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EMBEDDINGS OF CURVES ON QUADRICS OVER FINITE FIELDS

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Abstract: Some embeddings of curves on quadrics over finite fields are provided. Special emphasis is given to normal rational curves and linearly normal elliptic curves.

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1. Introduction

It is very well known that a twisted cubic of the finite projective three–space $PG(3, q)$ is the complete intersection of three hyperbolic smooth quadrics [5, p. 239]. Indeed, it is always possible "to draw" a twisted cubic on a hyperbolic quadric, but, as it is easy to see, it is impossible to embed a twisted cubic on a smooth elliptic quadric.

The generalization of a twisted cubic in higher dimensions is the normal rational curve [6, 27.5]. In this paper we are interested in the following problem. Let C be a normal rational curve in $PG(n, q)$, $n > 3$. Does there exists a smooth quadric Q containing it. Is Q hyperbolic or elliptic? We address these questions in Theorem 1.

Then we focus on linearly normal elliptic curves of $\mathbf{P}^5$ defined over $\mathbb{F}_q$ and prove that any such curve (up to linear transformations) is always contained in a hyperbolic quadric.

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We also provide other examples of embeddings of curves on quadrics.

For any algebraic set X defined over $\mathbb{F}_q$ let $X(q)$ denote the set of all its rational points with $\mathbb{F}_q$ as their residue field.

2. The Embeddings

We start with the following example.

Example 1. Let C be a smooth elliptic curve defined over $\mathbb{F}_q$.

$$q^e + 1 - 2\sqrt{q^e} \leq \sharp(C(q^e)) \leq q^e + 1 + 2\sqrt{q^e} \tag{1}$$

for all integers $e \geq 1$ (Hasse-Weil). Thus $C(q) \neq \emptyset$ if $q \geq 4$. Set $\bar{C} := C \times_{\mathbb{F}_q} \bar{\mathbb{F}}_q$. From now on we assume $C(q) \neq 0$. Thus $\mathrm{Pic}^t(C)(q^e) \cong C(q^e)$ for all integers t and hence we may apply Hasse-Weil inequalities to each $\mathrm{Pic}^t(C)(q^e)$. For all integers $e \geq 1$ set $\tau(e) := \sharp(C(q^e))$. By Hasse-Weil inequalities we have $\tau(4) - \tau(3) > \tau(1)^2$ if

$$q^4 + 1 - 2q^2 > q^3 + 1 + 2q^{3/2} + (q + 1 + 2\sqrt{q})^2$$

and in particular if $q \geq 55$. From now on we assume $\tau(4) > \tau(3) + \tau(1)^2$. Since $\tau(4) > \tau(3)$, there is $L \in \mathrm{Pic}^1(C)(q^4)\backslash\mathrm{Pic}^1(C)(q^4)$. Since $h^0(L) = 1$, there is a unique $P \in C(q^4)$ such that $L \cong \mathcal{O}_C(P)$. Since $L \in \mathrm{Pic}^1(C)(q^4)\backslash\mathrm{Pic}^1(C)(q^4)$, the Galois group $Z/4\mathbb{Z}$ of the extension of fields $[\mathbb{F}_{q^4} : \mathbb{F}_q]$ sends P into 4 distinct points of $C(q^4)$, say P, $\sigma(P)$, $\sigma^2(P)$ and $\sigma^3(P)$. Set $R := \mathcal{O}_C(P + \sigma(P) + \sigma^2(P) + \sigma^3(P))$. Notice that $R \in \mathrm{Pic}^4(C)(q)$. Since C is an elliptic curve, $h^0(R) = 4$ and R defines an embedding $\phi : C \to \mathbf{P}^3$ defined over $\mathbb{F}_q$ and with as image a degree 4 linearly normal elliptic curve. Hence $\phi(\bar{C})$ is a complete intersection of two quadric hypersurfaces. Since ϕ is defined over $\mathbb{F}_q$ this pencil Λ of quadrics is defined over $\mathbb{F}_q$. Fix any $Q \in \Lambda$ defined over $\mathbb{F}_q$. Q is irreducible. Q is a quadric cone if and only if there is $M \in \mathrm{Pic}^2(C)(q)$ such that $M^{\otimes 2} \cong R$. Since $\tau(4) > \tau(3) + 2\tau(1)$, we may find L such that Λ contains no singular surface defined over $\mathbb{F}_q$. Q is a hyperbolic quadric if and only if there are $M_1, M_2 \in \mathrm{Pic}^2(C)(q)$ such that $R \cong M_1 \otimes M_2$. Since $\tau(4) - \tau(3) > \tau(1)^2$, we may find L such that all quadrics of the pencil Λ defined over $\mathbb{F}_q$ are elliptic quadric surfaces. It is easy to find L such that some of the quadrics of the pencil Λ are singular, while all other quadrics are hyperbolic.

The following result is contained in [5, p. 27]. Here we provide a different proof.

Proposition 1. *Let $C \subset \mathbf{P}^3$ be a rational normal curve. Then C is contained in exactly $q + 1$ quadric cones and q^2 smooth quadrics. Every smooth quadric containing C and defined over $\mathbb{F}_q$ is hyperbolic.*

Proof. Since $\dim(|\mathcal{I}_C(2)|) = 2$, C is contained in exactly $q^2 + q + 1$ quadrics defined over $\mathbb{F}_q$. For every $P \in C(\mathbb{F}_q)$ the linear projection from P shows that C

is contained in exactly one conic quadric with P as its vertex. Conversely, every quadric cone T containing $C(\bar{\mathbb{F}}_q)$ is obtained from one $P \in C(\bar{\mathbb{F}}_q)$ taking the linear projection. Since T is defined over $\mathbb{F}_q$ if and only if $P \in C(\mathbb{F}_q)$ and $\sharp(C(\mathbb{F}_q)) = q+1$, C is contained in exactly $q + 1$ singular quadrics defined over $\mathbb{F}_q$ and q^2 smooth quadrics defined over over $\mathbb{F}_q$. Since every curve contained in an elliptic quadric surface has even degree, every smooth quadric containing C and defined over $\mathbb{F}_q$ is hyperbolic. □

Remark 1. Any integral non-degenerate space curve over $\bar{\mathbb{F}}_q$ contained in at least two quadric surfaces is either a rational normal curve or a linearly normal degree 4 smooth elliptic curve or a degree 4 singular curve whose normalization is $\mathbf{P}^1$. Here we will analyze the latter case. Let $C \subset \mathbf{P}^3$ be a geometrically integral singular degree 4 curve defined over $\mathbb{F}_q$. C has a unique singular point. Call it P. The normalization map $\pi : \mathbf{P}^1 \to C$ is defined over $\mathbb{F}_q$. Then $P \in C(\mathbb{F}_q)$ and one and only one of the following three cases must occur:

(a) C is a cuspidal curve, i.e. $\sharp(\pi^{-1}(P)) = 1$; in this case we have $\sharp(C(\mathbb{F}_q)) = q+1$;

(b) C is a nodal curve and the two points of $\pi^{-1}(P)$ are defined over $\mathbb{F}_q$; in this case we have $\sharp(C(\mathbb{F}_q)) = q$;

(c) C is a nodal curve and the two points of $\pi^{-1}(P)$ are not defined over $\mathbb{F}_q$, but only over $\mathbb{F}_{q^2}$; in this case we have $\sharp(C(\mathbb{F}_q)) = q$.

In all 3 cases the linear projection from P shows that C is contained in a quadric cone with P as its vertex.

Now we will see that $\mathbf{P}^3$ is rather exceptional from the point of view of rational normal curves defined over $\mathbb{F}_q$.

Theorem 1. *Fix an integer $n \geq 4$, a prime power q and a rational normal curve $C \subset \mathbf{P}^n$ defined over $\mathbb{F}_q$.*

(i) C is contained in a smooth quadric Q defined over $\mathbb{F}_q$.

(ii) If n is even, then C is contained in a smooth parabolic quadric defined over $\mathbb{F}_q$.

(iii) If n is odd, then C is contained in a smooth hyperbolic quadric defined over $\mathbb{F}_q$.

Proof. First assume q odd. Since all rational normal curves are projectively equivalent over $\mathbb{F}_q$, as well all smooth quadrics (case n even), all smooth hyperbolic quadrics (n odd) and all smooth elliptic quadrics (n odd), to prove part (i) (resp. (ii), resp. (iii)), it is sufficient to find a smooth quadric (resp. a smooth hyperbolic quadric, resp. a smooth elliptic quadric) Q containing a rational normal curve (see

[4], Ch. 5, or [6], Ch. 22). First assume n even and choose homogeneous coordinates z, x_i, y_i, $1 \le i \le n/2$, such that $Q = \{z^2 + \sum_i x_i y_i = 0\}$. Fix $c_i \in \mathbb{F}_q \backslash \{0\}$, $1 \le i \le n/2$, such that $\sum_i c_i = -1$. Let $C \subset \mathbf{P}^n$ be the rational normal curve with parametric equations $z = t^{n/2}$, $x_i = t^{i-1}$ and $y_i = c_i t^{n-i-1}$. By construction we have $C \subset Q$. Now assume n odd. We will check part (ii) and hence part (i), too. Choose homogeneous coordinates x_i, y_i, $1 \le i \le (n+1)/2$, such that the hyperbolic quadric Q has equation $\sum_i x_i y_i = 0$. Fix $c_i \in \mathbb{F}_q \backslash \{0\}$, $1 \le i \le (n+1)/2$, such that $\sum_i c_i = 0$ and take as C the rational normal curve with parametric equations $x_i = t^{i-1}$, $y_i = c_i t^{n-i+1}$, $1 \le i \le (n+1)/2$. Now assume n odd. We will prove part (iii). Fix a non-square $\nu \in \mathbb{F}_q \backslash \{0\}$ and choose homogeneous coordinates x_0, x_1, z_i, y_i, $1 \le i \le (n-1)/2$ such that a smooth elliptic quadric Q has $\nu x_0^2 + x_1^2 + \sum_i z_i y_i = 0$ as its equation (see [4], p. 100). First assume $n \equiv 1 \pmod 4$. Fix non-zero $c_j \in \mathbb{F}_q$, $2 \le j \le (n+1)/2$, such that $c_2 + \cdots + c_{(n+1)/2} = -1$. Take as rational normal curve $C \subset Q$ the rational normal curve with the following non-homogeneous parametrization: $x_0 = t^{(n+1)/2}$, $x_1 = t^{(n-1)/2}$, $z_1 = t^n$, $y_1 = -\nu t$, $z_j = t^{n+1-j}$ and $y_j = c_j t^{j-2}$, $2 \le j \le (n+1)/2$. Now assume $n = 7$. Take as C the rational normal curve with affine parametric equations $x_0 = t^4$, $x_1 = t^3$, $z_1 = t^7$, $y_1 = -\nu t + 1$, $z_2 = t^6$, $y_2 = -1$, $z_3 = t^5$ and $y_3 = -t^2$.Now assume $n \equiv 3 \pmod 4$ and $n \ge 11$. Fix non-zero $c_j \in \mathbb{F}_q$, $3 \le j \le (n-1)/2$, such that $c_3 + \cdots + c_{(n-1)/2} = 0$. Set $x_0 = t^{(n+1)/2}$, $x_1 = t^{(n-1)/2}$, $z_1 = t^n$, $y_1 = -\nu t$, $z_2 = t^{n-1}$, $y_2 = -1$, $z_j = t^{n+3-j}$, $y_j = c_j t^{j-3}$, $3 \le j \le (n-1)/2$, and take as C the rational normal curve with these affine parametric equations. □

Theorem 2. *Fix a prime power $q \ge 9$ and a linearly normal elliptic curve $C \subset \mathbf{P}^5$ defined over $\mathbb{F}_q$ together with its embedding in $\mathbf{P}^5$. No three points of C are collinear and $\sharp(C(\mathbb{F}_q)) \ge 3$. Fix any 3 points, say P_1, P_2, P_3, of $C(\mathbb{F}_q)$. Let M be the plane spaned by P_1, P_2, P_3. Then there exists a smooth hyperbolic quadric $Q \subset \mathbf{P}^5$ defined over $\mathbb{F}_q$ and containing $C \cup M$.*

Lemma 1. *Fix a prime power q and an integer $n > 0$ such that $q \ge n+2$. Fix any $\mathbb{F}_q$-linear subspace of the set of all homogeneous degree 2 polynomials in the variables $x_0, \dots, x_n$. Set $\bar{V} := V \otimes_{\mathbb{F}_q} \bar{\mathbb{F}}_q$ and assume the existence of a smooth quadric $Q \in |\bar{V}|$. Then there is a smooth quadric $A \in |V|$.*

Proof. Fix a basis $Q_1, \dots, Q_m$ of V. First assume q odd. With the choice of a basis the condition $\det(B) = 0$, $B \in V$ is described by a homogeneous degree $n+1$ polynomial Δ in m variables $y_1, \dots, y_m$ with coefficients depending only from the matrices $Q_1, \dots, Q_m$. Thus all the coefficients of Δ are in $\mathbb{F}_q$. The existence of a smooth $Q \in \bar{V}$ is equivalent to $\Delta \ne 0$. Since $q \ge n+2 > \deg(\Delta)$, there is $(z_1, \dots, z_m) \in \mathbb{F}_q^m \backslash \{0\}$ such that $\Delta(z_1, \dots, z_m) \ne 0$. The symmetric matrix $z_1 Q_1 + \cdots + z_m Q_m$ is non-singular. Now assume q even. If n is odd we use again Δ, while if n is even we consider formally $\Delta/2$ (see [6], Th. 22.2.1). □

Proof of Theorem 2. Notice that $\deg(C) = 6$. Since C has genus 1, the curve C is projectively normal and its homogeneous ideal is generated by quadrics (see [3], p. 102). The latter properties implies that no 3 points of C are collinear. We have $\sharp(C(\mathbb{F}_q)) \ge q - 2\sqrt{q}$. Since $q \ge 9$, we get $\sharp(C(\mathbb{F}_q)) \ge 3$. Set $T := C \cup M$. Every smooth quadric of $\mathbf{P}^5$ defined over $\mathbb{F}_q$ and containing M is hyperbolic. Hence it is sufficient to prove the existence of a smooth quadric of $\mathbf{P}^5$ defined over $\mathbb{F}_q$ and containing T. By Lemma 1 it is sufficient to prove the existence of a smooth quadric defined over $\bar{\mathbb{F}}_q$ and containing $T(\bar{\mathbb{F}}_q)$. Hence from now on in this proof we work over $\bar{\mathbb{F}}_q$ and we drop any mention of it. The projective normality of C implies $h^0(\mathbf{P}^5, \mathcal{I}_C(2)) = 21 - h^0(C, \mathcal{O}_C(2)) = 9$. Since $\sharp(C \cap M) = 3$, we have $h^0(\mathbf{P}^5, \mathcal{I}_C(2)) \ge 6$. Set $|W| := |\mathcal{I}_C|$. For any $P \in \mathbf{P}^5$, let $u_P : \mathbf{P}^5\backslash\{P\} \to \mathbf{P}^4$ denote the linear projection from P. Let $T_P \subset \mathbf{P}^4$ (resp. $T_P \subset \mathbf{P}^4$ denote the closure of $u_P(T\backslash\{P\})$ (resp. $u_P(T\backslash\{P\})$) in $\mathbf{P}^4$. For any $P \in \mathbf{P}^5$ set $|W_P| := \{Q \in |W| : Q$ is singular at $P\}$. Let $S^2(C)$ denote the secant variety of C. Hence $\dim(S^2(C)$. Furthermore, $S^2(C)$ does not contain M for the following reason. Assume $M \subset S^2(C)$. Then the linear projection of C from M would not be birational onto its image. However, the linear projection of $C\backslash\{P_1, P_2, P_3\}$ is isomorphic onto its image Γ and the closure of Γ in $\mathbf{P}^2$ is a smooth cubic curve $C_M \cong C$ and (up to this isomorphism) $\mathcal{O}_{C_M}(1) \cong \mathcal{O}_C(1)(-P_1 - P_2 - P_3)$; this is true because C is linearly normal and $\mathcal{O}_C(1)(-P_1 - P_2 - P_3)$ is very ample and non-special.

(a) Fix $P \in \mathbf{P}^5\backslash S^2(C)$. Hence $C_P \cong C$ is a degree 6 smooth elliptic curve. Since C is defined by the 2×2 minors of a matrix, $h^1(\mathbf{P}^4, \mathcal{I}_{C_P}(2)) = 0$ (see [8], Th. 3.1), i.e. $h^0(\mathbf{P}^4, \mathcal{I}_{C_P}(2)) = 3$.

(b) Fix $P \in \mathbf{P}^4\backslash M$ and take any $Q \in |W|$ singular at P. Thus Q contains the 3-dimensional linear space $[P; M]$ spanned by M and P. This implies $\dim(\mathrm{Sing}(Q)) \ge 1$ and that $\mathrm{Sing}(Q)$ contains a line joining P with one of the points of M.

(c) Fix $i \in \{1, 2, 3\}$. Here we will check the existence of a quadric $Q \in |W|$ which is smooth at P_i. Assume that this is not the case, i.e. assume that every element of $|W|$ is a cone with vertex containing P_i. Then $h^0(\mathbf{P}^5, \mathcal{I}_T(2)) = h^0(\mathbf{P}^4, \mathcal{I}_{T_{P_i}}(2))$. C_{P_i} is a degree 5 smooth linearly normal elliptic curve, while T_P is the union of C_{P_i} and the secant line spanned by the points $u_{P_i}(P_j), u_{P_i}(P_k)$ with $\{P_i, P_j, P_k\} = \{P_1, P_2, P_3\}$. We have $h^0(C_{P_i}, \mathcal{O}_{C_{P_i}}(2)) = 10$ and hence $h^0(T_{P_i}, \mathcal{O}_{T_{P_i}}(2)) = 11$. C_{P_i} is projectively normal and hence $h^0(\mathbf{P}^4, \mathcal{I}_{C_{P_i}}(2)) = 15 - 10 = 5$. Hence $h^0(\mathbf{P}^4, \mathcal{I}_{T_{P_i}}(2)) \le 5 < h^0(\mathbf{P}^5, \mathcal{I}_C(2))$, concluding the proof of the existence of $Q \in |W|$ smooth at P_i.

(d) Here we claim (and prove) the existence of $Q \in |W|$ such that Q is smooth at all points of $M \cap (S^2(C)\backslash C)$, i.e. at all $P \in M \cap S^2(C)$ such that $P \notin \{P_1, P_2, P_3\}$. Since $\dim(M \cap S^2(C)) = 1$, it is sufficient to prove that $\dim(|W_P|) \le \dim(|W|)$ for all $P \in M \cap S^2(C)$ such that $P \notin \{P_1, P_2, P_3\}$. Fix anty such P. Since C has no trisecant line, C_P is a degree 6 curve with arithmetic genus 2. It is very easy to check that C_P is projectively normal (we would need much less) and hence $h^0(\mathbf{P}^4, \mathcal{I}_{C_P}(2)) = 15 - 12 - 1 + 2 = 4$. Hence $h^0(\mathbf{P}^4, \mathcal{I}_{T_P}(2)) \leq 4$, proving the claim.

(e) Here we claim (and we will check) the existence of $Q \in |W|$ which is smooth at each point of $M \backslash S^2(C) \cap M$. Since $\dim(M) = 2$ and $\dim(M \cap S^2(C)) = 1$, to prove this assertion it is sufficient to prove $\dim(|W_P|) \leq \dim(|W|) - 3$ for all $P \in \backslash S^2(C) \cap M$. Fix P. By part (a) we have $h^0(\mathbf{P}^4, \mathcal{I}_{C_P}(2)) = 3$. Hence $h^0(\mathbf{P}^4, \mathcal{I}_{C_P}(2)) \leq 3$. Hence $\dim(|W_P|) \leq \dim(|W|)$, proving the claim.

(f) By parts (c), (d) and (e) a general $Q \in |W|$ is smooth at each point of M. By part (b) any such quadric Q must be smooth everywhere. □

Example 2. Let C be a non-hyperelliptic genus 4 curve defined over $\mathbb{F}_q$. Set $\bar{C} := C \times_{\mathbb{F}_q} \bar{\mathbb{F}}_q$. Since C is not hyperelliptic the canonical map is an embedding $\phi : C \to \mathbf{P}^3$ and the curve $\phi(C)$ has degree 6. It is well-known that $\phi(\bar{C})$ is the complete intersection of a quadric surface Q and a cubic surface. The quadric surface is uniquely determined and hence it is defined over $\mathbb{F}_q$. It is well-known that $\bar{C}$ is a trigonal curve and that it has one or two g^1_3's (both cases may occur for certain curves C). The curve $\bar{C}$ has a unique g^1_3 if and only if Q is singular, i.e. it is a quadric cone. In this case the g^1_3 is induced by the ruling of the quadric cone Q. Over $\mathbb{F}_q$ the following three cases may occur.

(a) The g^1_3 on $\bar{C}$ is unique, i.e. Q is a quadric cone. In this case the g^1_3 is defined over $\mathbb{F}_q$, because the ruling of a quadric cone is defined over $\mathbb{F}_q$.

(b) $\bar{C}$ has two g^1_3 (i.e. Q is smooth) and both g^1_3's are defined over $\mathbb{F}_q$. In this case the g^1_3 are defined by the two ruling of $\bar{Q} := Q \times_{\mathbb{F}_q} \bar{\mathbb{F}}_q$. Since the g^1_3's are defined over $\mathbb{F}_q$, the two rulings of $\bar{Q}$ are defined over $\mathbb{F}_q$. Thus Q is a hyperbolic quadric. Conversely, if Q is a hyperbolic quadric, then the two rulings of $\bar{Q}$ are defined over $\mathbb{F}_q$. Since these two rulings induces the g^1_3's of $\bar{C}$, both g^1_3's are defined over $\mathbb{F}_q$.

(c) The two g^1_3's are not defined over $\mathbb{F}_q$. By (a) and (b) this is equivalent to the ellipticity of the smooth quadric Q. In this case the two g^1_3's of $\bar{C}$ are defined over $\mathbb{F}_{q^2}$ and they are conjugate for the action of the Galois group $\mathbb{Z}/2\mathbb{Z}$ of the field extension $[\mathbb{F}_{q^2} : \mathbb{F}_q]$.

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