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FREE SURFACE FLOW MODELING BETWEEN TWO PLATES

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Abstract: Two dimensional free surface flow, that is uniform far upstream between two semi-infinite horizontal plates, is analysed analytically using the method of the free streamline theory. The flow is assumed to be steady. The fluid is treated as inviscid and incompressible. The effects of gravity and surface tension are not taken into account. We use the method of Kirchhoff based on the hodograph method and Schwartz-Christoffel transformation technique to obtain the exact solution.

AMS Subject Classification: 76B07, 76D45, 76M40, 76B07
Key Words: free surface, flow, potential function, streamline theory

1. Introduction

We consider a steady two dimensional potential and irrotational flow of a fluid flowing between two semi-infinite horizontal plates in a domain bounded below by an infinite rigid wall and above by a free surface. The fluid is assumed to be inviscid, incompressible and the flow is irrotationnal. Far upstream the flow is uniform with a constant velocity U and a constant depth D. The classical problem of a free streamline flow of an ideal fluid has been studied by many authors [3-9]. Toison et al. [10] used an iterative method and Vanden-Broeck [11-12] used a series truncation. The first work in this type of problem is characterized by the use of the Schwartz-Christoffel formula. The latter can treat the flows of border, which combine rectilinear wall and unknown free surface.

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2. Problem Formulation

Let us consider a two-dimensional flow of inviscid, irrotational and incompressible fluid between two semi-infinite horizontal plates. Due to the symmetry of the flow with respect to y=D, one can restrict the study of the problem in the lower half-plan (Fig.1). The effects of gravity and surface tension are not taken into account. A system of cartesian coordinates is defined with the x-axis along the horizontal wall A‘O and the y-axis is perpendicular to the latter at point O. As $x \to \infty$, the flow approaches a uniform stream with a constant velocity U and depth D.

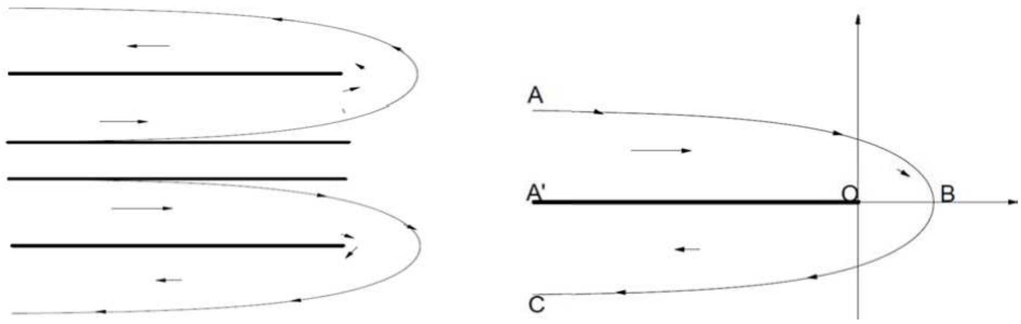

Figure 1: Sketch of the flow and of the system of coordinates

The flow is limited superiorly by the thread of current ABC (the free surface). One indicates by $\xi = u - iv$ the complex velocity such that u and v are its components and $f = \phi + \mathrm{i}\psi$ the complex function where ϕ and ψ are the potential and the stream functions respectively. We have

$$\frac{df}{dz} = u - iv \qquad and \qquad q = \sqrt{u^2 + v^2}.$$

where q is the module of the velocity. The function f transforms the z-plan into an infinite band (see Fig. 2). The mathematical problem consists in determining the potential function ϕ which checks the following conditions:

$$\begin{cases} \Delta\phi = 0 & \text{in the flow domain} \\ \frac{1}{2}[(\frac{\delta\phi}{\delta x})^2 + (\frac{\delta\phi}{\delta y})^2] + \frac{p}{\rho} = Cte & \text{on the free surface of unknown form} \\ \frac{\delta\phi}{\delta x} = 0 & \text{at } B \\ \frac{\delta\phi}{\delta y} = 0 & \text{at } A'O \end{cases}$$

3. Problem Resolution

To solve the above problem, we initially use the method of the free surface streamline theory introduced by Kirchhoff [1-2], based on the hodograph transformation to find the form of the free surface. The complex transformation is defined by:

$$\omega = \log(\frac{U}{\frac{df}{dz}}) = \log(\frac{U}{u-iv}) = \log(\frac{U}{q}) + i\theta. \quad (1)$$

Where $z = x + iy$, q and θ are the module speed and the angle between the velocity vector and the x-axis, respectively. By this last transformation, the field occupied by the fluid in the z-plan is transformed into an infinite band in the Ω-plan (see Fig.3). The conform transformation of a semi-infinite band in the plan to the lower half-plan of another complex λ-plan, is given by the theorem of Schwartz-Christoffel, by respecting the direction and the orientation of the flow (see Fig. 4). This transformation is given by:

$$\frac{d\Omega}{d\lambda} = k(\lambda+1)^{\frac{-1}{2}}(\lambda-1)^{\frac{-1}{2}}$$

$$\lambda = ch\Omega \quad (2)$$

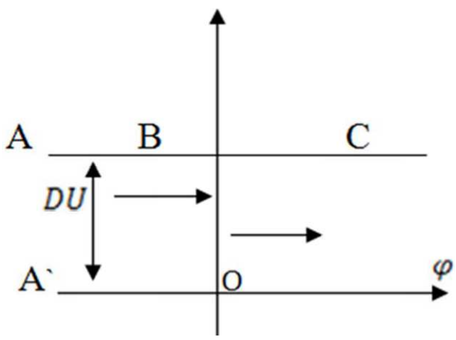

Figure 2: The flow domain in the f-plan

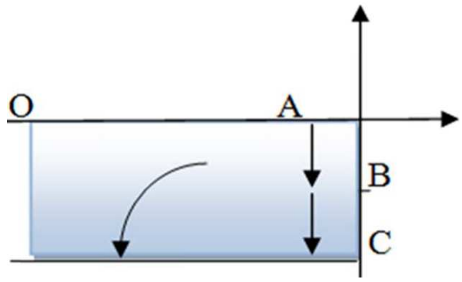

Figure 3: The flow domain in the Ω-plan.

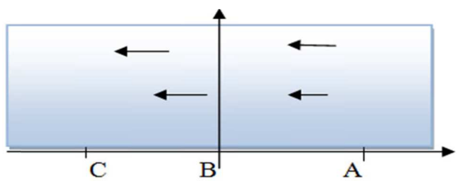

Figure 4: The flow domain in the λ-plan.

The transformation which transforms the interior of the infinite band of the$f$-plan towards the lower half-plan of the λ-plan is:

$$\frac{df}{d\lambda} = k(\lambda - 1)^{-1}(\lambda + 1)^{-1}$$

After calculations, we find a relation between λ and f:

$$\lambda = \frac{1 + e^{(\frac{\pi f}{DU})}}{1 - e^{(\frac{\pi f}{DU})}} \tag{3}$$

Using the relation $U \frac{dz}{d\lambda}$=$U \frac{dz}{df} \frac{df}{d\lambda}$, we obtain

$$\frac{dz}{d\lambda} = \frac{2D}{\pi}(\frac{\lambda}{\lambda^2 - 1} + i\frac{1}{\sqrt{1 - \lambda^2}}) \tag{4}$$

By integrating (4)

$$z - z_0 = \frac{D}{\pi}(log(\lambda^2 - 1) + 2iarcsin\lambda) \tag{5}$$

Where z_0 is a constant, we have z_0= (x_0, y_0) = (x(λ), y(λ)) = (x_0, 0) at$B$. Finally the solution is as follows:

$$\begin{cases} x = x_0 + \frac{D}{\pi} log(sin(\theta))^2 \\ \\ y = \frac{2D}{\pi}(\theta + \frac{\pi}{2}) \end{cases} \quad -\pi \leq \theta \leq 0$$

The form of the free surface is given in Figure 5.

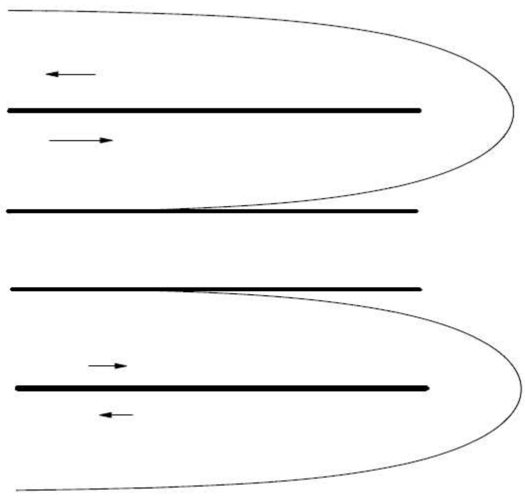

Figure 5: The form of the free surface

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