International Electronic Journal of Pure and Applied Mathematics – IEJPAM, Volume 4, No. 3 (2012)

International Electronic Journal of Pure and Applied Mathematics

Volume 4 No. 3 2012, 143-147

FINITELY GENERATED s_1IDEALS IN COMMUTATIVE RINGS

Sarab A. Al-Taha

Department of Mathematics
Faculty of Science
Al Zaytoonah University
Amman, JORDAN

Abstract: Let R be a commutative ring with identity, and let a be a nonzero element of R.

The principal ideal $I = \langle a \rangle$is called s_1 ideal of R if, and only if the following condition holds true:

If $ab = a$, $b \in R$, there exists $a' \in R$, $a\prime \neq 0$ such that $a'b = 0$, see [1].

Now, let a_1, $a_2, \ldots,$ a_n be any n nonzero elements in R. This paper deals with a new definition for the finitely generated ideal $I = <a_1, a_2, \ldots, a_n>$ of the ring R that we call finitely generated s_1 ideal. We prove the following result among others, if $A^- = B^-$, where $A=(a_1, a_2, \ldots, a_n) \in R^n$ and $B=(b_1, b_2, \ldots, b_n) \in R^n$ then $<a_1, a_2, \ldots, a_n>$ is $s_1$ideal of R if, and only if $<b_1, b_2, \ldots, b_n>$ is $s_1$ideal of R.

AMS Subject Classification: 13C99
Key Words: commutative rings, finitely generated ideals

1. Introduction

Throughout this work we use the following notations:

R: Commutative ring with identity

R^n: Set of all n-tuples with components in a ring R and operations defined componentwise.

$M_n(R)$: Set of all $n \times n$ matrices whose entries belong to R

$<a_1, a_2, \ldots, a_n>$: The finitely generated ideal in the ring R with generators $(a_1, a_2, \ldots, a_n) \in R^n$.

Let $A=(a_1, a_2, \ldots, a_n)$ be an orderd n-tuples in R^n we denote:

$$A\text{-}=\{x=(x_1, x_2, \ldots, x_n) \in R^n : AX^T = \sum_{i=1}^{n} a\text{ixi} = 0\}.$$

Received: August 20, 2011

International Electronic Journal of Pure and Applied Mathematics – IEJPAM, Volume 4, No. 3 (2012)

This paper deals with a new definition for a finitely generated ideal $I=<a_1, a_2, \ldots, a_n>$, $a_i \in R$ and $a_i \neq 0$ $i=1,2,\ldots,n$ that we call s_1 ideal of R if, and only if the following holds:

If $AM = A$, where

$$A = (a_1, a_2, \ldots, a_n) \in R^n \text{ and } M = (m_{ij}) \in M_n(R), \tag{1}$$

then there exists $A' = (a'_1, a'_2, \ldots, a'_n) \in R^n, a'_i \neq 0$, $i=1,2,\ldots,n$ such that:

$$A'M = 0. \tag{2}$$

This definition will serve as our main tool throughout this work. First we give two examples the first one is for a principal ideal which is $s_1$ideal and the second one is for a principal ideal which is not $s_1$ideal

1. $I=<2>$ is $s_1$ideal of z_6. [1]

2. $I=<4>$ is not $s_1$ideal of z_{12}

2. Main Results

In this section we state down the following results:

Theorem 1. *If $A^- = B$- where $A= (a_1, a_2, \ldots, a_n) \in R^n$ and $B= (b_1, b_2, \ldots, b_n) \in R$, then $<a_1, a_2, \ldots, a_n>$ is $s_1$ideal of a ring R if, and only if $<b_1, b_2, \ldots, b_n>$ is s_1 ideal of R.*

Theorem 2. *If $I=<a_1, a_2, \ldots, a_n>$, $J=<b>$ are two ideals of a ring R, b is not a zero divisor of R, then $IJ= <a_1 b, a_2 b, \ldots, a_n b>$ is $s_1$ideal of R if, and only if $I=<a_1, a_2, \ldots, a_n>$ is s_1 ideal of R.*

Theorem 3. *If $<a_1, a_2, \ldots, a_n>$ is s_1 ideal of a ring R, then $<a_i>$ is s_1 ideal of R for each i=1,2,...,n.*

Theorem 4. *Let R and S be any two commutative rings with identity, let φ be an isomorphism from R into S then $<a_1, a_2, \ldots, a_n>$ is s_1 ideal of R if, and only if $<\varphi(a_1), \varphi(a_2), \ldots \varphi(a_n)>$ is $s_1$ideal of S.*

Theorem 5. *Let R be a commutative ring with identity if $<a_1, a_2, \ldots, a_n>$ is s_1 ideal of R[x]*,then $<a_1, a_2, \ldots, a_n>$ is $s_1$ideal of R.

3. Proofs

In this section we prove our main results:

Proof of Theorem 1. Assume that $<a_1,a_2,\ldots,a_n>$ is s_1 ideal of a ring R with A- = B-, to prove that $<b_1,b_2,\ldots,b_n>$ is s_1 ideal of R, let BM=B, where $B=(b_1,b_2,\ldots,b_n) \in R^n$ and $M=(m_{ij}) \in M_n(R)$ i,j=1,2,...,n

It follows that B(M-I)=0, where I is the identity matrix. This means that each column of (M-I) belong to B-, but B-=A- this implies that each column of (M-I) belong to A-.

Thus A (M-I) =0, which means that: AM=A. Since $<a_1,a_2,\ldots,a_n>$ is s_1 ideal of the ring R, using (2) there exists

$A' = (a_1',a_2',\ldots,a_n') \in R^n$ such that:

$$A'M = 0. \tag{3}$$

It follows that $<b_1, b_2,\ldots,b_2>$ is s_1 of a ring R. Using exactly the same way, we can prove that if $<b_1,b_2,\ldots,b_n>$ is s_1 ideal of R then $<a_1, a_2, \ldots, a_n>$ is s_1 ideal of R.

Proof of Theorem 2. To prove that the ideal $IJ=<a_1b,a_2b,\ldots,a_nb>$ (see [2]) is $s_1$ideal of a ring R if, and only if $I=<a, a, \ldots,a>$ is s_1 ideal of R, it is enough to prove that E-=A- where $E=(a_1 b,a_2b,\ldots,a_n b)\in R^n$ and $A=(a_1,a_2,\ldots,a_n) \in R^n$

Now let $x=(x_1, x_2\ldots,x_n) \in E+$, this means that

$$\sum_{i=1}^{n} ba_i x_i = o$$

which implies that

$$b\sum_{i=1}^{n} a_i x_i = 0,$$

since b is not a zero divisor of R we get (see [3])

$$\sum_{i=1}^{n} a_i x_i = 0,$$

hence $x\in A+$.

In the same way we prove that if $x\in A+$ then $x\in E+$, using theorem (1) we get the result.

Proof of Theorem 3. Assume that$<a_1,a_2,\ldots,a_n>$ is $s_1$ideal of a ring R, to prove that $<a_i>$ is s_1 ideal of R, i=1,2,...,n.

Let $a_i b_i = a_i$, i=1,2,...,n

International Electronic Journal of Pure and Applied Mathematics – IEJPAM, Volume 4, No. 3 (2012)

We get

$$(a_1, a_2, \ldots, a_n) \begin{bmatrix} b1\,0\,0\ldots0 \\ 0\ b2\,0..0 \\ \ldots\ldots\ldots\ldots \\ 0\ldots\quad 0\,bn \end{bmatrix} = (a_1, a_2, \ldots, a_n).$$

Since $<a_1,a_2,\ldots,a_n>$ is s_1 ideal of R, using (2), there exists $(a_{1\prime}, a_2\prime,\ldots,a_n\prime) \in R^n$, $a_i\prime \neq 0$, i=1,2,...,n such that:

$$(a_1', a_2\prime, \ldots, a_n\prime) \begin{bmatrix} b1\,0\ldots0 \\ 0\ b2\,0..0 \\ \ldots\ldots\ldots\ldots \\ 0\ldots\ldots0\,bn \end{bmatrix} = 0.$$

Thus $a_i\prime b_i = 0$, i = 1,2,...n and therefore $<a_i>$ is s_1 ideal of R for each i=1,2,...,n.

Proof of Theorem 4. To prove that $<\varphi(a_1),\varphi(a_2),\ldots,\varphi(a_n)>$ is s_1 ideal of S, let $(\varphi(a_1),\varphi(a_2),\ldots,\varphi(a_n)(m_{ij}) = (\varphi(a_1),\varphi(a_2),\ldots,\varphi(a_n))$, where $(m_{ij}) \in M_n(S)$.This implies that

$$(a_1, a_2, \ldots, a_n)(\varphi^{-1}m_{ij}) = (a_1, a_2, \ldots, a_n), (\varphi^{-1}m_{ij}) \in M_n(R),$$

see [4].

Since $<a_1,a_2,\ldots,a_n>$ is $s_1$ideal of R, using (2), there exists $(a_1\prime,a_2\prime,\ldots,a_n\prime) \in R^n$, $a_i\prime \neq 0$ i=1,2,...,n

Such that:

$$(a_1\prime, a_2\prime, \ldots, a_n\prime)(\varphi^{-1}m_{ij}) = 0,$$

hence

$$(\varphi(a_1\prime), \varphi(a_2\prime), \ldots, \varphi(a_{n\prime})(m_{ij}) = 0,$$

which means that $<\varphi(a_1),\varphi(a_2),\ldots,\varphi(a_n)>$ is s_1 ideal of S.

In the same way we can prove that if $<\varphi(a_1),\varphi(a_2),\ldots,\varphi(a_n)>$ is s_1 ideal of S then

$$<a_1, a_2, \ldots, a_n>$$

is s_1 ideal of R.

Proof of Theorem 5. To prove that$<a_1,a_2,\ldots,a_n>$ is s_1 ideal of R, let

$$(a_1, a_2, \ldots, a_n)(m_{ij}) = (a_1, a_2, \ldots, a_n),$$

where $(m_{ij}) \in M_n(R)$, i,j=1,2,...,n.

Since $<a_1,a_2,\ldots,a_n>$ is s_1 ideal of R[x], using (2) there exists

$$(f_1(x), f_2(x), \ldots, f_n(x)) \in (R[x])^n$$

International Electronic Journal of Pure and Applied Mathematics – IEJPAM, Volume 4, No. 3 (2012)

such that:

$$(f_1(x), f_2(x), \ldots, f_{\mathrm{n}}(x))(m_{\mathrm{ij}}) = 0,$$

where

$$f_{\mathrm{i}}(x) = a_0^{\mathrm{i}} + a_1^{\mathrm{i}}x + \cdots + a_{\mathrm{n}}^{\mathrm{i}}x, \quad i = 1, 2, \ldots, n$$

This means that we there exists $(a_0^1, a_0^2, \ldots, a_0^n) \in R^n$ with

$$(a_0^1, a_0^2, \ldots, a_0^{\mathrm{n}})(m_{\mathrm{ij}}) = 0,$$

see [5].

Hence we get that $<a_1, a_2, \ldots, a_n>$ is s_1 ideal of R.

References

[1] Sarab A. Al-Taha, S_1 ideals in commutative rings, *Jour. of Inst. and Computer Sciences*, **24**, No. 1 (2011), 23-27.

[2] Joseph A. Gallian, *Contemporary Abstract Algebra*, Houghton Mifflin Company (2006).

[3] Robert Gilmer, *Multiplicative Ideal Theory*, Marcel Dekker, New York (1972).

[4] Irving Kaplansky, *Commutative Rings*, The University of Chicago Press (1974).

[5] T.S. Blyth, E.F. Robertson, *Groups, Rings and Fields*, Cambridge University Press (1984).