International Electronic Journal of Pure and Applied Mathematics
Volume 4 No. 3 2012, 137-142

INVESTIGATION OF BENZ SURFACES INDUCED BY A UNIT SPHERE

Grozio Stanilov[1], Liudmila Filipova[2 §]
[1]Sofia University "St. Kliment Ohridski
5, J. Boucher, Blvd., Sofia, BULGARIA
[2]Branch Plovdiv
Technical University – Sofia
25, Tsanko Diustabanov, Str., Plovdiv, 4000, BULGARIA

Abstract: Following the method in a paper of the first author in this paper the authors investigate the family of Benz surfaces induced from the unit sphere. It is given a condition when such con –like surface is rounded down. All calculations are done by *Maple*™.

AMS Subject Classification: 51004, 51M15, 51M04, 52A15, 52A40
Key Words: Benz surface, Gaussian curvature, meridian

*

In the paper [1] is given an algorithms for introducing a family of surfaces arising from given surface. It consists of the following:

If the given surface in the 3-dimensional Euclidean space is represented in the form

$$x1 := X1(u,v),\ x2 := X2(u,v),\ x3 := X3(u,v)$$

including the *Maple*™ package

with(VectorCalculus):

and putting

x:=<x1,x2,x3>;

we calculate the coefficients of the first fundamental form to the given surface and its unite normal vector field:

Received: May 12, 2011

§Correspondence author

International Electronic Journal of Pure and Applied Mathematics – IEJPAM, Volume 4, No. 3 (2012)


```
g11:=simplify(DotProduct(diff(x,u),diff(x,u)));
g12:=simplify(DotProduct(diff(x,u),diff(x,v)));
g22:=simplify(DotProduct(diff(x,v),diff(x,v)));
g:=simplify(g11*g22-g12^2);
n:=CrossProduct(diff(x,u),diff(x,v))/sqrt(g);
```

Then we find the functions:

```
e:=DotProduct(diff(x,u),diff(x,u,u)+2*m*diff(x,u,v)+m^2*diff(x,v,v));
f:=DotProduct(diff(x,v),diff(x,u,u)+2*m*diff(x,u,v)+m^2*diff(x,v,v));
II(m):=DotProduct(n,diff(x,u,u)+2*m*diff(x,u,v)+m^2*diff(x,v,v));
k1:=g12+m*g22;k2:=-(g11+m*g12);k3:=-(k1*e+k2*f)/II(m);
```

and the vector-valued parameterization of the family of Benz surfaces (m is an arbitrary function) is

```
B:=k1*diff(x,u)+k2*diff(x,v)+k3*n;
```

Introducing the vector base

```
>e1:=<1,0,0>;e2:=<0,1,0>;e3:=<0,0,1>;
```

one can find the coordinate representation

```
B1:=DotProduct(B,e1);B2:=DotProduct(B,e2);B3:=DotProduct(B,e3);
```

In this paper we apply this method to the unite sphere under the parameterization

$$X1(u,v) := \sin(u)\cos(v),\ X2(u,v) := \sin(u)\sin(v),\ X3(u,v) := cos(u)$$

$u \in (0,\pi)$, $v \in (0,2\pi)$.

Using this formulas the family of Benz surfaces get the parameterization:

$$B1s := \frac{(m\sin(u)\cos(u)\cos(v) + \cos(u)^2\sin(v)m^2 - \sin(v)m^2 - \sin(v))\sin(u)}{-m^2 - 1 + \cos(u)^2 m^2}$$

$$B2s := \frac{(m\sin(u)\cos(u)\sin(v) + \cos(v)^2 m^2 + \cos(v)\cos(v)^2 m^2 - \cos(v))\sin(u)}{-m^2 - 1 + \cos(u)^2 m^2}$$

$$B3s := -\frac{(-1 + \cos(u)^2 m^2 - m^2 - \cos(u)^2)\sin(u)m}{-m^2 - 1 + \cos(u)^2 m^2}$$

Before to state our problem we bring some visualizations: see Figure 1, for $m = 2$ for $m = 0.1$.

We remark that for small values as $m = 0.1$ the Benz surface is rounded down and it is not so for bigger values of parameter$m$. Then we state our

Problem. To find the m-intervals where the Benz surfaces are rounded down.

To solve this problem we remark that in such cases the Benz surface changes its protuberance.

International Electronic Journal of Pure and Applied Mathematics – IEJPAM, Volume 4, No. 3 (2012)

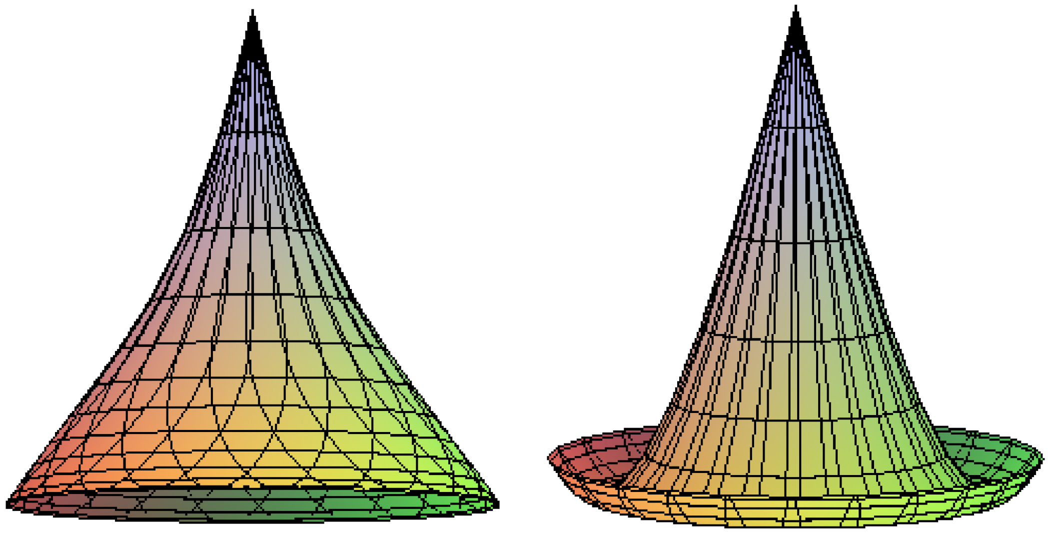

Figure 1

This is related to the Gaussian curvature. Because of this reason we find this curvature:

$$
\begin{aligned}
Ks :=&(\sin(u)^2+\cos(u)^2+\sin(u)^2m^2)^6 \\
&(\cos(u)^4m^4-\cos(u)^4m^2+3\cos(u)^2-1+\cos(u)^2m^2-2\cos(u)^2m^4+m^4)(\\
&-4\cos(u)^6m^{10}+6\cos(u)^4m^{10}-4\cos(u)^2m^{10}+\cos(u)^8m^{10}+m^{10} \\
&-2\cos(u)^8m^8 \\
&-\cos(u)^6m^8+15\cos(u)^4m^8-19\cos(u)^2m^8+7m^8+\cos(u)^8m^6 \\
&-4\cos(u)^6m^6 \\
&+3\cos(u)m^6+2\cos(u)^2m^6-2m^6+9\cos(u)^6m^4-18\cos(u)^4m^4 \\
&+34\cos(u)^2m^4-25m^4-6\cos(u)^4m^2+17\cos(u)^2m^2-23m^2-6)m^2/(\\
&(\sin(u)^2+\cos(u)^2)^6(21\sin(u)^6\cos(u)^6m^8+70\sin(u)^8\cos(u)^4m^6 \\
&+14\sin(u)^{10}\cos(u)^2m^{10}+12\sin(u)^8\cos(u)^4m^{10}+16\sin(u)^2\cos(u)^{10}m^4 \\
&+43\sin(u)^{10}\cos(u)^2m^4+61\cos(u)^8\sin(u)^4m^4+85\sin(u)^8\cos(u)^4m^4 \\
&+5\sin(u)^{10}m^{12}\cos(u)^2+26\sin(u)^{10}\cos(u)^2m^2+72\sin(u)^6\cos(u)^6m^2 \\
&+24\sin(u)^{10}\cos(u)^2m^8+22\cos(u)^{10}m^2\sin(u)^2+97\cos(u)^6\sin(u)^6m^4 \\
&+6\sin(u)^{10}\cos(u)^2+10\sin(u)^{12}m^4+5\sin(u)^{12}m^2+38\sin(u)^{10}\cos(u)^2m^6 \\
&+62\sin(u)^6\cos(u)^6m^6+41\sin(u)^8\cos(u)^4m^8+53\cos(u)^8\sin(u)^4m^2 \\
&+4\sin(u)^{12}m^8+15\sin(u)^8\cos(u)^4+4\cos(u)^{12}m^2+58\sin(u)^8\cos(u)^4m^2
\end{aligned}
$$

International Electronic Journal of Pure and Applied Mathematics – IEJPAM, Volume 4, No. 3 (2012)

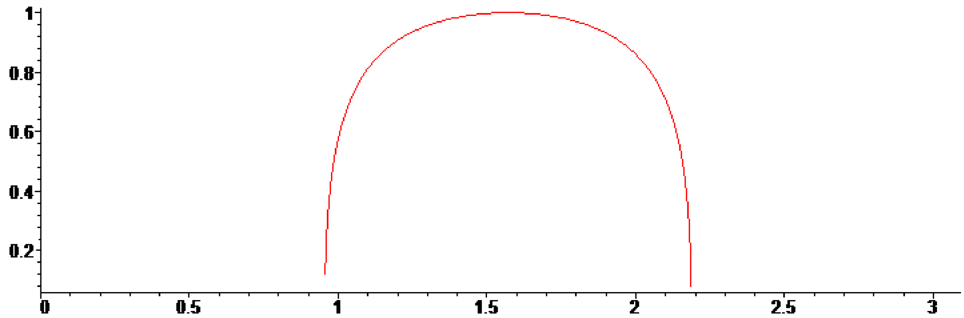

Figure 2

$$
\begin{aligned}
&+21\sin(u)^4\cos(u)^8m^6+\sin(u)^{12}m^{14}+3\sin(u)^{12}m^{12}+3\sin(u)^{12}m^{10}\\
&+9\sin(u)^{12}m^6+\sin(u)^{12}+\cos(u)^{12}+20\sin(u)^6\cos(u)^6+15\sin(u)^4\cos(u)^8\\
&+6\cos(u)^{10}\sin(u)^2)(1-4\cos(u)^2m^2+5m^2-23\cos(u)^8m^{10}+20\cos(u)^4n\\
&+3\cos(u)^4m^2+17\cos(u)^6m^8+10\cos(u)^8m^6+m^{14}\cos(u)^{12}+\cos(u)^{12}m^{10}\\
&-17\cos(u)^2m^4+32\cos(u)^6m^{10}-13\cos(u)^4m^{10}-4\cos(u)^2m^{10}+3\cos(u)^8\\
&-19\cos(u)^4m^8-18\cos(u)^6m^6+15\cos(u)^4m^6-16\cos(u)^2m^6-13\cos(u)^6\\
&+10m^4+3m^{10}+4m^8+9m^6+3m^{12}+m^{14}+15m^{14}\cos(u)^4-6m^{14}\cos(u)^{10}\\
&-20m^{14}\cos(u)6+15m^{14}\cos(u)^8-5\cos(u)^{10}m^8-6m^{14}\cos(u)^2\\
&-13m^{12}\cos(u)^2+20m^{12}\cos(u)^4-10m^{12}\cos(u)6-5m^{12}\cos(u)^8\\
&+7m^{12}\cos(u)^{10}+4\cos(u)^{10}m^{10}-2m^{12}\cos(u)-2)(-m^2-1+\cos(u)^2m^2)^2)
\end{aligned}
$$

We need to discovery when it is zero:

$$Ks(m,u)=0.$$

It is quiet difficult problem. We solve at first the equation

$$\mathbf{solve}(\{Ks=0\},\{m\}); \tag{1}$$

We see only real solutions (up to the sign of the parameter) of this equation are $m = 0$and the function

$$M := \frac{1}{2}\frac{\sqrt{-2+2\sin(u)^2+2\sqrt{\sin(u)^4+10\sin(u)^2-7}}}{\sin(u)} \tag{2}$$

The graphic of this function is shown in Figure 2.

It shows the parameter $m \in (0,1)$ and there is some boundary for parameter u. To find an expression for this parameter we solve the equation

$$\mathbf{solve}(\{Ks=0\},\{u\}); \tag{3}$$

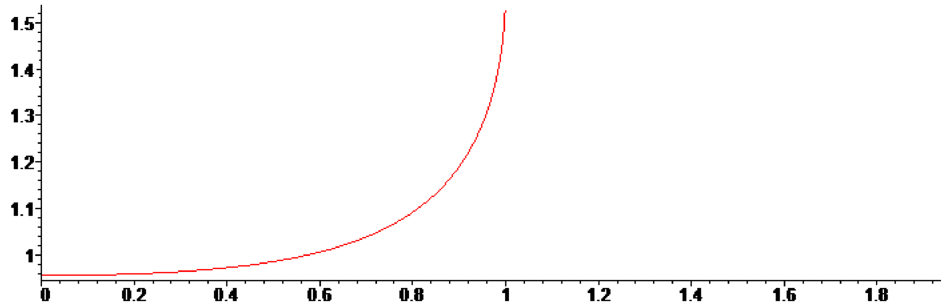

Figure 3

One of the solutions is:

$$U := \arctan\left(-\frac{\sqrt{-(2m^2-2)\left(\sqrt{-7m^4+2m^2+9}+m^2-3\right)}}{m(m^2-1)}\right.$$
$$\left.\frac{\sqrt{2}\sqrt{\dfrac{-m^2+2m^4-3+\sqrt{-7m^4+2m^2+9}}{m^2-1}}}{m}\right).$$

The graphic of this function is shown in Figure 3.

It shows the same restrictions. The following relations are checked up:

simplify(eval(Ks,[u=U]))=0;simplify(eval(Ks,[m=M]))=0;

So we can formulate our results.

Theorem 1. *In the region (m, u) determined by the relation (4) (or (3)) the Gaussian curvatures of the Benz surfaces induced from the unite sphere is zero.*

As a consequence we have (up to the sign of the parameter):

Theorem 2. *The investigated Benz surfaces are rounded down exactly in the case $m \in (0, 1)$.*

For the whole proof of the second theorem it is to take in view that for fixed $m \in (0, 1)$ the Gaussian curvature changes the sign along the corresponding meridian in the point where the curvature is zero.

Now we show some pictures – see Figure 4. The first picture is in the case $m = 0.5, u \approx 0.980586203$ -the corresponding value u is calculated by (4). The second picture is the boundary case$m = 1, u = \frac{\pi}{2}$.

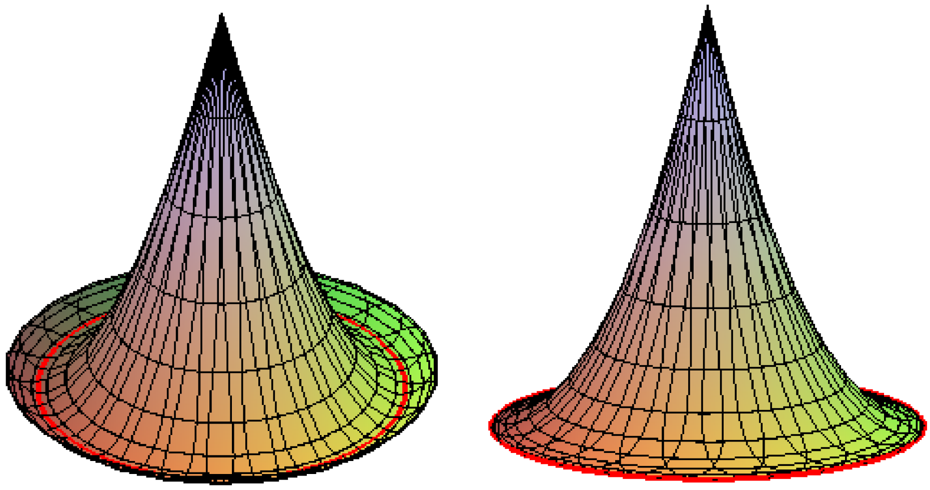

Figure 4

References

[1] G. Stanilov, Benz surfaces treated by Maple, *Journal of Geometry*, Admitted for Publication.