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BOUNDARY-FITTED CONFORMAL MAPPING IN A MULTIPLY-CONNECTED REGION

Yoshihiro Mochimaru
Department of International Development Engineering
Tokyo Institute of Technology
2-12-1, Ookayama, Meguro-ku, Tokyo, 152-8550, JAPAN

Abstract: Formation of a boundary-fitted conformal mapping is proposed as a boundary value problem in a multiply-connected region, which includes various parameters to be matched.

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Key Words: conformal mapping, univalence

1. Introduction

Conformal mapping is very useful if the region in a multiply-connected region can be mapped into any standard typical configuration as a boundary value problem, although in some cases isolated boundaries may be successively mapped into interior line segment(s) [1], resulting in not pure boundary value problem(s). Necessarily, univalence except zero measure is required [2], whereas specification of only boundary configuration does not lead to a unique conformal mapping [3]. Especially, numerical construction of a conformal mapping function specifying at least only finitely many points on the boundary to satisfy a given mapping condition frequently breaks down on univalence [4] as a small variation of a parameter in a mapping analytical function lacks in univalence [5].

In the following, boundary-fitted conformal mapping functions with parameters are proposed in multiply-connected regions.

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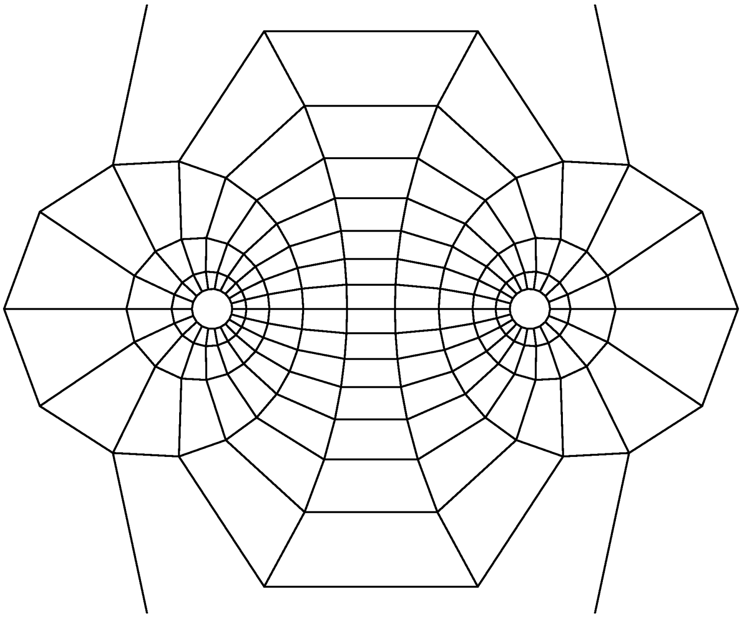

Figure 1: Complex plane z, coordinate surfaces

2. Construction of Boundary-Fitted Conformal Mapping

2.1. Two Nearly-Circular Cylinders in Infinite Extension

$$z^2 - a^2 + \frac{b^2}{z^2 - a^2} = \left(a - \frac{b}{a}\right)^2 \operatorname{sn}^2(u,k) - \left(a^2 + \frac{b^2}{a^2}\right), \tag{1}$$

$$z \equiv x + iy; \quad x, y : \text{real},$$

$$u \equiv \frac{K'}{\pi}(\alpha + i\beta); \quad \alpha, \beta : \text{real},$$

$$\frac{1}{k^2} = 1 + \frac{4b}{(a - b/a)^2}; \quad a > \sqrt{b} > 0,$$

$$|\alpha| \le \pi K/K', \quad |\beta| \le \pi, \quad \left|\frac{z^2 - a^2}{b}\right| \ge 1,$$

$K \equiv K(k) = \int_0^{\pi/2} 1/\sqrt{1 - k^2 \sin^2\theta}\; d\theta$, $K' \equiv K'(k) \equiv K(\sqrt{1-k^2})$.

$$z \sim \left(a - \frac{b}{a}\right) \operatorname{sn}(u,k) \text{ as } |\operatorname{sn}(u,k)| \to +\infty .$$

Hereafter moduli, say k, in elliptic functions are assumed to be $0 < k < 1$.

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2.2. A Single Nearly Circular Cylinder in a Rectangular Cavity

$$e^{\alpha + i\beta} = \frac{1 + i\{a \ \mathrm{sn}(z,k) + b\}}{1 - i\{a \ \mathrm{sn}(z,k) + b\}} \ , \tag{2}$$

$$a > 0 \ , \ b, \alpha, \beta : \mathrm{real} \ , \ \alpha_0 \le \alpha \le 0, \ |\beta| \le \pi \ ,$$

$$|\Re(z)| \le K(k) \ , \ 0 \le \Im(z) \le K'(k) \ .$$

The center of the cylinder , z_0 , is

$$z_0 = \mathrm{sn}^{-1}\left(\frac{i-b}{a}, k\right) \ .$$

$\mathrm{sn}^{-1}(,k)$ is the principal value of the inverse Jacobian elliptic function. The cylinder surface is given by

$$|z - z_0| \approx \frac{2a}{[(1+a^2)\,(a^2+k^2)]^{1/2}} \ e^{\alpha_0} \ .$$

In case of $a = b = 1, k = 1/\sqrt{2}$, $z_0 \approx -0.506 + 1.07i$.

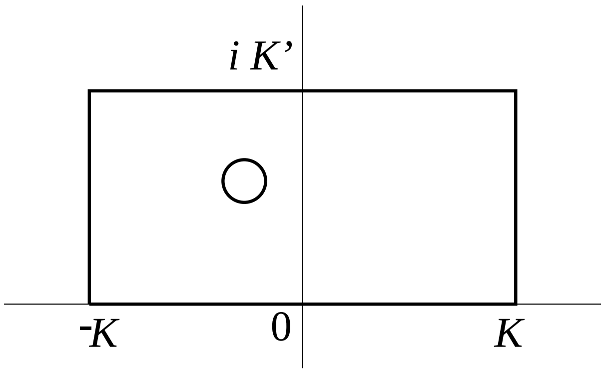

Figure 2: Complex plane

2.3. Two Nearly Circular Cylinders in a Nearly Rectangular Cavity

$$\frac{\cosh(\alpha + i\beta)}{\cosh \alpha_0} = \frac{1 + i \ \mathrm{sn}(\omega,k)}{1 - i \ \mathrm{sn}(\omega,k)} \ , \tag{3}$$

$$\alpha, \beta : \mathrm{real} \ ,$$

$$|\Re(\omega)| \le K(k) \ , \ 0 \le \Im(\omega) \le K'(k) \ ,$$

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$$\omega \equiv i\left(z + \frac{\delta_1^2}{z-a} + \frac{\delta_2^2}{z-b}\right) \ ,$$
$$b > a > 0 \ ; \ \delta_1, \delta_2 > 0 \ ,$$
$$b - a > \sqrt{2}(\delta_1 + \delta_2) \ , \ a \gg \delta_1 \ , \ b \gg \delta_2 \ ,$$
$$0 \le \alpha \le \alpha_0 \ , \ \cosh \alpha_0 \gg 1 \ , \ |\beta| \le \pi \ .$$

The centers of the cylinders are given by

$$z = a \ \text{ and } \ z = b.$$

The cylinder surfaces are given by

$$\frac{|z-a|}{\delta_1} \approx \left[1 - \left(\frac{\delta_2}{a-b}\right)^2\right]^{1/2} ,$$

$$\frac{|z-b|}{\delta_2} \approx \left[1 - \left(\frac{\delta_1}{a-b}\right)^2\right]^{1/2}$$

respectively, and for reducing to a boundary-value problem,

$$\int_0^{\varphi} \left\{1 - (1-k^2)\sin^2\theta\right\}^{-1/2} d\theta \lesssim a - 2\delta_1 \ ,$$

$$\int_0^{\pi/2-\varphi} \left\{1 - (1-k^2)\sin^2\theta\right\}^{-1/2} d\theta \gtrsim b + 2\delta_2 \ ,$$

$$\varphi \equiv \tan^{-1}\left(\tanh^2 \frac{\alpha_0}{2}\right) \ .$$

2.4. Around a Rectangular Cylinder

$$w = \frac{K(k^*)}{K(k)}$$
$$\times \left[\mathrm{sn}^{-1}\left\{\left(-i\tanh\frac{\alpha+i\beta}{2} - b\right)\Big/ a \ , \ k\right\} - \frac{K'(k)}{2} i\right] \ , \qquad (4)$$
$$\alpha, \beta : \text{real} \ \ , \ \ -\infty < \alpha \le 0 \ , \ |\beta| \le \pi \ ,$$
$$z = Z(w,k^*) + \frac{\mathrm{cn}(w,k^*)\mathrm{dn}(w,k^*)}{\mathrm{sn}(w,k^*)} + m\ w \ , \qquad (5)$$
$$\frac{1}{2}\frac{K'(k)}{K(k)} = \frac{K'(k^*)}{K(k^*)} \ ,$$

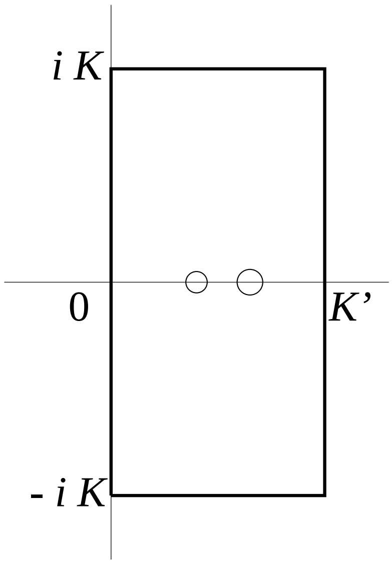

Figure 3: Typical complex plane

i.e. $k^* = 2\sqrt{k}/(1+k)$.

$$m \equiv \frac{E(k^*)}{K(k^*)} - (1 - k^{*2}) \ ,$$

where $Z(,)$ is a Jacobi zeta function, and $E()$ is a complete elliptic integral of the second kind. The transformation Eq.(5) regarding a rectangle is found in [6]. The aspect ratio of the cross section of the cylinder, i.e. the ratio of the width to the height, is

$$\frac{E(k^*) - (1 - k^{*2})K(k^*)}{E(k^{*\prime}) - k^{*2}K(k^{*\prime})} \ ,$$

where $k^{*\prime} \equiv \sqrt{1 - k^{*2}} = (1-k)/(1+k)$.

2.5. Rectangular Cylinder in a Nearly-Circular Cylinder

$$\mathrm{e}^{\alpha + i\beta} = \frac{1 + i\ \eta(u)}{1 - i\ \eta(u)} \ , \tag{6}$$

$$\eta \equiv a\,\{\mathrm{sn}(u,k) + i\ \mathrm{cn}(u,k)\} + b \ ,$$

$$a > 0, b, \alpha, \beta : \mathrm{real} \ ,$$

$$|\Re(u)| \leq K(k) \ , \ |\Im(u)| \leq K'(k) \ ,$$

$$z = Z(u,k) + \frac{\mathrm{cn}(u,k)\mathrm{dn}(u,k)}{\mathrm{sn}(u,k)} + m\ u \ , \tag{7}$$

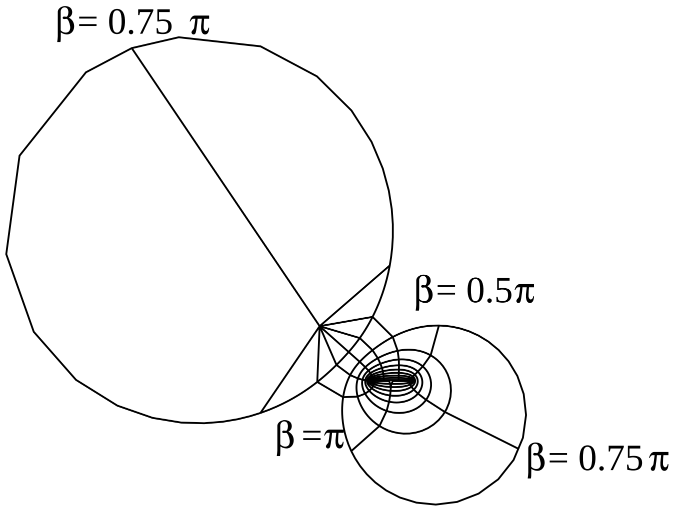

Figure 4: Coordinate surfaces in the complex plane z at $k = 1/\sqrt{2}, a = 1, b = 0.2$

$$m \equiv \frac{E(k)}{K(k)} - (1 - k^2) \ ,$$

$$\alpha_0 \leq \alpha \leq 0 \ , \ |\beta| \leq \pi \ .$$

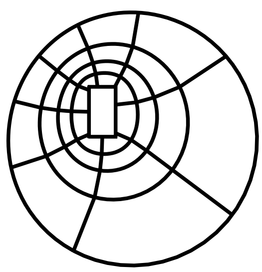

Figure 5: Coordinate surfaces in the complex plane z at $k = 0.6, a = 1.1, b = 0.1, \alpha_0 = \ln 0.2$

2.6. Nearly-Circular Cylinder over a Semi-Rectangular Ridge

$$
\begin{aligned}
\mathrm{e}^{\alpha+i\ \beta} &= \frac{1+i\ \eta}{1-i\ \eta}\ , \qquad (8)\\
\eta &\equiv a\ \mathrm{sn}(u,k)+b\ ,\\
a &> 0, b, \alpha, \beta : \text{real}\ ,\\
|\Re(u)| &\le K(k), 0 \le \Im(u) \le K'(k)\ ,\\
z &= -\left[Z(u,k)+\frac{\mathrm{cn}(u,k)\mathrm{dn}(u,k)}{\mathrm{sn}(u,k)}+m\ u\right]\ , \qquad (9)\\
m &= \frac{E(k)}{K(k)}-(1-k^2)\ ,\\
\alpha_0 &\le \alpha \le 0\ ,\ |\beta| \le \pi\ .
\end{aligned}
$$

The aspect ratio of the ridge, i.e. the ratio of the width to the height, is $2\{E(k)-(1-k^2)K(k)\}/\{E(k')-k^2K(k')\}$, $k' \equiv \sqrt{1-k^2}$.

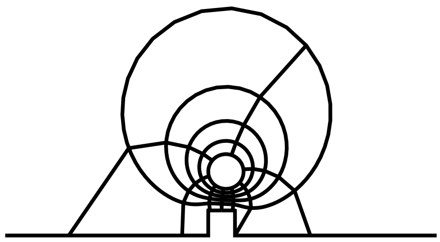

Figure 6: Figure 6 Coordinate surfaces in the complex plane z at $k = 0.6, a = 1.1, b = 0.1, \alpha_0 = \ln 0.2$

2.7. Nearly-Elliptic Cylinder over a Semi-Rectangular Ridge

$$
\begin{aligned}
\frac{\cosh(\alpha+i\beta)}{\cosh\alpha_0} &= \frac{1+i\ \mathrm{sn}(w,k)}{1-i\ \mathrm{sn}(w,k)}\ , \qquad (10)\\
\alpha, \beta : \text{real}\ ,\ &|\Re(w)| \le K(k)\ ,\ 0 \le \Im(w) \le K'(k)\ ,\\
z &= -\left[Z(w,k)+\frac{\mathrm{cn}(w,k)\mathrm{dn}(w,k)}{\mathrm{sn}(w,k)}+m\ w\right]\ , \qquad (11)\\
m &\equiv \frac{E(k)}{K(k)}-(1-k^2)\ ,
\end{aligned}
$$

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$$0 < \alpha_m \le \alpha \le \alpha_0 \ , \cosh \alpha_0 \gg 1 \ , \ |\beta| \le \pi \ ,$$

$$0 \le \Im(w) \le K'(k) \ , \ |\Re(w)| \le K(k) \ .$$

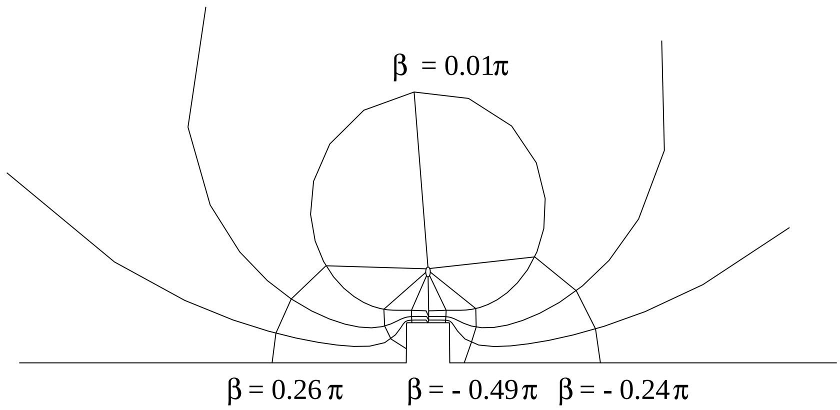

Figure 7: Coordinate surfaces in the complex plane z at $k = 0.6, \alpha_0 = 4, \alpha_m = 0.5$

2.8. Nearly-Circular Cylinder over Semi-Circular Ridges

$$\mathrm{e}^{\alpha + i\beta} = \frac{1 + i(a\ w + b)}{1 - i(a\ w + b)} \ , \tag{12}$$

$$w = z + \frac{\delta_1^2}{z - c} + \frac{\delta_2^2}{z - d} \ , \tag{13}$$

$$a > 0 \ , \ b, c, d, \alpha, \beta : \text{real} \ ,$$

$$\delta_1 > 0, \delta_2 > 0 \ ,$$

$$\alpha_0 \le \alpha \le 0 \ , \ |\beta| \le \pi \ .$$

2.9. Nearly-Circular Cylinder over a Nearly Semi-Elliptic Ridge

$$\mathrm{e}^{\alpha + i\beta} = \frac{1 + i(a\ w + b)}{1 - i(a\ w + b)} \ , \tag{14}$$

$$w = z + \cfrac{\delta}{\cfrac{z - c}{\delta} - \epsilon \cfrac{\delta}{z - c}} \ , \tag{15}$$

$$a > 0; \ b, \ c : \text{real}; \ \delta > 0 \ , \ \epsilon \ge 0 \ ,$$

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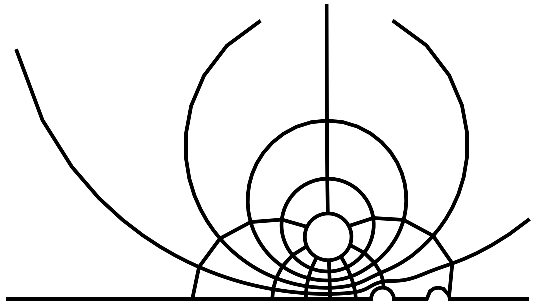

Figure 8: Coordinate surfaces in the complex plane z at $a = 1, b = -1, c = 2, d = 3, \delta_1 = 0.2, \delta_2 = 0.2, \alpha_0 = \ln 0.2$

$$\alpha_0 \leq \alpha \leq 0 \ , \ |\beta| \leq \pi \ .$$

The part corresponding to a nearly semi-elliptic ridge for small $|\epsilon|$ is expressed by

$$\frac{z-c}{\delta} \approx \left(1 + \frac{\epsilon}{2} - \frac{\epsilon^2}{8}\right) \mathrm{e}^{i\phi} + \left(\epsilon - \frac{\epsilon^2}{2}\right) \mathrm{e}^{-i\phi} \ , \tag{16}$$

where ϕ is a real parameter, and $-\pi \leq \phi \leq \pi$.

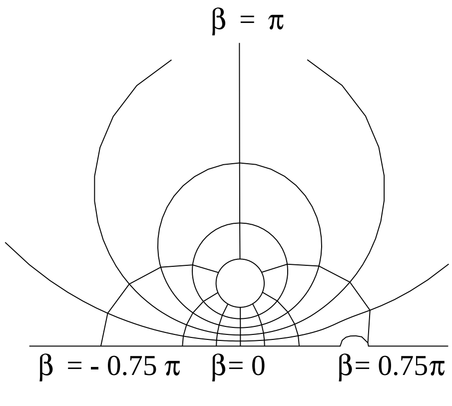

Figure 9: Coordinate surfaces in the complex plane z at $a = 1, \ b = 1, c = 1, \delta = 0.2, \epsilon = 1, \alpha_0 = \ln 0.2$

3. Discussions

Geometrical configurations such as an aspect ratio of a rectangle may be covered for all. However, values of parameters for relative locations of isolated boundaries or just for shape factors may be limited for keeping univalence.

4. Conclusions

Possibility of finding a boundary-fitted conformal mapping in a multiply-connected region is shown for various geometrical configurations.

References

[1] Y. Mochimaru, New features of a spectral finite difference scheme, In: *Proc. 2003 INHA Symposium on Nonlinear Analysis and Related Topics* (2003), 47-52.

[2] P. Henrici, *Applied and Computational Complex Analysis*, **1**, Wiley-Interscience (1974).

[3] Y. Mochimaru, Non-uniqueness of conformal mapping and possibility of a fixed point mapping, *International Journal of Pure and Applied Mathematics*, **76** (2012).

[4] M. Hartmann, G. Opfer, Uniform approximation as a numerical tool for constructing conformal maps, *J. Computational and Applied Mathematics*, **14** (1986), 193-206.

[5] Y. Mochimaru, Existence/non-existence of conformal maps for a family of functions, *International Journal of Pure and Applied Mathematics*, **43** (2008), 345-350.

[6] C. Darwin, Some conformal transformations involving elliptic functions, *Phil. Mag., Ser.7*, **41** (1950), 1-11.