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BENZ SURFACES OVER A ROTATIONAL PARABOLOID

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Abstract: The authors investigate the family of Benz surfaces induced from the rotation paraboloid. It is given the now visualizations. All calculations are done by *Maple*™.

Key Words: Benz surface, Gaussian curvature

*

We introduce the following surface:

$$x1 := \cos(u) + \sin(v),$$
$$x2 := \cos(v) + \sin(u),$$
$$x3 := (\cos(u) + \sin(v))^2 + (\cos(v) + \sin(u))^2 ,$$

where $u \in (0, 2\pi)$, $v \in (0, 2\pi)$. It is shown by Figure 1

Evidently it is a part of rotational paraboloid; we can call it CUP.

To define the family of the Benz surfaces we introduce the *Maple*™ package:

> with(VectorCalculus): x:=<x1,x2,x3>;

$$x := (\cos(u) + \sin(v))\, e_x + (\cos(v) + \sin(u))\, e_y + \left((\cos(u) + \sin(v))^2 + (\cos(v) + \sin(u))^2\right) e_z$$

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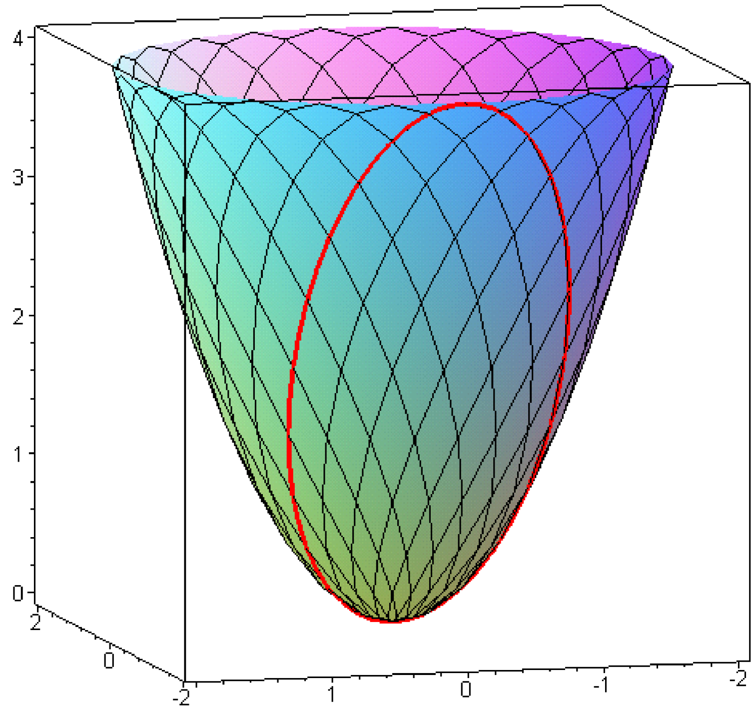

Figure 1

We calculate the coefficients of first fundamental form of the given surface

```
g11:=DotProduct(diff(x,u),diff(x,u)):
g12:=DotProduct(diff(x,u),diff(x,v)):
g22:=DotProduct(diff(x,v),diff(x,v)):
g:=simplify(g11*g22-g12^2):
```

Then we calculate the unit normal vector field

```
n:=CrossProduct(diff(x,u),diff(x,v))/sqrt(g):
```

For the family of Benz surfaces we have the following formulas:

```
e:=DotProduct(diff(x,u),diff(x,u,u)+2*m*diff(x,u,v)+m^2*diff(x,v,v)):
f:=DotProduct(diff(x,v),diff(x,u,u)+2*m*diff(x,u,v)+m^2*diff(x,v,v)):
II(m):=DotProduct(n,diff(x,u,u)+2*m*diff(x,u,v)+m^2*diff(x,v,v)):
k1:=g12+m*g22:k2:=-(g11+m*g12):k3:=-(k1*e+k2*f)/II(m):
```

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The vector parameterization of the Benz surfaces is given by:

B:=k1*diff(x,u)+k2*diff(x,v)+k3*n:

To find its coordinate parameterization we need the coordinate basis:

e1:=<1,0,0>;e2:=<0,1,0>;e3:=<0,0,1>;

$$e1 := \boldsymbol{e_x},\ e2 := \boldsymbol{e_y},\ e3 := \boldsymbol{e_z}$$

So, we get:

>B1:=simplify(DotProduct(B,e1));B2:=simplify(DotProduct(B,e2));

>B3:=simplify(DotProduct(B,e3));

$$\begin{aligned}
B1 :=&(-32\cos(u)^3\cos(v)^3 m\sin(v) - 32\cos(u)^3\cos(v)^3 m^2\sin(v) \\
&+ 22\cos(u)^3\cos(v)m^2\sin(v) + 6\cos(u)^3\cos(v)m\sin(v) \\
&+ 38\cos(u)^2\cos(v)^2 m^2\sin(u) - 8\cos(v)^2 m^2\sin(u) - 24\cos(v)m\cos(u)^4 \\
&+ \cos(u)^2\sin(u)m - 18\cos(u)\sin(u)\sin(v)m^2 + \cos(v)^3 m - 9\sin(u)\cos(v)^2 \\
&+ 9\cos(v)m^2 + 9\sin(u)m - 16\cos(u)^2\cos(v) - \cos(v)m + 16\cos(u)^2\cos(v)^3 \\
&- 26\cos(v)^3 m\cos(u)^2 + 8\cos(v) + 9\sin(u) + 9\cos(u)^2\cos(v)m3 \\
&+ 32\cos(v)^3 m\cos(u)^4 - 24m^2\cos(v)\cos(u)^4 + 8\cos(v)^4 m^2\sin(u) \\
&+ 43m^2\cos(v)\cos(u)^2 - 8\cos(u)^2 m^3\sin(u) - 17\cos(v)^2 m\sin(u) \\
&+ 18\cos(v)m\cos(u)2 + 26\cos(u)\sin(u)\sin(v)m^2\cos(v)^2 \\
&+ 8\cos(u)^3\sin(u)\sin(v)m^2 + 24\cos(u)m^2\sin(v)\cos(v)^3 \\
&+ 16\cos(u)^3 m^3\sin(v)\cos(v) - \cos(u)m\sin(u)\sin(v) \\
&+ 8\cos(u)^3 m\sin(u)\sin(v) + 24\cos(u)m\sin(v)\cos(v)^3 \\
&- 16\cos(u)\sin(u)\sin(v)\cos(v)^2 - 16\cos(u)m\sin(v)\cos(v) \\
&+ 16\cos(u)^2 m^3\sin(u)\cos(v)^2 - 8\cos(v)^3 - 9\cos(u)m^3\sin(u)\sin(v) \\
&- 15\cos(u)m^2\sin(v)\cos(v) - 8\cos(u)m^3\sin(v)\cos(v) - 42\cos(u)^2\cos(v)^3 \\
&- 32\cos(u)^3\cos(v)^2 m\sin(u)\sin(v) - 32\cos(u)^3\cos(v)^2\sin(u)\sin(v)m^2 \\
&+ 8\cos(v)^4\sin(u)m - 7\cos(u)^2\sin(u)m^2 + 32\cos(u)^4\cos(v)^3 m^2 \\
&+ 8\cos(u)\sin(u)\sin(v) - 32\cos(u)^2\cos(v)^4 m^2\sin(u) \\
&- 32\cos(u)^2\cos(v)4\sin(u)m + 22\cos(u)^2\cos(v)^2 m\sin(u) \\
&+ 10\cos(u)\cos(v)^2 m\sin(u)\sin(v) - 9m^2\cos(v) - 9\sin(v)\cos(u)\cos(v)) \\
&/(2m\cos(u)\sin(v) + 2m\sin(u)\cos(v) - m^2 - 1)
\end{aligned}$$

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$$
\begin{aligned}
B2 :=&(-32\cos(u)^3\cos(v)^4 m + 42\cos(u)^3 m\cos(v)^2 - 8\cos(v)^3\sin(u)m^2\sin(v)\\
&- 8\cos(v)^3\sin(v)m\sin(u) + 9\cos(u)\cos(v)m^3\sin(u) + \cos(u)m^2 - 9m^3\sin(v)\\
&- \cos(u)m^2 - 9\cos(u)^3 m + 8\cos(u)^3 m^3 + 8\cos(v)^2\sin(v) - 8\cos(u)m^3\\
&- 9\cos(u)\cos(v)^2 + 9\cos(u)m^3\sin(v) + 8\cos(u)^2 m\sin(v)\\
&+ 18\cos(v)\sin(u)\sin(v)m - 43\cos(u)\cos(v)^2 m - 16\cos(u)\cos(v)^3\sin(u)\\
&+ 17\cos(u)^2 m^2\sin(v) + 16\cos(u)m^3\cos(v)^2 - 16\cos(u)^3 m^3\cos(v)^2\\
&+ 9\cos(v)\sin(u)\sin(v) + 24\cos(u)\cos(v)^4 m + 8\sin(u)\cos(u)\cos(v)\\
&+ 24\cos(u)\cos(v)^4 m^2 + 26\cos(u)^3\cos(v)^2 m^2 - 32\cos(u)^3\cos(v)^4 m^2\\
&- 8\cos(u)^4 m\sin(v) - 8\cos(u)^4 m^2\sin(v) - 16\cos(v)^2\sin(v)\cos(u)^2\\
&+ 9m\cos(u) - \cos(v)^2 m^2\sin(v) - 6\cos(u)\cos(v)^3 m^2\sin(u)\\
&- 26\cos(u)^2\cos(v)\sin(u)\sin(v)m - 8\cos(v)m^3\sin(u)\sin(v)\\
&+ 16\cos(u)\cos(v)\sin(u)m^2 + \cos(v)\sin(u)m^2\sin(v)\\
&- 24\cos(u)^3\cos(v)\sin(u)m + 32\cos(u)^2\cos(v)^3\sin(u)m^2\sin(v)\\
&- 10\cos(u)^2\cos(v)\sin(u)m^2\sin(v) - 18\cos(u)\cos(v)^2 m^2\\
&+ 15\cos(u)\cos(v)\sin(u)m - 38\cos(u)^2\cos(v)^2 m\sin(v)\\
&- 22\cos(u)^2\cos(v)^2 m^2\sin(v) + 32\cos(u)^2\cos(v)^3\sin(v)m\sin(u)\\
&+ 7\cos(v)^2 m\sin(v) - 24\cos(u)^3\cos(v)\sin(u)m^2 - 9m^2\sin(v)\\
&- 22\cos(u)\cos(v)^3\sin(u)m + 32\cos(u)^4\cos(v)^2 m^2\sin(v)\\
&+ 32\cos(u)^4\cos(v)^2 m\sin(v) + 32\cos(u)^3\cos(v)^3\sin(u)m\\
&+ 32\cos(u)^3\cos(v)^3 m^2\sin(u) + 16\cos(u)^2\cos(v)m^3\sin(v)\sin(u))\\
&/(2m\cos(u)\sin(v) + 2m\sin(u)\cos(v) - m^2 - 1)
\end{aligned}
$$

$$
\begin{aligned}
B3 :=&\frac{1}{2}(-18\cos(u)\cos(v)^2 m^2\sin(u) - 18\cos(u)^2\cos(v)m^2\sin(v)\\
&+ 8\sin(u)\cos(u) + 8\cos(v)\sin(v) - 24\cos(u)\cos(v)^3 m^2\\
&- 8\cos(u)^2\sin(v)\sin(u)m + 8\cos(u)^2\sin(v)\sin(u)m^2\\
&+ 16m^3\sin(u)\cos(u)\cos(v)^2 + 16m^3\sin(v)\cos(v)\cos(u)^2\\
&- 8\sin(u)\cos(v)^2 m\sin(v) + 32\sin(u)\cos(v)^2\cos(u)^2 m\sin(v)\\
&- 32\sin(u)\cos(v)^2\cos(u)^2 m^2\sin(v) + 32\cos(u)^3\cos(v)^3 m^2\\
&- 8m^3\sin(v)\cos(v) - 8m^3\sin(u)\cos(u) - 9m\sin(u)\sin(v)\\
&+ 9\cos(v)m^2\sin(v) - 9\cos(v)m\sin(v) + 16\cos(u)m^2\cos(v)\\
&- 16\cos(u)^2\cos(v)\sin(v) - 16\cos(u)\cos(v)^2\sin(u) - 16\cos(v)m\cos(u)\\
&- 32m\cos(u)^3\cos(v)^3 + 24\cos(v)^3 m\cos(u) + 24\cos(v)m\cos(u)^3
\end{aligned}
$$

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Figure 2

$$
\begin{aligned}
&-24\cos(u)^3m^2\cos(v)-9m\sin(u)\cos(u)+9m^3\cos(u)\cos(v)\\
&+9\sin(u)m^2\cos(u)+8\cos(v)^2\sin(u)\sin(v)m^2+18\cos(u)\cos(v)^2m\sin(u)\\
&+18\cos(u)^2\cos(v)m\sin(v)-9\cos(u)\cos(v)+9\sin(u)\sin(v))\\
&/(2m\cos(u)\sin(v)+2m\sin(u)\cos(v)-m^2-1)
\end{aligned}
$$

We give now some visualizations:
1. If $m=-1$ the Benz surface is a segment (see Figure 2):

```
plot3d([eval(B1,[m=-1]),eval(B2,[m=-1]),eval(B3,[m=-1])],
u=0..2*Pi,v=0..2*Pi,grid=[30,30]);
```

2. If $m=0$ the Benz surface seems as a cone (see Figure 3):

```
with(plots):
q1:=plot3d([eval(B1,[m=0]),eval(B2,[m=0]),eval(B3,[m=0])],
u=0..2*Pi,v=0..2*Pi,grid=[30,30]):
q2:=spacecurve([eval(B1,[m=0,v=0]),eval(B2,[m=0,v=0]),
eval(B3,[m=0,v=0])],u=0..2*Pi,thickness=2,color=red):
opts:=thickness=3,color=red,numpoints=100:
q3:=animate(spacecurve,[[eval(B1,[m=0]),eval(B2,[m=0]),
eval(B3,[m=0])],opts,u=0..2*Pi],v=0..2*Pi):
q4:=spacecurve([eval(B1,[m=0,u=0]),eval(B2,[m=0,u=0]),
eval(B3,[m=0,u=0])],v=0..2*Pi,thickness=3,color=blue):
display([q1,q2,q3,q4]);
```

3. If $m=1$ Benz surface seems as a circle (see Figure 4):
We shall prove the following classification theorem.

Theorem. *The family of Benz surfaces (parameter m) induced from the CUP consists from the following surfaces:*

1. If $m=-1$ the Benz surface is a segment;

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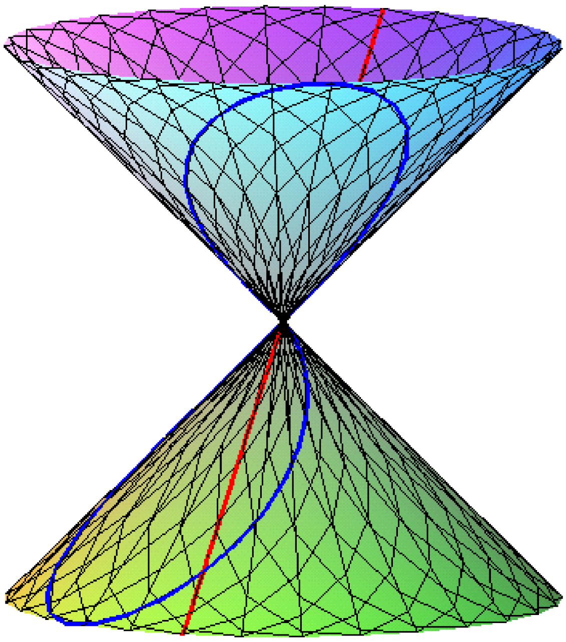

Figure 3

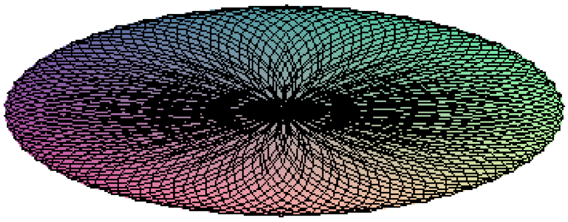

Figure 4

2. If $m = 1$ the Benz surface is a circle of radius

$$R = -\frac{(-115+\sqrt{5}\sqrt{469})\sqrt{5}\sqrt{2}\sqrt{45+\sqrt{5}\sqrt{469}}(55+\sqrt{5}\sqrt{469})}{16000}.$$

It is approximately 13.15371036.

3. If $m = 0$ the Benz surface is a rotational cone.

4. In all other cases the Benz surface seems as cone, but it is not cone.

Proof. In the first case the assertion follows from the equalities:

eval(B1,[m=-1])=0;eval(B2,[m=-1])=0;

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and for the function **F:=eval(B3,[m=-1]);** holds

$$F := (18\sin(u)\sin(v) + 14\cos(u)\cos(v) + 34\sin(u)\cos(u) + 34\cos(v)\sin(v) + 64\cos(u)^3\cos(v)^3 - 48\cos(v)^3\cos(u) + 16\cos(u)^2\sin(v)\sin(u) - 64\sin(u)\cos(v)^2\cos(u)^2\sin(v) - 68\cos(u)^2\cos(v)\sin(v) - 68\cos(u)\cos(v)^2\sin(u) - 48\cos(u)^3\cos(v) + 16\sin(u)\cos(v)^2\sin(v)) /(2 + 2\sin(u)\cos(v) + 2\cos(u)\sin(v)).$$

In the second case we have **eval(B3,[m=1])=0,** so the surface is a plane piece.
From the picture we conclude the surface seems as a circle. We confirm it in two steps.

Let us consider the function

R2:=factor(eval(B2,[m=1])):

Solving the equation

solve({r2=0},{u});

$$\{u = v + \frac{\pi}{2}\}, \{u = \arctan((9 - 680\sin(v)^2 - 1336\sin(v)^4 + 4608\sin(v)^6 + (-788\sin(v)^3 + 37\sin(v) + 1040\sin(v)^5)(-9\sin(v) + \sqrt{17\sin(v)^2 - 17})) /(-\cos(v) + 68\cos(v)\sin(v)^2 + 256\cos(v)\sin(v)^4 + (-4\cos(v)\sin(v) + 72\cos(v)\sin(v)^3)(-9\sin(v) + \sqrt{17\sin(v)^2 - 17})), - 9\sin(v) + \sqrt{17\sin(v)^2 - 17})\}, \{u = \arctan((9 - 680\sin(v)^2 - 1336\sin(v)^4 + 4608\sin(v)^6 + (-788\sin(v)^3 + 37\sin(v) + 1040\sin(v)^5)(-9\sin(v) - \sqrt{17\sin(v)^2 - 17})) /(-\cos(v) + 68\cos(v)\sin(v)^2 + 256\cos(v)\sin(v)^4 + (-4\cos(v)\sin(v) + 72\cos(v)\sin(v)^3)(-9\sin(v) - \sqrt{17\sin(v)^2 - 17})), - 9\sin(v) - \sqrt{17\sin(v)^2 - 17})\}$$

We can define the function

s2:=simplify(eval(B1,[m=1,u=1/2*Pi+v]));

$$s2 := -4\sin(v)^2\cos(v)(16\cos(v)^2 + 1);$$

To find its extreme we solve the equation

>solve({diff(s2,v)},{v});

$$\{v = 0\},\ \{v = \arctan\frac{\sqrt{1150 - 10\sqrt{2345}}}{\sqrt{450 + 10\sqrt{2345}}}\},\ \{v = -\arctan\frac{\sqrt{1150 - 10\sqrt{2345}}}{\sqrt{450 + 10\sqrt{2345}}}\},$$

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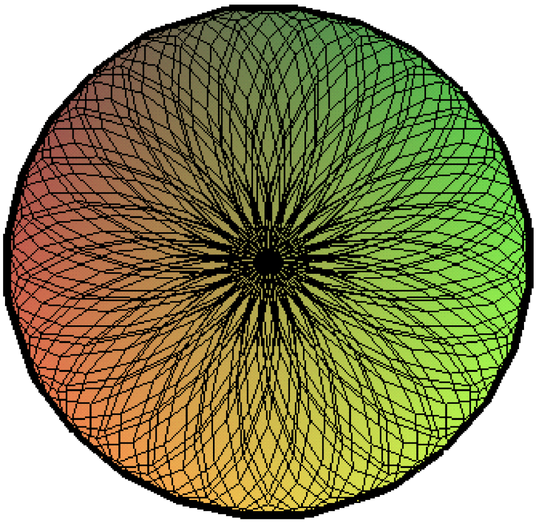

Figure 5

$$\{v = -\arctan\frac{\sqrt{1150-10\sqrt{2345}}}{\sqrt{450+10\sqrt{2345}}}+\pi\},\ \{v = \arctan\frac{\sqrt{1150-10\sqrt{2345}}}{\sqrt{450+10\sqrt{2345}}}-\pi\},$$

$$\{v = \arctan(\sqrt{1150+10\sqrt{2345}}, \sqrt{450-10\sqrt{2345}})\},$$

$$\{v = \arctan-\frac{\sqrt{1150+10\sqrt{2345}}}{40}, \frac{\sqrt{450-10\sqrt{2345}}}{40}\},$$

$$\{v = \arctan\frac{\sqrt{1150+10\sqrt{2345}}}{40}, -\frac{\sqrt{450-10\sqrt{2345}}}{40}\},$$

$$\{v = \arctan-\frac{\sqrt{1150+10\sqrt{2345}}}{40}, -\frac{\sqrt{450-10\sqrt{2345}}}{40}\},$$

We find its maximum if

v2:=-arctan((1150
-10*2345^(1/2))^(1/2)/(450+10*2345^(1/2))^(1/2))+Pi ;

$$v2 := -\arctan\frac{\sqrt{1150-10\sqrt{2345}}}{\sqrt{450+10\sqrt{2345}}}+\pi$$

Then we calculate it:

s20:=simplify(eval(s2,[v = v2]));

$$s20 := -\frac{(-115+\sqrt{5}\sqrt{469})\sqrt{5}\sqrt{2}\sqrt{45+\sqrt{5}\sqrt{469}}(55+\sqrt{5}\sqrt{469})}{16000}.$$

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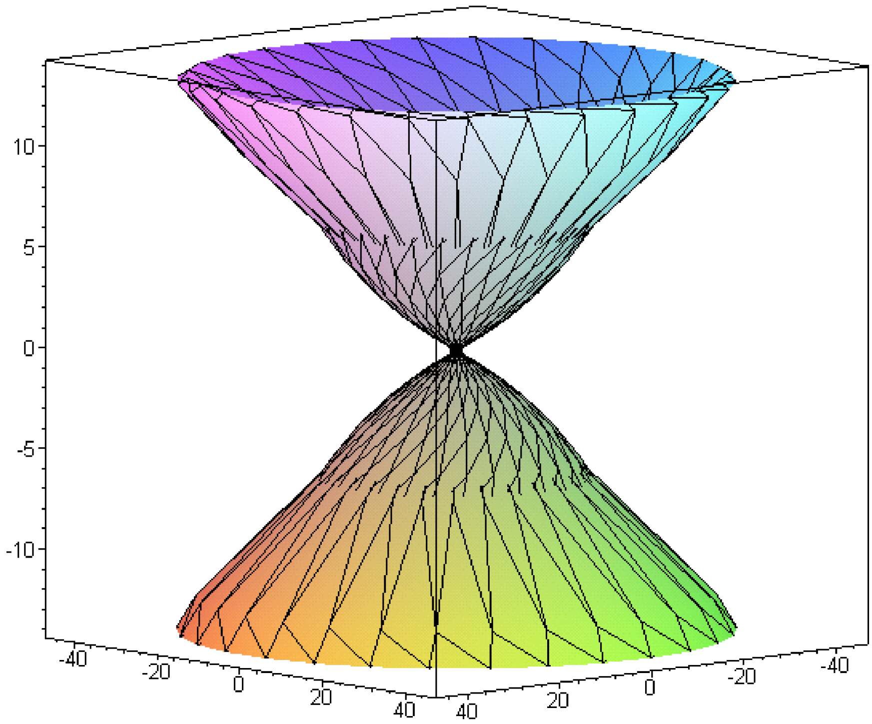

Figure 6

It is approximately

evalf(s20);

13.15371036

So the second part is proved. The corresponding Benz surface is again visualized by the following picture, where is given the graphic of the boundary circle, see Figure 5.

To prove the part 3.we shall prove that in this case the curvature of the lines v =const is zero. To do this let:

b1:=diff(B,u):
b2:=diff(B,u,u):
b:=CrossProduct(b1,b2):

We calculate easy:

simplify(eval(b,[m=0]));

$$0\mathbf{e_0}$$

There are not another values for m except 0,-1, for which this equality holds. So

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the part 3 is proved.

Figure 6 illustrates the last case:

```
>plot3d([eval(B1,[m=2]),eval(B2,[m=2]),eval(B3,[m=2])],
    u=0..2*Pi,v=0..2*Pi,grid=[30,30]);
```

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