, b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R}_+ \times [0, b] \times \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are given continuous functions, and $\Gamma(\cdot)$ is the (Euler's) Gamma function defined by

$$\Gamma(\xi) = \int_0^\infty t^{\xi-1} e^{-t} dt; \ \xi > 0.$$

This paper deals with the existence and the stability of solutions to the following nonlinear fractional order Riemann-Liouville Volterra-Stieltjes quadratic integral equations of the form,

$$\begin{aligned} u(t,x) &= f(t,x,u(t,x),u(\alpha(t),x)) + \tfrac{1}{\Gamma(r)} \int_0^{\beta(t)} (\beta(t)-s)^{r-1} \\ &\quad \times h(t,x,s,u(s,x),u(\gamma(s),x)) d_s g(t,s); \ (t,x) \in J := \mathbb{R}_+ \times [0,b], \end{aligned} \tag{3}$$

where $b > 0$, $r \in (0, \infty)$, $\alpha, \ \beta, \ \gamma : \mathbb{R}_+ \to \mathbb{R}_+$, $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $g : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$, $h : J_1 \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are given continuous functions and $J_1 = \{(t, x, s) \in J \times \mathbb{R}_+ : s \leq t\}$.

Our investigations are conducted in Banach spaces with an application of Schauder's fixed point theorem for the existence of solutions of the equation (3). Also, we prove that all solutions are locally asymptotically stable. Finally, we present an example illustrating the applicability of the imposed conditions.

2. Preliminaries

In this section, we introduce notations, definitions, and preliminary facts which are used throughout this paper. By $L^1([0,a]\times[0,b])$, for $a,b>0$, we denote the space of Lebesgue-integrable functions $u:[0,a]\times[0,b]\to\mathbb{R}$ with the norm

$$\|u\|_1=\int_0^a\int_0^b|u(t,x)|dxdt.$$

By $BC:=BC(J)$ we denote the Banach space of all bounded and continuous functions from J into $\mathbb{R}$ equipped with the standard norm

$$\|u\|_{BC}=\sup_{(t,x)\in J}|u(t,x)|.$$

For $u_0\in BC$ and $\eta\in(0,\infty)$, we denote by $B(u_0,\eta)$, the closed ball in BC centered at u_0 with radius η.

Definition 2.1. (see [27]) Let $r\in(0,\infty)$ and $u\in L^1([0,a]\times[0,b])$. The partial Riemann-Liouville integral of order r of $u(t,x)$ with respect to t is defined by the expression

$$I^r_{0,t}u(t,x)=\frac{1}{\Gamma(r)}\int_0^t(t-s)^{r-1}u(s,x)ds,\ \text{for almost all } (t,x)\in[0,a]\times[0,b].$$

Analogously, we define the integral

$$I^r_{0,x}u(t,x)=\frac{1}{\Gamma(r)}\int_0^x(x-s)^{r-1}u(t,s)ds,\ \text{for almost all } (t,x)\in[0,a]\times[0,b].$$

Example 2.2. Let $\lambda,\omega\in(-1,\infty)$ and $r\in(0,\infty)$, then

$$I^r_{0,t}t^\lambda x^\omega=\frac{\Gamma(1+\lambda)}{\Gamma(1+\lambda+r)}t^{\lambda+r}x^\omega,\ \text{for almost all } (t,x)\in[0,a]\times[0,b].$$

If u is a real function defined on the interval $[a,b]$, then the symbol $\bigvee_a^b u$ denotes the variation of u on $[a,b]$. We say that u is of bounded variation on the interval $[a,b]$ whenever $\bigvee_a^b u$ is finite. If $w:[a,b]\times[c,b]\to\mathbb{R}$, then the symbol $\bigvee_{t=p}^q w(t,s)$ indicates the variation of the function $t\to w(t,s)$ on the interval $[p,q]\subset[a,b]$, where s is arbitrarily fixed in $[c,d]$. In the same way we define $\bigvee_{s=p}^q w(t,s)$. For the properties of functions of bounded variation we refer to [25].

If u and φ are two real functions defined on the interval $[a,b]$, then under some conditions (see [25]) we can define the Stieltjes integral (in the Riemann-Stieltjes sense)

$$\int_a^b u(t)d\varphi(t)$$

of the function u with respect to φ. In this case we say that u is Stieltjes integrable on $[a, b]$ with respect to φ. Several conditions are known guaranteeing Stieltjes integrability [25, 28]. One of the most frequently used requires that u is continuous and φ is of bounded variation on $[a, b]$.

In what follows we will use a few properties of the Stieltjes integral contained in the below given lemmas:

Lemma 2.3. (see [24]) *If u is Stieltjes integrable on the interval $[a, b]$ with respect to a function φ of bounded variation, then*

$$\left|\int_a^b u(t)d\varphi(t)\right| \leq \int_a^b |u(t)|d\left(\bigvee_a^t \varphi\right).$$

Lemma 2.4. (see [24]) *Let u and v be Stieltjes integrable functions on the interval $[a, b]$ with respect to a nondecreasing function φ such that $u(t) \leq v(t)$ for $t \in [a, b]$. Then*

$$\int_a^b u(t)d\varphi(t) \leq \int_a^b v(t)d\varphi(t).$$

In the sequel we will also consider Stieltjes integrals of the form

$$\int_a^b u(t)d_s g(t, s),$$

and Riemann-Liouville Stieltjes integrals of fractional order of the form

$$\frac{1}{\Gamma(r)}\int_0^t (t-s)^{r-1}u(s)d_s g(t, s),$$

where $g : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R},\ r \in (0, \infty)$ and the symbol d_s indicates the integration with respect to s.

Let $\emptyset \neq \Omega \subset BC$, and let $G : \Omega \to \Omega$, and consider the solutions of equation

$$(Gu)(t, x) = u(t, x). \tag{4}$$

Inspired by the definition of the attractivity of solutions of integral equations (for instance [11]), we introduce the following concept of attractivity of solutions for equation (4).

Definition 2.5. Solutions of equation (4) are locally attractive if there exists a ball $B(u_0, \eta)$ in the space BC such that, for arbitrary solutions $v = v(t, x)$ and $w = w(t, x)$ of equations (4) belonging to $B(u_0, \eta) \cap \Omega$, we have that, for each $x \in [0, b]$,

$$\lim_{t\to\infty}(v(t, x) - w(t, x)) = 0. \tag{5}$$

When the limit (5) is uniform with respect to $B(u_0, \eta) \cap \Omega$, solutions of equation (4) are said to be uniformly locally attractive (or equivalently that solutions of (4) are locally asymptotically stable).

Lemma 2.6. (see [7], p. 62) *Let $D \subset BC$. Then D is relatively compact in BC if the following conditions hold:*

(a) D is uniformly bounded in BC.

(b) The functions belonging to D are almost equicontinuous on $\mathbb{R}_+ \times [0, b]$, i.e., *equicontinuous on every compact subset of $\mathbb{R}_+ \times [0, b]$.*

(c) The functions from D are equiconvergent, that is, given $\epsilon > 0$, $x \in [0, b]$ there corresponds $T(\epsilon, x) > 0$ such that $|u(t, x) - \lim_{t\to\infty} u(t, x)| < \epsilon$ for any $t \geq T(\epsilon, x)$ and $u \in D$.

3. Main Results

In this section, we are concerned with the existence and the local asymptotic stability of solutions for the equation (3). Let us start by defining what we mean by a solution of the equation (3).

Definition 3.1. We mean by a solution of equation (3), every function $u \in BC$ such that u satisfies equation (3) on J.

The following hypotheses will be used in the sequel.

(H_1) The functions $\alpha, \beta, \gamma : \mathbb{R}_+ \to \mathbb{R}_+$ are continuous and $\lim_{t\to\infty} \alpha(t) = \infty$.

(H_2) The function f is continuous and there exist positive constants M and L such that $M + L < 1$ and

$$|f(t, x, u_1, u_2) - f(t, x, v_1, v_2)| \leq M|u_1 - v_1| + L|u_2 - v_2|,$$

for $(t, x) \in J$ and $u_1, u_2, v_1, v_2 \in \mathbb{R}$.

(H_3) The function $t \to f(t, x, 0, 0)$ is bounded on J with

$$f^* = \sup_{(t,x)\in\mathbb{R}_+\times[0,b]} f(t, x, 0, 0) \quad and \quad \lim_{t\to\infty} |f(t, x, 0, 0)| = 0, \ x \in [0, b].$$

(H_4) For all $t_1, t_2 \in \mathbb{R}_+$ such that $t_1 < t_2$ the function $s \mapsto g(t_2, s) - g(t_1, s)$ is nondecreasing on $\mathbb{R}_+$.

(H_5) The function $s \mapsto g(0, s)$ is nondecreasing on $\mathbb{R}_+$.

(H_6) The functions $s \mapsto g(t, s)$ and $t \mapsto g(t, s)$ are continuous on $\mathbb{R}_+$ for each fixed $t \in \mathbb{R}_+$ or $s \in \mathbb{R}_+$, respectively.

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(H_7) The function h is continuous and there exist continuous functions $p_1, p_2 : J_1 \to \mathbb{R}_+$, $\Phi, \Psi : \mathbb{R}_+ \to \mathbb{R}_+$ such that Φ and Ψ are nondecreasing and

$$|h(t,x,s,u,v)| \le p_1(t,x,s)\Phi(|u|) + p_2(t,x,s)\Psi(|v|);\ (t,x,s) \in J_1,\ u,v \in \mathbb{R}.$$

Moreover, assume that $\lim\limits_{t\to\infty} \int_0^{\beta(t)} (\beta(t)-s)^{r-1} p_i(t,x,s) d_s g(t,s) = 0;\ i = 1,2.$

Remark 3.2. Set

$$p_i^* := \sup_{(t,x)\in J'} \frac{1}{\Gamma(r)} \int_0^{\beta(t)} (\beta(t)-s)^{r-1} p_i(t,x,s) d_s \left(\bigvee_{k=0}^{s} g(t,k) \right);\ i = 1,2.$$

From hypothesis (H_7), we infer that p_i^* is finite, for $i = 1,2$.

Theorem 3.3. *Assume that hypotheses $(H_1) - (H_7)$ hold. If there exists a constant $\eta > 0$, such that*

$$f^* + p_1^*\Phi(\eta) + p_2^*\Psi(\eta) \le \eta(1 - M - L), \tag{6}$$

then the equation (3) has at least one solution in the space BC. Moreover, solutions of equation (3) are locally asymptotically stable.

Proof. Let us define the operator N such that, for any $u \in BC$,

$$\begin{aligned} (Nu)(t,x) &= f(t,x,u(t,x),u(\alpha(t),x)) + \tfrac{1}{\Gamma(r)} \int_0^{\beta(t)} (\beta(t)-s)^{r-1} \\ &\quad \times h(t,x,s,u(s,x),u(\gamma(s),x)) d_s g(t,s);\ (t,x) \in J. \end{aligned} \tag{7}$$

From the assumptions of this theorem, we infer that $N(u)$ is continuous on J.

Now we prove that $N(u) \in BC$ for any $u \in BC$. For arbitrarily fixed $(t,x) \in J$, we have

$$\begin{aligned} |(Nu)(t,x)| &= \Big| f(t,x,u(t,x),u(\alpha(t),x)) \\ &\quad + \frac{1}{\Gamma(r)} \int_0^{\beta(t)} (\beta(t)-s)^{r-1} h(t,x,s,u(s,x),u(\gamma(s),x)) d_s g(t,s) \Big| \\ &\le \Big| f(t,x,u(t,x),u(\alpha(t),x)) - f(t,x,0,0) + f(t,x,0,0) \Big| \\ &\quad + \Big| \frac{1}{\Gamma(r)} \int_0^{\beta(t)} (\beta(t)-s)^{r-1} h(t,x,s,u(s,x),u(\gamma(s),x)) d_s g(t,s) \Big| \\ &\le M|u(t,x)| + L|u(\alpha(t),x)| + \Big| f(t,x,0,0) \Big| + \int_0^{\beta(t)} \frac{(\beta(t)-s)^{r-1}}{\Gamma(r)} \\ &\quad \times \Big(p_1(t,x,s)\Phi(|u(s,x)|) + p_2(t,x,s)\Psi(|u(\gamma(s),x)|) \Big) d_s \left(\bigvee_{k=0}^{s} g(t,k) \right) \end{aligned}$$

$$\leq f^* + (M+L)\|u\|_{BC} + p_1^*\Phi(\|u\|_{BC}) + p_2^*\Psi(\|u\|_{BC}).$$

Thus

$$\|N(u)\| \leq f^* + (M+L)\|u\|_{BC} + p_1^*\Phi(\|u\|_{BC}) + p_2^*\Psi(\|u\|_{BC}). \tag{8}$$

Hence $N(u) \in BC$. From (6) and (8), we infer that N transforms the ball $B_\eta := B(0,\eta)$ into itself. We shall show that $N : B_\eta \to B_\eta$ satisfies the assumptions of Schauder's fixed point theorem [21]. The proof will be given in several steps and cases.

Step 1. *N is continuous.* Let $\{u_n\}_{n\in\mathbb{N}}$ be a sequence such that $u_n \to u$ in B_η. Then, for each $(t,x) \in J$, we have

$$\begin{aligned}
|(Nu_n)(t,x) - (Nu)(t,x)| \leq\ & |f(t,x,u_n(t,x),u_n(\alpha(t),x)) \\
& -f(t,x,u(t,x),u(\alpha(t),x))| \\
& +\frac{1}{\Gamma(r)}\int_0^{\beta(t)}(\beta(t)-s)^{r-1}|h(t,x,s,u_n(s,x),u_n(\gamma(s),x)) \\
& -h(t,x,s,u(s,x),u(\gamma(s),x))|d_s g(t,s) \\
\leq\ & (M+L)\|u_n-u\|_{BC} \\
& +\frac{1}{\Gamma(r)}\int_0^{\beta(t)}(\beta(t)-s)^{r-1}|h(t,x,s,u_n(s,x),u_n(\gamma(s),x)) \\
& -h(t,x,s,u(s,x),u(\gamma(s),x))|d_s \left(\bigvee_{k=0}^s g(t,k)\right).
\end{aligned} \tag{9}$$

Case 1. If $(t,x) \in [0,a]\times[0,b];\ a>0$, then, since $u_n \to u$ as $n\to\infty$ and g,h are continuous, (9) gives

$$\|N(u_n) - N(u)\|_{BC} \to 0 \quad \text{as } n\to\infty.$$

Case 2. If $(t,x) \in (a,\infty)\times[0,b];\ a>0$, then from (H_4) and (9), for each $(t,x)\in\mathbb{R}_+\times[0,b]$, we get

$$\begin{aligned}
|(Nu_n)(t,x) - (Nu)(t,x)| \leq\ & (M+L)\|u_n-u\| \\
& +\int_0^{\beta(t)}\frac{(\beta(t)-s)^{r-1}}{\Gamma(r)}\Big(p_1(t,x,s)(\Phi(|u_n(s,x)|) \\
& +\Phi(|u(s,x)|)) + p_2(t,x,s)(\Psi(|u_n(\gamma(s),x)|) \\
& +\Psi(|u(\gamma(s),x)|))\Big)d_s\left(\bigvee_{k=0}^s g(t,k)\right) \\
\leq\ & (M+L)\|u_n-u\| \\
& +\frac{\Phi(\eta)}{\Gamma(r)}\int_0^{\beta(t)}(\beta(t)-s)^{r-1}p_1(t,x,s)d_s\left(\bigvee_{k=0}^s g(t,k)\right) \\
& +\frac{\Psi(\eta)}{\Gamma(r)}\int_0^{\beta(t)}(\beta(t)-s)^{r-1}p_2(t,x,s)d_s\left(\bigvee_{k=0}^s g(t,k)\right).
\end{aligned} \tag{10}$$

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From (H_7) and since $u_n \to u$ as $n \to \infty$ and $t \to \infty$, then (10) gives

$$\|N(u_n) - N(u)\|_{BC} \to 0 \quad \text{as } n \to \infty.$$

Step 2. *$N(B_\eta)$ is uniformly bounded.* This is clear since $N(B_\eta) \subset B_\eta$ and B_η is bounded.

Step 3. *$N(B_\eta)$ is equicontinuous on every compact subset $[0,a] \times [0,b]$ of J,* $a > 0$.

Let $(t_1, x_1), (t_2, x_2) \in [0,a] \times [0,b]$, $t_1 < t_2$, $x_1 < x_2$ and let $u \in B_\eta$. Also without loss of generality, suppose that $\beta(t_1) \leq \beta(t_2)$. Thus we have

$$\begin{aligned}
&(Nu)(t_2,x_2) - (Nu)(t_1,x_1)| \\
&\leq |f(t_2,x_2,u(t_2,x_2),u(\alpha(t_2),x_2)) - f(t_2,x_2,u(t_1,x_1),u(\alpha(t_1),x_1))| \\
&+|f(t_2,x_2,u(t_1,x_1),u(\alpha(t_1),x_1)) - f(t_1,x_1,u(t_1,x_1),u(\alpha(t_1),x_1))| \\
&+\frac{1}{\Gamma(r)}\Big|\int_0^{\beta(t_2)}(\beta(t_2)-s)^{r-1} \\
&\times[h(t_2,x_2,s,u(t_2,x_2),u(\gamma(s),x_2)) - h(t_1,x_1,s,u(t_1,x_1),u(\gamma(s),x_1))]d_s g(t,s)\Big| \\
&+\frac{1}{\Gamma(r)}\Big|\int_0^{\beta(t_2)}(\beta(t_2)-s)^{r-1}h(t_1,x_1,s,u(t_1,x_1),u(\gamma(s),x_1))d_s g(t,s) \\
&-\int_0^{\beta(t_1)}(\beta(t_2)-s)^{r-1}h(t_1,x_1,s,u(t_1,x_1),u(\gamma(s),x_1))d_s g(t,s)\Big| \\
&+\frac{1}{\Gamma(r)}\Big|\int_0^{\beta(t_1)}(\beta(t_2)-s)^{r-1}h(t_1,x_1,s,u(t_1,x_1),u(\gamma(s),x_1))d_s g(t,s) \\
&-\int_0^{\beta(t_1)}(\beta(t_1)-s)^{r-1}h(t_1,x_1,s,u(t_1,x_1),u(\gamma(s),x_1))d_s g(t,s)\Big| \\
&\leq M|u(\alpha(t_2),x_2) - u(\alpha(t_1),x_1)| + L|u(t_2,x_2) - u(t_1,x_1)| \\
&+|f(t_2,x_2,u(t_1,x_1),u(\alpha(t_1),x_1)) - f(t_1,x_1,u(t_1,x_1),u(\alpha(t_1),x_1))| \\
&+\frac{1}{\Gamma(r)}\int_0^{\beta(t_2)}(\beta(t_2)-s)^{r-1} \\
&\times\Big|h(t_2,x_2,s,u(t_2,x_2),u(\gamma(s),x_2)) - h(t_1,x_1,s,u(t_1,x_1),u(\gamma(s),x_1))\Big| \\
&d_s\left(\bigvee_{k=0}^{s} g(t,k)\right) \\
&+\frac{1}{\Gamma(r)}\int_{\beta(t_1)}^{\beta(t_2)}(\beta(t_2)-s)^{r-1}\Big|h(t_1,x_1,s,u(t_1,x_1),u(\gamma(s),x_1))\Big|d_s\left(\bigvee_{k=0}^{s} g(t,k)\right) \\
&+\frac{1}{\Gamma(r)}\int_0^{\beta(t_1)}\left|(\beta(t_2)-s)^{r-1} - (\beta(t_1)-s)^{r-1}\right|
\end{aligned}$$

$$
\begin{aligned}
&\times\Big|h(t_1,x_1,s,u(t_1,x_1),u(\gamma(s),x_1))\Big|d_s\left(\bigvee_{k=0}^{s}g(t,k)\right)\\
&\le M|u(\alpha(t_2),x_2)-u(\alpha(t_1),x_1)|+L|u(t_2,x_2)-u(t_1,x_1)|\\
&+|f(t_2,x_2,u(\alpha(t_1),x_1))-f(t_1,x_1,u(\alpha(t_1),x_1))|\\
&+\frac{1}{\Gamma(r)}\int_0^{\beta(t_2)}(\beta(t_2)-s)^{r-1}\\
&\times\Big|h(t_2,x_2,s,u(s,x_2),u(\gamma(s),x_2))-h(t_1,x_1,s,u(s,x_1),u(\gamma(s),x_1))\Big|\\
&d_s\left(\bigvee_{k=0}^{s}g(t,k)\right)\\
&+\frac{\Phi(\eta)}{\Gamma(r)}\int_{\beta(t_1)}^{\beta(t_2)}(\beta(t_2)-s)^{r-1}p_1(t_1,x_1,s)d_s\left(\bigvee_{k=0}^{s}g(t,k)\right)\\
&+\frac{\Psi(\eta)}{\Gamma(r)}\int_{\beta(t_1)}^{\beta(t_2)}(\beta(t_2)-s)^{r-1}p_2(t_1,x_1,s)d_sg(t_1,s)\\
&+\frac{\Phi(\eta)}{\Gamma(r)}\int_0^{\beta(t_1)}\Big|(\beta(t_2)-s)^{r-1}-(\beta(t_1)-s)^{r-1}\Big|p_1(t_1,x_1,s)d_s\left(\bigvee_{k=0}^{s}g(t,k)\right)\\
&+\frac{\Psi(\eta)}{\Gamma(r)}\int_0^{\beta(t_1)}\Big|(\beta(t_2)-s)^{r-1}-(\beta(t_1)-s)^{r-1}\Big|p_2(t_1,x_1,s)d_s\left(\bigvee_{k=0}^{s}g(t,k)\right).
\end{aligned}
$$

From continuity of α,β,f,g,h and as $t_1\longrightarrow t_2$ and $x_1\longrightarrow x_2$, the right-hand side of the above inequality tends to zero.

Step 4. $N(B_\eta)$ *is equiconvergent.* Let $(t,x)\in J$ and $u\in B_\eta$, then we have

$$
\begin{aligned}
|u(t,x)| \le{}& \Big|f(t,x,u(t,x),u(\alpha(t),x))-f(t,x,0,0)+f(t,x,0,0)\Big|\\
&+\Big|\frac{1}{\Gamma(r)}\int_0^{\beta(t)}(\beta(t)-s)^{r-1}g(t,x,s,u(s,x),u(\gamma(s),x))d_sg(t,s)\Big|\\
\le{}& M|u(t,x)|+L|u(\alpha(t),x)|+|f(t,x,0,0)|\\
&+\frac{\Phi(\eta)}{\Gamma(r)}\int_0^{\beta(t)}(\beta(t)-s)^{r-1}p_1(t,x,s)d_s\left(\bigvee_{k=0}^{s}g(t,k)\right)\\
&+\frac{\Psi(\eta)}{\Gamma(r)}\int_0^{\beta(t)}(\beta(t)-s)^{r-1}p_2(t,x,s)d_s\left(\bigvee_{k=0}^{s}g(t,k)\right).
\end{aligned}
$$

Then, since $\alpha(t)\longrightarrow\infty$ as $t\longrightarrow\infty$, we deduce that

$$
\lim_{t\to\infty}|u(t,x)|\le\lim_{t\to\infty}\frac{1}{(1-M-L)\Gamma(r)}\Big[|f(t,x,0,0)|
$$

$$+\Phi(\eta)\int_0^{\beta(t)}(\beta(t)-s)^{r-1}p_1(t,x,s)d_s\left(\bigvee_{k=0}^{s}g(t,k)\right)$$
$$+\Psi(\eta)\int_0^{\beta(t)}(\beta(t)-s)^{r-1}p_2(t,x,s)d_s\left(\bigvee_{k=0}^{s}g(t,k)\right)\Bigg].$$

Thus, for each $x\in[0,b]$, we get

$$|u(t,x)|\longrightarrow 0,\ as\ t\longrightarrow+\infty.$$

Hence,

$$|u(t,x)-u(+\infty,x)|\longrightarrow 0,\ as\ t\longrightarrow+\infty.$$

As a consequence of Steps 1 to 4 together with the Lemma 2.6, we can conclude that $N:B_\eta\to B_\eta$ is continuous and compact. From an application of Schauder's theorem [21], we deduce that N has a fixed point u which is a solution of the equation (3).

Step 5. *The local asymptotic stability of solutions.* Now we investigate the stability for solutions of equation (3). Let us assume that u_0 is a solution of equation (3) with the conditions of this theorem. Consider the ball $B(u_0,\eta^*)$ with $\eta^*=\frac{M^*}{1-M-L}$, where

$$M^*:=\frac{1}{\Gamma(r)}\sup_{(t,x)\in J}\Bigg\{\int_0^{\beta(t)}(\beta(t)-s)^{r-1}|h(t,x,s,u(s,x),u(\gamma(s),x))$$
$$-h(t,x,s,u_0(s,x),u_0(\gamma(s),x))|d_s\left(\bigvee_{k=0}^{s}g(t,k)\right);\ u\in BC\Bigg\}.$$

Taking $u\in B(u_0,\eta^*)$, we have

$$\begin{aligned}|(Nu)(t,x)-u_0(t,x)| &= |(Nu)(t,x)-(Nu_0)(t,x)|\\ &\le |f(t,x,u(t,x),u(\alpha(t),x))-f(t,x,u_0(t,x),u_0(\alpha(t),x))|\\ &\quad+\frac{1}{\Gamma(r)}\int_0^{\beta(t)}(\beta(t)-s)^{r-1}|h(t,x,s,u(s,x),u(\gamma(s),x))\\ &\quad-h(t,x,s,u_0(s,x),u_0(\gamma(s),x))|d_sg(t,s)\\ &\le (M+L)\|u-u_0\|\\ &\quad+\frac{1}{\Gamma(r)}\int_0^{\beta(t)}(\beta(t)-s)^{r-1}|h(t,x,s,u(s,x),u(\gamma(s),x))\\ &\quad-h(t,x,s,u_0(s,x),u_0(\gamma(s,x)))|d_s\left(\bigvee_{k=0}^{s}g(t,k)\right)\\ &\le \eta^*(M+L)+M^*=\eta^*.\end{aligned}$$

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Thus we observe that N is a continuous function such that $N(B(u_0,\eta^*)) \subset B(u_0,\eta^*)$. Moreover, if u is a solution of equation (3), then

$$\begin{aligned} |u(t,x)-u_0(t,x)| &= |\ (Nu)(t,x)-(Nu_0)(t,x)| \\ &\leq\ |f(t,x,u(t,x),u(\alpha(t),x))-f(t,x,u_0(t,x),u_0(\alpha(t),x))| \\ &\quad +\tfrac{1}{\Gamma(r)}\int_0^{\beta(t)}(\beta(t)-s)^{r-1}|h(t,x,s,u(s,x),u(\gamma(s),x)) \\ &\quad -h(t,x,s,u_0(s,x),u_0(\gamma(s),x))|d_s\left(\bigvee_{k=0}^{s} g(t,k)\right) \\ &\leq\ M|u(t,x)-u_0(t,x)|+L|u(\alpha(t),x)-u_0(\alpha(t),x)| \\ &\quad +\tfrac{\Phi(\eta)+\Phi(\eta+\eta^*)}{\Gamma(r)}\int_0^{\beta(t)}(\beta(t)-s)^{r-1}p_1(t,x,s)d_s\left(\bigvee_{k=0}^{s} g(t,k)\right) \\ &\quad +\tfrac{\Psi(\eta)+\Psi(\eta+\eta^*)}{\Gamma(r)}\int_0^{\beta(t)}(\beta(t)-s)^{r-1}p_2(t,x,s)d_s\left(\bigvee_{k=0}^{s} g(t,k)\right). \end{aligned} \tag{11}$$

By using (11) and the fact that $\alpha(t)\longrightarrow\infty$ as $t\longrightarrow\infty$, we deduce that

$$\begin{aligned} &\lim_{t\to\infty}|u(t,x)-u_0(t,x)| \leq \lim_{t\to\infty}\frac{1}{(1-M-L)\Gamma(r)} \\ \times &\left[(\Phi(\eta)+\Phi(\eta+\eta^*))\int_0^{\beta(t)}(\beta(t)-s)^{r-1}p_1(t,x,s)d_s\left(\bigvee_{k=0}^{s} g(t,k)\right)\right. \\ &\left. +\ (\Psi(\eta)+\Psi(\eta+\eta^*))\int_0^{\beta(t)}(\beta(t)-s)^{r-1}p_2(t,x,s)d_s\left(\bigvee_{k=0}^{s} g(t,k)\right)\right] \\ = &\ \ 0. \end{aligned}$$

Hence,

$$\lim_{t\to\infty}|u(t,x)-u_0(t,x)|=0.$$

Consequently, all solutions of equation (3) are locally asymptotically stable.

4. An Example

As an application of our results we consider the following nonlinear fractional order Riemann-Liouville Volterra-Stieltjes quadratic integral equation of the form

$$\begin{aligned} u(t,x) &= f(t,x,u(t,x),u(\alpha(t),x))+\tfrac{1}{\Gamma(r)}\int_0^{\beta(t)}(\beta(t)-s)^{r-1} \\ &\quad \times h(t,x,s,u(s,x),u(\gamma(s),x))d_s g(t,s),\ (t,x)\in J:=\mathbb{R}_+\times[0,1], \end{aligned} \tag{12}$$

where $r=\frac{1}{4},\ \alpha(t)=\beta(t)=\gamma(t)=t;\ t\in\mathbb{R}_+$,

$$f(t,x,u,v)=\frac{xe^{-t}|uv|}{8(1+t^2)(1+|u|+2|v|},\ (t,x)\in J \ \text{ and } u,v\in\mathbb{R},$$

$$g(t,s)=s,\ (t,s)\in\mathbb{R}_+^2,$$

$$\begin{cases} h(t,x,s,u,v) = \dfrac{cxs^{\frac{-3}{4}}(1+|u|)\sin\sqrt{t}\sin s}{(1+t^2)(2+|u|)}; & (t,x,s)\in J_1,\ s\neq 0 \ and \ u,v\in\mathbb{R}, \\ h(t,x,0,u,v)=0; & (t,x)\in J \ and \ u,v\in\mathbb{R}, \end{cases}$$

$$c = \frac{\sqrt{\pi}}{8e\Gamma(\frac{1}{4})} \ and \ \ J_1 = \{(t,x,s)\in J\times\mathbb{R}_+ : s\leq t\}.$$

First, we can see that $\lim_{t\to 0}\alpha(t) = 0$. Then the assumption (H_1) is satisfied. Next, the function f is a continuous, and

$$|f(t,x,u_1,u_2) - f(t,x,v_1,v_2)| \leq \frac{1}{8}(|u_1-v_1| + 2|u_2-v_2|);\ (t,x)\in J,\ u,v\in\mathbb{R}.$$

Then, the assumption (H_2) is satisfied with $M=\frac{1}{8},\ L=\frac{1}{4}$, and (H_3) is satisfied with $f^*=0$. Also, we can easily see that the function g satisfies the hypotheses $(H_4)-(H_6)$.
The function h satisfies the assumption (H_7). Indeed, h is continuous and

$$|h(t,x,s,u,v)| \leq p_1(t,x,s)\Phi(|u|) + p_2(t,x,s)\Psi(|v|);\ (t,x,s)\in J_1,\ u,v\in\mathbb{R},$$

where $\Phi,\Psi:\mathbb{R}_+\to\mathbb{R}_+ :\ \Phi(w)=w,\ \Psi(w)=1$, and

$$\begin{cases} p_i(t,x,s) = \dfrac{cxs^{\frac{-3}{4}}\sin\sqrt{t}\sin s}{1+t^2}; & (t,x,s)\in J_1,\ s\neq 0,\ i=1,2, \\ p_i(t,x,0)=0; & (t,x)\in J,\ i=1,2. \end{cases}$$

Then, for $i=1,2$, we have

$$\begin{aligned} \left|\int_0^t (t-s)^{r-1}p_i(t,x,s)d_sg(t,s)\right| &\leq \int_0^t (t-s)^{\frac{-3}{4}}cxs^{\frac{-3}{4}}|\sin\sqrt{t}\sin s|d_s\left(\bigvee_{k=0}^{s} g(t,k)\right) \\ &\leq cx|\sin\sqrt{t}|\int_0^t (t-s)^{\frac{-3}{4}}s^{\frac{-3}{4}}ds \\ &\leq \frac{cx\Gamma^2(\frac{1}{4})}{\sqrt{\pi}}\left|\frac{\sin\sqrt{t}}{\sqrt{t}}\right| \\ &\leq \frac{cx\Gamma^2(\frac{1}{4})}{\sqrt{\pi t}} \longrightarrow 0 \ as \ t\to\infty, \end{aligned}$$

and

$$\begin{aligned} p_i^* &:= \sup_{(t,x)\in J}\frac{1}{\Gamma(r)}\int_0^t (t-s)^{r-1}p_i(t,x,s)d_s\left(\bigvee_{k=0}^{s} g(t,k)\right) \\ &\leq \sup_{(t,x)\in J}\frac{cx\Gamma(\frac{1}{4})}{\sqrt{\pi}}\left|\frac{\sin\sqrt{t}}{\sqrt{t}}\right| \end{aligned}$$

$$= \frac{c\Gamma(\frac{1}{4})}{\sqrt{\pi}} = \frac{1}{8e}.$$

Finally, let us notice that the condition (6) is satisfied with $\eta = 1$. Indeed, the inequality

$$f^* + p_1^*\Phi(\eta) + p_2^*\Psi(\eta) \leq \eta(1 - M - L),$$

implies $\eta \geq \frac{1}{5e-1}$, which is satisfied for $\eta = 1$. Hence by Theorem 3.3, the equation (12) has a solution defined on $\mathbb{R}_+ \times [0, 1]$ and solutions of this equation are locally asymptotically stable.

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