International Electronic Journal of Pure and Applied Mathematics

Volume 4 No. 2 2012, 119-136

RANDOM FRACTIONAL DIFFERENTIAL EQUATIONS

Vasile Lupulescu[1], Sotiris K. Ntouyas[2 §]

[1]Constantin Brancusi University
Targu-Jiu, ROMANIA
[2]Department of Mathematics
University of Ioannina
451 10, Ioannina, GREECE

Abstract: In this paper we study the existence and uniqueness of solutions for initial value problems for random fractional differential equations.

AMS Subject Classification: 6A33, 47H40, 60H25
Key Words: random fractional differential equations, existence, uniqueness

1. Introduction

Differential equations are a powerful tool for modeling and studying several real-world phenomena. In formulating a differential equation, the values of the coefficients, parameters and initial conditions are usually represented by the mean of the values obtained as a result of some experimental determinations. Hence, the physical constants and parameters may be considered to be random variables whose values are determined by some probability distribution or law. The same thing can be said about coefficients and nonhomogeneous terms (or forcing functions) of equations, which may be random variables or random functions. For significant results and further references about the differential and integral equations with random parameters we refer to the monographs and papers [2], [5], [9], [12], [13], [14], [15].

In this paper we prove some existence and uniqueness results for initial value problems for fractional differential equations with random parameters (or random fractional differential equations) of the following form

$$\begin{cases} D^{\alpha}x(t,\omega) = f(t,x(t,\omega),\omega), \\ x(0,\omega) = x_0(\omega), \end{cases} \tag{1.1}$$

where x is a random function, x_0 is a random variable, $\omega \in \Omega$ (a probability space),

Received: March 23, 2012

§Correspondence author

International Electronic Journal of Pure and Applied Mathematics – IEJPAM, Volume 4, No. 2 (2012)

$D^\alpha x$ is some sort of fractional derivative (sample path, mean square, etc.) of x with respect to the variable $t \in [0,T]$ with $T > 0$, and $f : [0,\infty) \times \mathbb{R} \times \Omega \to \mathbb{R}$ is a given function.

One basic tool in the study of the initial value problems for random fractional differential equations (1.1) is to treat it as a fractional differential equation in some appropriate Banach space $\mathcal{X}$. Formally, this may be done by writing (1.1) simply as

$$\begin{cases} \mathcal{D}^\alpha X(t) = F(t, X(t)), \\ X(0) = X_0. \end{cases} \tag{1.2}$$

In other words, we identify the real valued function $(t,\omega) \mapsto x(t,\omega)$ on $[0,T] \times \Omega$ with $\mathcal{X}$-valued function $t \mapsto x(t,\cdot)$ on $[0,T]$, where $\mathcal{X}$ may be, at least formally, an arbitrary Banach space of real functions on Ω, and $F : [0,T] \times \mathcal{X} \to \mathcal{X}$ is defined by $F(t,X)(\cdot) = f(t, x(t,\cdot), \cdot)$. In general, the problems (1.1) and (1.2) need not be equivalent. First, problem (1.1) may have a solution x which may not be regarded as a vector function with values in $\mathcal{X}$ (for instance, because the solution $x(t,\cdot)$ does not belong to $\mathcal{X}$ for some $t \in [0,T]$). Second, the solutions of (1.2) need not be solutions of (1.1) (for instance, because the derivatives $D^\alpha X$ in (1.1) and $\mathcal{D}^\alpha X$ in (1.2) are not meant in the same sense).

2. Preliminaries

Let $(\Omega, \mathcal{A}, P)$ be a complete probability space and let $([0,T], \mathcal{L}, \lambda)$ be a Lebesgue-measure space where $T > 0$. A function $x : \Omega \to \mathbb{R}$ is called a random variable if $\{\omega \in \Omega; x(\Omega) < a\} \in \mathcal{A}$ for all $a \in \mathbb{R}$. Let $1 \leq p < \infty$. A random variable $x : \Omega \to \mathbb{R}$ is said to be p-integrable if $\int_\Omega |x(\omega)|^p \, dP(\omega) < \infty$. Let $\mathcal{L}^p(\Omega)$ be the space of all p-integrable random variables. Then $\mathcal{L}^p(\Omega)$ is a vector space and the function $x \mapsto \|x\|_{\mathcal{L}^p(\Omega)}$ defined by

$$\|x\|_{\mathcal{L}^p(\Omega)} = \left(\int_\Omega |x(\omega)|^p \, dP(\omega) \right)^{1/p}$$

is a seminorm on $\mathcal{L}^p(\Omega)$. Let $L^p(\Omega)$ be the space of all equivalence classes of random variables that are p-integrable. If $x \in \mathcal{L}^p(\Omega)$ we denote by $\widehat{x}$ its equivalence class; that is, $y \in \widehat{x}$ if and only if $y(\omega) = x(\omega)$ for P-a.e. $\omega \in \Omega$. Moreover, we have that $\|y\|_{\mathcal{L}^p(\Omega)} = \|x\|_{\mathcal{L}^p(\Omega)}$. Thus we can define a norm $\|\cdot\|_{L^p(\Omega)}$ on $L^p(\Omega)$ by means of the formula $\|\widehat{x}\|_{L^p(\Omega)} = \|x\|_{\mathcal{L}^p(\Omega)}$. Then $L^p(\Omega)$ is a separable Banach space with respect to the norm $\|\cdot\|_{L^p(\Omega)}$.

Since, for $1 \leq p < \infty$, $L^p(\Omega)$ is a separable Banach space, then all elementary properties of the calculus (as continuity, differentiability, and integrability) for abstract functions defined on a real interval with values into a separable Banach space remain also true for the functions defined on a real interval with values into $L^p(\Omega)$.

International Electronic Journal of Pure and Applied Mathematics – IEJPAM, Volume 4, No. 2 (2012)

Thereby, if $X : [0,T] \to L^p(\Omega)$ is a strongly measurable (we also say L^p-measurable) then the function $t \mapsto \|X(t)\|_{L^p(\Omega)}$ is Lebesgue measurable on $[0,T]$. Also, a L^p-measurable function $X : [0,T] \to L^p(\Omega)$ is Bochner integrable (we also say L^p-integrable) on $[0,T]$ if and only if the function $t \mapsto \|X(t)\|_{L^p(\Omega)}$ is Lebesgue integrable on $[0,T]$.

If $X : [0,T] \to L^p(\Omega)$ is a measurable function then for each fixed $t \in [0,T]$, $X(t) \in L^p(\Omega)$ is an equivalence class. If for each $t \in [0,T]$ we select a particular function $x(t,\cdot) \in X(t)$ then we obtain a function $x(\cdot,\cdot) : [0,T] \times \Omega \to \mathbb{R}$ such that $\widehat{x}(t,\cdot) = X(t)$ for each $t \in [0,T]$. This resulting function is called *a representation* of X. More precisely we have the following result.

Lemma 2.1. *(a)* (see [4, Lemma III.11.17]) *Let $X : [0,T] \to L^p(\Omega)$ be Bochner integrable on $[0,T]$. Then there exists a $\lambda \times P$-measurable function $x(\cdot,\cdot) : [0,T] \times \Omega \to \mathbb{R}$, which is uniquely determined except for a set of $\lambda \times P$-measure zero, such that $\widehat{x}(t,\cdot) = X(t)$ for λ-a.e. $t \in [0,T]$. Moreover, $t \mapsto x(t,\omega)$ is Lebesgue integrable function on $[0,T]$ for a.e. P-a.e. $\omega \in \Omega$ and the integral $\int_0^T x(s,\omega)ds$, as a function of ω, is equal to the element $\int_0^T X(s)ds$ of $L^p(\Omega)$, that is,*

$$\int_0^T x(s,\cdot)ds = \left(\int_0^T X(s)ds\right)(\cdot). \tag{2.1}$$

(b) (see [4, Lemma III.11.16]) *Let $x(\cdot,\cdot) : [0,T] \times \Omega \to \mathbb{R}$ be a $\lambda \times P$-measurable function such that $x(t,\cdot) \in \mathcal{L}^p(\Omega)$ for λ-a.e. $t \in [0,T]$. Then the function $X : [0,T] \to L^p(\Omega)$, defined by $X(t) = \widehat{x}(t,\cdot)$, is L^p-measurable on $[0,T]$.*

The sample path fractional integral. Let $x(\cdot,\cdot) : [0,T] \times \Omega \to \mathbb{R}$ be a $\lambda \times P$-measurable function. We say that $x(\cdot,\cdot)$ *is sample path Lebesgue integrable on* $[0,T]$ if $x(\cdot,\omega) : [0,T] \to \mathbb{R}$ is Legesgue integrable on $[0,T]$ for each $\omega \in \Omega$.

Let $\alpha > 0$. If $x(\cdot,\cdot) : [0,T] \times \Omega \to \mathbb{R}$ is sample path Lebesgue integrable on $[0,T]$, then we can consider the fractional integral

$$I^\alpha x(t,\omega) = \int_0^t g_\alpha(t-s)x(s,\omega)ds, \tag{2.2}$$

which will be called *the sample path fractional integral of* x. The function $g_\alpha : \mathbb{R} \to [0,\infty)$ is defined by

$$g_\alpha(s) = \begin{cases} \dfrac{s^{\alpha-1}}{\Gamma(\alpha)} & \text{if } s > 0, \\ 0 & \text{if } s \le 0, \end{cases}$$

where Γ is the Euler's Gamma function.

Remark 2.2. In fact, $I^\alpha x(t,\omega) = (g_\alpha * x)(t,\omega)$ for each $\omega \in \Omega$, where $g_\alpha * x$ denote the convolution product (see [1]). By the properties of convolution product (see [1]), it follows that, if $x(\cdot,\omega) : [0,T] \to \mathbb{R}$ is Lebesgue integrable on $[0,T]$ for each $\omega \in \Omega$, then $t \mapsto I^\alpha x(t,\omega)$ is also Lebesgue integrable on $[0,T]$ for each $\omega \in \Omega$.

International Electronic Journal of Pure and Applied Mathematics – IEJPAM, Volume 4, No. 2 (2012)

A function $x(\cdot,\cdot) : [0,T] \times \Omega \to \mathbb{R}$ is said to be a *Carathéodory function* if $t \mapsto x(t,\omega)$ is continuous for each $\omega \in \Omega$, and $\omega \mapsto x(t,\omega)$ is P-measurable for each $t \in [0,T]$. We recall that a Carathéodory function is a $\lambda \times P$-measurable function (see [7]).

Proposition 2.3. *If $x(\cdot,\cdot) : [0,T] \times \Omega \to \mathbb{R}$ is a Carathéodory function, then the function $(t,\omega) \mapsto I^\alpha x(t,\omega)$ is also a Carathéodory function.*

Proof. Since $t \mapsto x(t,\omega)$ is continuous for each $\omega \in \Omega$, then by the properties of convolution product (see [1]), the function $t \mapsto \int_0^t g_\alpha(t-s)x(s,\omega)ds$ is continuous for each $\omega \in \Omega$. Obviously, $\omega \mapsto \int_0^t g_\alpha(t-s)x(s,\omega)ds$ is measurable for each $t \in [0,T]$. It follows that $(t,\omega) \mapsto I^\alpha x(t,\omega)$ is a Carathéodory function. □

The sample path fractional derivative. A function $x(\cdot,\cdot) : [0,T] \times \Omega \to \mathbb{R}$ is said to have *a sample path derivative at* $t \in (0,T)$ if the function $t \mapsto x(t,\omega)$ has a derivative at t for each $\omega \in \Omega$. We will denote by $\frac{d}{dt}x(t,\omega)$ or by $x'(t,\omega)$ the sample path derivative of $x(\cdot,\omega)$ at t. We say that $x(\cdot,\cdot) : [0,T] \times \Omega \to \mathbb{R}$ is sample path differentiable on $[0,T]$ if $x(\cdot,\cdot)$ has a sample path derivative for each $t \in (0,T)$ and possesses a one-sided sample path derivative at the end points 0 and T.

If $x(\cdot,\cdot) : [0,T] \times \Omega \to \mathbb{R}$ is sample path absolutely continuous on $[0,T]$ (that is, $t \mapsto x(t,\omega)$ is absolutely continuous on $[0,T]$ for each $\omega \in \Omega$), then the sample path derivative $x'(t,\omega)$ exists for λ-a.e. $t \in [0,T]$ and

$$x(t,\omega) = x(0,\omega) + \int_0^t x'(s,\omega)ds, \quad t \in [0,T], \tag{2.3}$$

for each $\omega \in \Omega$.

Let $x(\cdot,\cdot) : [0,T] \times \Omega \to \mathbb{R}$ be a $\lambda \times P$-measurable function such that $x(\cdot,\omega) : [0,T] \to \mathbb{R}$ is Lebesgue integrable on $[0,T]$ for each $\omega \in \Omega$, and let $\alpha \in (0,1]$. For each $\omega \in \Omega$ we define the sample path fractional derivative of x by

$$D^\alpha x(t,\omega) = I^{1-\alpha}x'(t,\omega) = \int_0^t g_{1-\alpha}(t-s)x'(s,\omega)ds. \tag{2.4}$$

Obviously, the sample path fractional derivative $D^\alpha x(t,\omega)$ exists for λ-a.e. $t \in [0,T]$ if $x(\cdot,\cdot) : [0,T] \times \Omega \to \mathbb{R}$ is sample path absolutely continuous on $[0,T]$. In particular, if $x(\cdot,\cdot) : [0,T] \times \Omega \to \mathbb{R}$ is sample path differentiable on $[0,T]$ and $t \mapsto x(t,\omega)$ is continuous on $[0,T]$, then $D^\alpha x(t,\omega)$ exists for every $t \in [0,T]$ and $t \mapsto D^\alpha x(t,\omega)$ is continuous on $[0,T]$.

Remark 2.4. The following properties are well known (see [8], [11]). If $x(\cdot,\cdot) : [0,T] \times \Omega \to \mathbb{R}$ is a Carathéodory function on $[0,T]$ then (see [8, Lemma 2.21])

$$D^\alpha I^\alpha x(t,\omega) = x(t,\omega) \tag{2.5}$$

for all $t \in [0,T]$ and each $\omega \in \Omega$. If $x(\cdot,\cdot) : [0,T] \times \Omega \to \mathbb{R}$ is sample path absolutely continuous on $[0,T]$ then (see [8, Lemma 2.22])

$$I^\alpha D^\alpha x(t,\omega) = x(t,\omega) - x(0,\omega) \tag{2.6}$$

for λ-a.e. $t \in [0,T]$ and each $\omega \in \Omega$. In particular, if $x(\cdot,\cdot) : [0,T] \times \Omega \to \mathbb{R}$ is sample path differentiable on $[0,T]$ and $t \mapsto x(t,\omega)$ is continuous on $[0,T]$, then (2.6) holds for all $t \in [0,T]$ and each $\omega \in \Omega$.

The L^p-fractional integral. Let $X : [0,T] \to L^p(\Omega)$ be Bochner integrable on $[0,T]$, and let $\alpha > 0$. Then the *L^p-fractional integral of X* is defined by

$$\mathcal{I}^\alpha X(t) = \int_0^t g_\alpha(t-s)X(s)ds. \tag{2.7}$$

The L^p-fractional integral of X is well defined and it exists for λ-a.e. $t \in [0,T]$ (see [10]).

Proposition 2.5. *Let $X : [0,T] \to L^p(\Omega)$ be Bochner integrable on $[0,T]$. Then there exists a $\lambda \times P$-measurable function $x(\cdot,\cdot) : [0,T] \times \Omega \to \mathbb{R}$, which is uniquely determined except for a set of $\lambda \times P$-measure zero, and such that $\widehat{x}(t,\cdot) = X(t)$ for λ-a.e. $t \in [0,T]$. Moreover, the sample path fractional integral $y(t,\omega) = \int_0^t g_\alpha(t-s)x(s,\omega)ds$ exists for P-a.e. $\omega \in \Omega$ and $\widehat{y}(t,\cdot) = \mathcal{I}^\alpha X(t)$; that is,*

$$\left[\mathcal{I}^\alpha X(t)\right](\omega) = \int_0^t g_\alpha(t-s)x(s,\omega)ds \quad \text{for } P-\text{a.e. } \omega \in \Omega. \tag{2.8}$$

Proof. If $X : [0,T] \to L^p(\Omega)$ is Bochner integrable on $[0,T]$, then using Lemma 2.1 it follows that there exists a $\lambda \times P$-measurable function $x(\cdot,\cdot) : [0,T] \times \Omega \to \mathbb{R}$, which is uniquely determined except for a set of $\lambda \times P$-measure zero, such that $\widehat{x}(s,\cdot) = X(s)$ for λ-a.e. $s \in [0,T]$, and $x(\cdot,\omega)$ is Lebesgue integrable on $[0,T]$ for P-a.e. $\omega \in \Omega$. Since the function $s \mapsto x(s,\omega)$ is Lebesgue integrable on $[0,T]$ for P-a.e. $\omega \in \Omega$, then for $s < t$, the function $s \mapsto g_\alpha(t-s)x(s,\omega)$ is also Lebesgue integrable on $[0,T]$ for P-a.e. $\omega \in \Omega$. It follows that the sample fractional integral $y(t,\omega) = \int_0^t g_\alpha(t-s)x(s,\omega)ds$ exists for P-a.e. $\omega \in \Omega$. Further, $x(s,\cdot) \in \widehat{x}(s,\cdot)$ for λ-a.e. $s \in [0,T]$ implies that $\widehat{y}(t,\cdot) = \int_0^t g_\alpha(t-s)\widehat{x}(s,\cdot)ds$, and so we obtain (2.8). □

Remark 2.6. If $X : [0,T] \to L^p(\Omega)$ is Bochner integrable on $[0,T]$ and $\alpha, \beta > 0$, then the L^p-fractional integral has the property

$$I^\alpha I^\beta X(t) = I^{\alpha+\beta} X(t) \text{ for } \quad \lambda - \text{a.e. } \; t \in [0,T]. \tag{2.9}$$

Indeed, we have that

$$I^\alpha I^\beta X(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \int_0^s \frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)} X(\tau)d\tau ds,$$

International Electronic Journal of Pure and Applied Mathematics – IEJPAM, Volume 4, No. 2 (2012)

from which by interchanging the order of integration by Fubini's theorem for Bochner integral [4, Theorem III.11.9] and setting $s=\tau+r(t-\tau)$, we have

$$\begin{aligned} I^{\alpha}I^{\beta}X(t) &= \int_0^t\int_\tau^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)}X(\tau)dsd\tau \\ &= \int_0^t \frac{(t-\tau)^{\alpha+\beta-1}}{\Gamma(\alpha)\Gamma(\beta)}d\tau\int_0^1 r^{\beta-1}(1-r)^{\alpha-1}X(\tau)dr \\ &= \int_0^t \frac{(t-\tau)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}X(\tau)d\tau = I^{\alpha+\beta}X(t). \end{aligned}$$

The L^p-fractional derivative. A function $X:[0,T]\to L^p(\Omega)$ is called *L^p-continuous at $t\in(0,T)$ if*

$$\lim_{h\to 0}\|X(t+h)-X(t)\|_{L^p(\Omega)}=0.$$

If X is L^p-continuous for each $t\in(0,T)$ and if it is one-sided continuous at the end points 0 and T, then X is said to be L^p-continuous on $[0,T]$. A function $X:[0,T]\to L^p(\Omega)$ is said to have a *L^p-derivative (or a strong derivative)* $X'(t)$ at $t\in(0,T)$ if

$$\lim_{h\to 0}\left\|\frac{X(t+h)-X(t)}{h}-X'(t)\right\|_{L^p(\Omega)}=0.$$

If X has a L^p-derivative for each $t\in(0,T)$ and possesses a one-sided L^p-derivative at the end points 0 and T, then X is said to be L^p-differentiable on $[0,T]$.

Remark 2.7. Let $x(\cdot,\cdot):[0,T]\times\Omega\to\mathbb{R}$ be a function such that $x'(t,\omega)$ exists for P-a.e. $\omega\in\Omega$, and $x(t,\cdot)\in\mathcal{L}^p(\Omega)$ for each $t\in[0,T]$. Then, in general, $\widehat{x}'(t,\cdot)$ is to be distinguished from $\widehat{x'}(t,\cdot)$. However, when $\widehat{x}'(t,\cdot)$ exists and $x'(t,\omega)$ exists for P-a.e. $\omega\in\Omega$, then $x'(t,\cdot)\in\mathcal{L}^p(\Omega)$ and $\widehat{x}'(t,\cdot)=\widehat{x'}(t,\cdot)$ (see [13]).

A function $X:[0,T]\to L^p(\Omega)$ is called *L^p-absolutely continuous on* $[0,T]$, if for any $\varepsilon>0$ there exists a $\delta>0$ such that for any finite disjoint family $\{[a_k,b_k];1\le k\le n\}$ of subintervals $[a_k,b_k]$ of $[0,T]$ with $\sum_{k=1}^n(b_k-a_k)<\delta$ we have

$$\sum_{k=1}^n\|X(b_k)-X(a_k)\|_{L^p(\Omega)}<\varepsilon.$$

Remark 2.8. Let $Y:[0,T]\to L^p(\Omega)$ be a Bochner integrable function on $[0,T]$. Then it is well known that

$$X(t)=X(0)+\int_0^t Y(s)ds \quad\text{for all}\quad t\in[0,T]$$

if and only if $X:[0,T]\to L^p(\Omega)$ is L^p-absolutely continuous on $[0,T]$ and a.e. L^p-differentiable on $[0,T]$ with L^p-derivative Y. In fact, if $X:[0,T]\to L^p(\Omega)$ is L^p-absolutely continuous on $[0,T]$ and a.e. L^p-differentiable on $[0,T]$ with L^p-derivative Y, then we have that

$$\mathcal{I}^1Y(t)=X(t)-X(0).$$

Let $X : [0,T] \to L^p(\Omega)$ be a L^p-absolutely continuous function on $[0,T]$. We define the *L^p-fractional derivative of X of order $\alpha \in (0,1]$ by*

$$\mathcal{D}^\alpha X(t) = \mathcal{I}^{1-\alpha} X'(t) = \int_0^t g_{1-\alpha}(t-s) X'(s) ds,$$

where X' the is L^p-derivative of X.

Proposition 2.9. *Let $X : [0,T] \to L^p(\Omega)$ be a L^p-continuously differentiable function on $[0,T]$. Then there exists a $\lambda \times P$-measurable function $x(\cdot,\cdot) : [0,T] \times \Omega \to \mathbb{R}$ such that $s \mapsto x(s,\omega)$ is absolutely continuous for each $\omega \in \Omega$ and $\widehat{x}(s,\cdot) = X(s)$ for each $s \in [0,T]$. Moreover, the sample path fractional derivative $y(s,\omega) = \int_0^t g_\alpha(t-s)x'(s,\omega)ds$ exists for P-a.e. $\omega \in \Omega$ and $\widehat{y}(t,\cdot) = \mathcal{D}^\alpha X(t)$; that is,*

$$[\mathcal{D}^\alpha X(t)](\omega) = \int_0^t g_\alpha(t-s) x'(s,\omega) ds \quad \textit{for} \quad P\textit{-a.e. } \omega \in \Omega. \tag{2.10}$$

Proof. If $X : [0,T] \to L^p(\Omega)$ is a L^p-continuously differentiable function on $[0,T]$, then using Theorem 3.4.2 from [6] it follows that there exists a $\lambda \times P$-measurable function $x(\cdot,\cdot) : [0,T] \times \Omega \to \mathbb{R}$ such that $s \mapsto x(s,\omega)$ is absolutely continuous for each $\omega \in \Omega$, $\widehat{x}(s,\cdot) = X(s)$ for each $s \in [0,T]$ and $\widehat{x}'(s,\cdot) = X'(s)$ for λ-a.e. $s \in [0,T]$. Since the function $s \mapsto x'(s,\omega)$ is Lebesgue integrable on $[0,T]$ for P-a.e. $\omega \in \Omega$, then for $s < t$, the function $s \mapsto g_\alpha(t-s)x'(s,\omega)$ is also Lebesgue integrable on $[0,T]$ for P-a.e. $\omega \in \Omega$. It follows that the sample fractional derivative $y(t,\omega) = \int_0^t g_\alpha(t-s)x'(s,\omega)ds$ exists for P-a.e. $\omega \in \Omega$. Further, $x'(s,\cdot) \in \widehat{x}'(s,\cdot)$ for λ-a.e. $s \in [0,T]$ implies that $\widehat{y}(t,\cdot) = \int_0^t g_\alpha(t-s)\widehat{x}'(s,\cdot)ds$, and so we obtain (2.10). ☐

Proposition 2.10. *Let $X : [0,T] \to L^p(\Omega)$ be a L^p-absolutely continuous function on $[0,T]$ and a.e. L^p-differentiable on $[0,T]$. If $\alpha \in (0,1]$ then*

(a) $\mathcal{I}^\alpha \mathcal{D}^\alpha X(t) = X(t) - X(0)$ for λ-a.e. $t \in [0,T]$,

(b) $\mathcal{D}^\alpha \mathcal{I}^\alpha X(t) = X(t)$ for λ-a.e. $t \in [0,T]$.

In particular, if X is L^p-differentiable on $[0,T]$ then (a) and (b) hold for all $t \in [0,T]$.

Proof. First, using Remarks 2.6 and 2.8, we observe that

$$\begin{aligned} \mathcal{I}^{\alpha+1} X'(t) &= \mathcal{I}^\alpha \mathcal{I}^1 X'(t) = \mathcal{I}^\alpha [X(t) - X(0)] \\ &= \mathcal{I}^\alpha X(t) - \frac{t^\alpha}{\Gamma(\alpha+1)} X(0), \end{aligned}$$

from which we obtain

$$\mathcal{I}^{\alpha}X(t) = \frac{t^{\alpha}}{\Gamma(\alpha+1)}X(0) + \mathcal{I}^{\alpha+1}X'(t).$$

Then it follows that

$$\mathcal{I}^{\alpha}\mathcal{D}^{\alpha}X(t) = \mathcal{I}^{\alpha}\ \mathcal{I}^{1-\alpha}X'(t) = \mathcal{I}^{1}X'(t) = X(t) - X(0).$$

Also, we have that

$$\begin{aligned}\mathcal{D}^{\alpha}\mathcal{I}^{\alpha}X(t) =& \mathcal{I}^{1-\alpha}\frac{d}{dt}\mathcal{I}^{\alpha}X(t) = \\ =& \mathcal{I}^{1-\alpha}\frac{d}{dt}\left[\frac{t^{\alpha}}{\Gamma(\alpha+1)}X(0) + \mathcal{I}^{\alpha+1}X'(t)\right] \\ =& \mathcal{I}^{1-\alpha}\left[\frac{t^{\alpha-1}}{\Gamma(\alpha)}X(0) + \mathcal{I}^{\alpha}X'(t)\right] \\ =& X(0) + \mathcal{I}^{1}X'(t) = X(t).\end{aligned}$$

□

3. Sample and L^p-Solutions

Le us consider the following fractional initial value problem (L^p-problem, for short)

$$\begin{cases} \mathcal{D}^{\alpha}X(t) = F(t, X(t)), \\ X(0) = X_0, \end{cases} \tag{3.1}$$

where $\alpha \in (0,1]$ and $F : [0,T] \times L^p(\Omega) \to L^p(\Omega)$. The domain of F is permitted to vary with t. When we write $F : [0,T] \times D^p(t) \to L^p(\Omega)$ we mean that the domain of F is the set $\{(t,X); t \in [0,T],\ X \in D^p(t)\}$, where $D^p(\cdot)$ maps $[0,T]$ into subsets of $L^p(\Omega)$.

Definition 3.1. Let $F : [0,T] \times D^p(t) \to L^p(\Omega)$ and $X_0 \in D^p(0)$ be given. A function $X : [0,T] \to L^p(\Omega)$ is said to be a L^p-solution for (3.1) if and only if:

(a) $X(t) \in D^p(t)$ for all $t \in [0,T]$;

(b) X is L^p-continuous on $[0,T]$ and a.e. L^p-differentiable on $[0,T]$;

(c) X satisfies (3.1) for a.e. $t \in [0,T]$.

Remark 3.2. If $t \mapsto F(t, X(t))$ is Bochner integrable on $[0,T]$, then we observe that $X : [0,T] \to L^p(\Omega)$ is a solution for (3.1) if and only if

$$X(t) = X_0 + \mathcal{I}^{\alpha}F(t, X(t));$$

that is,

$$X(t) = X_0 + \int_0^t g_{\alpha}(t-s)F(s, X(s))ds. \tag{3.2}$$

Now consider the sample fractional initial vale problem (S-problem, for short)

$$\begin{cases} D^\alpha x(t,\omega) = f(t, x(t,\omega), \omega), \\ x(0,\omega) = x_0(\omega), \end{cases} \tag{3.3}$$

where $f : [0,T] \times \mathbb{R} \times \Omega \to \mathbb{R}$ and $x_0 : \Omega \to \mathbb{R}$ is given random variable.

The S-problem (3.3) can be formulated as a L^p-problem (3.1). To do this, we simply set $F(t, X(t)) = \widehat{f}(t, x(t,\cdot), \cdot)$, $X_0 = \widehat{x}_0$, and consider $F(t, X(t))$ as an element of $L^p(\Omega)$.

Definition 3.3. A $\lambda \times P$-measurable function $x : [0,T] \times \Omega \to \mathbb{R}$ is said to be a sample solution for S-problem (3.3) if and only if:

(a) $x(t,\omega)$ is continuous on $[0,T]$ for P-a.e. $\omega \in \Omega$;

(b) $x(t,\omega)$ satisfies (3.3) for a.e. $t \in [0,T]$ and P-a.e. $\omega \in \Omega$.

Remark 3.4. If $(t,\omega) \mapsto f(t, x(t,\omega), \omega)$ is $\lambda \times P$-measurable and

$$t \mapsto f(t, x(t,\omega), \omega)$$

is Lebesgue integrable on $[0,T]$ for P-a.e. $\omega \in \Omega$, then we observe that $x : [0,T] \times \Omega \to \mathbb{R}$ is a solution for (3.3) if and only if

$$x(t,\omega) = x_0(\omega) + \int_0^t g_\alpha(t-s) f(s, x(s,\omega), \omega) ds \tag{3.4}$$

for P-a.e. $\omega \in \Omega$.

The following result establishes the relationship between the sample and L^p-problems.

Theorem 3.5. *(a) If $X : [0,T] \to L^p(\Omega)$ is a L^p-solution for (3.1) then there exists a sample solution $x : [0,T] \times \Omega \to \mathbb{R}$ for (3.3) such that $X(t) = \widehat{x}(t,\cdot)$ for a.e. $t \in [0,T]$.*

(b) If $x : [0,T] \times \Omega \to \mathbb{R}$ is a sample solution for (3.3) and $(t,\omega) \mapsto f(t, x(t,\omega), \omega)$ is $\lambda \times P$-measurable, then the equivalence class $X(t) = \widehat{x}(t,\cdot)$ is a L^p-solution for (3.1) if and only if

$$\int_0^t \|F(s, X(s))\|_{L^p(\Omega)} \, ds < \infty \quad \text{for} \quad t \in [0,T], \tag{3.5}$$

and $x_0 \in \mathcal{L}^p(\Omega)$.

Proof. (a) Using Lemma 2.1 it follows that there exists a $\lambda \times P$-measurable function $x(\cdot,\cdot) : [0,T] \times \Omega \to \mathbb{R}$, which is uniquely determined except for a set of $\lambda \times P$-measure zero, such that $\widehat{x}(t,\cdot) = X(t)$ for λ-a.e. $t \in [0,T]$, and $x(\cdot,\omega)$

is Lebesgue integrable on $[0,T]$ for P-a.e. $\omega \in \Omega$. Since $X : [0,T] \to L^p(\Omega)$ is a L^p-solution for (3.1) then it satisfies the integral equation (3.2). Therefore, by Proposition 2.5, it follows that

$$\begin{aligned}\widehat{x}(t,\cdot) &= X(t) = X_0 + \int_0^t g_\alpha(t-s)F(s,X(s))ds \\ &= \widehat{x}_0(\cdot) + \int_0^t g_\alpha(t-s)\widehat{f}(s,x(s,\cdot),\cdot)ds,\end{aligned}$$

which implies that

$$x(t,\omega) = x_0(\omega) + \int_0^t g_\alpha(t-s)f(s,x(s,\omega),\omega)ds$$

for P-a.e. $\omega \in \Omega$. Consequently, $x(\cdot,\cdot)$ is a solution for (3.3).

(b) Let $x(\cdot,\cdot) : [0,T] \times \Omega \to \mathbb{R}$ be a solution for (3.3) and $x_0 \in \mathcal{L}^p(\Omega)$. First, we observe that the function $(t,\omega) \mapsto f(t,x(t,\omega),\omega)$ is $\lambda \times P$-measurable and $F(t,X(t)) = \widehat{f}(t,x(t,\cdot),\cdot) \in L^p(\Omega)$ for λ-a.e. $t \in [0,T]$. Then by Lemma 2.1 it follows that the function $t \mapsto F(t,X(t))$ is L^p-measurable. This, together with (3.5), implies that $t \mapsto F(t,X(t))$ is Bochner integrable. Hence, by Proposition 2.5 we have

$$\int_0^t g_\alpha(t-s)F(s,X(s))ds = \int_0^t g_\alpha(t-s)\widehat{f}(s,x(s,\cdot),\cdot)ds$$

and thus

$$X_0 + \int_0^t g_\alpha(t-s)F(s,X(s))ds = \widehat{x}_0 + \int_0^t g_\alpha(t-s)\widehat{f}(s,x(s,\cdot),\cdot)ds.$$

Therefore, $X(t) = \widehat{x}(t,\cdot)$ is a L^p-solution for (3.1). The converse is obvious. □

4. Existence and Uniqueness

In the following, we consider the S-problem (3.3) and suppose that:

(H1) The function $(t,x) \mapsto f(t,x,\omega)$ is continuous for each $\omega \in \Omega$.

(H2) The function $\omega \mapsto f(t,x,\omega)$ is measurable for each $(t,x) \in [0,T] \times \mathbb{R}$.

(H3) There exists a Carathéodory function $k : [0,T] \times \Omega \to \mathbb{R}_+$ such that

$$|f(t,x,\omega) - f(t,y,\omega)| \le k(t,\omega)\,|x-y|$$

for every $t \in [0,T]$ and P-a.e. $\omega \in \Omega$.

International Electronic Journal of Pure and Applied Mathematics – IEJPAM, Volume 4, No. 2 (2012)

(H4) $|f(t, x_0(\omega), \omega)| \le M < \infty$ for every $t \in [0, T]$ and P-a.e. $\omega \in \Omega$, where M is a positive constant.

To prove the existence and uniqueness of the solution for the problem (3.3) we need the following result.

Lemma 4.1. *Let $y_n : [0, T] \times \Omega \to \mathbb{R}$ be a sequence of Carathéodory positive functions and let M, b be positive constants such that $y_0(t, \omega) \le M$ and for every $n \ge 1$,*

$$y_n(t, \omega) \le M + b \int_0^t g_\alpha(t - s) y_{n-1}(t, \omega) ds, \quad t \in [0, T], \quad \omega \in \Omega.$$

Then for every $n \ge 1$,

$$y_n(t, \omega) \le M E_\alpha(bT^\alpha), \quad \omega \in \Omega,$$

where $E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha + 1)}$ is the Mittag-Leffler function (see [8]).

Proof. Indeed, we have

$$\begin{aligned} y_1(t, \omega) &\le M + \frac{b}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha - 1} y_0(t, \omega) ds \\ &\le M + \frac{Mb}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha - 1} ds = M \left[1 + \frac{b}{\Gamma(\alpha + 1)} t^\alpha\right] \\ &\le M \left[1 + \frac{b}{\Gamma(\alpha + 1)} T^\alpha\right], \end{aligned}$$

and

$$\begin{aligned} y_2(t, \omega) &\le M + \frac{b}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha - 1} y_1(t, \omega) ds \\ &\le M + \frac{Mb}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha - 1} \left[1 + \frac{b)}{\Gamma(\alpha + 1)} s^\alpha\right] ds \\ &= M \left[1 + \frac{b}{\Gamma(\alpha + 1)} t^\alpha + \frac{b^2}{\Gamma(\alpha)\Gamma(\alpha + 1)} \int_0^t (t - s)^{\alpha - 1} s^\alpha ds\right] \\ &\le M \left[1 + \frac{b}{\Gamma(\alpha + 1)} t^\alpha + \frac{b^2}{\Gamma(2\alpha + 1)} t^{2\alpha}\right] \end{aligned}$$

$$\leq \quad M\left[1+\frac{b}{\Gamma(\alpha+1)}T^{\alpha}+\frac{b^2}{\Gamma(2\alpha+1)}T^{2\alpha}\right].$$

Thus, by mathematical induction, for every $n \geq 1$

$$y_n(t,\omega) \leq M\sum_{k=0}^{n}\frac{(bt^{\alpha})^k}{\Gamma(k\alpha+1)} \leq M\sum_{k=0}^{n}\frac{(bT^{\alpha})^k}{\Gamma(k\alpha+1)},$$

and therefore $y_n(t,\omega) \leq ME_{\alpha}(bT^{\alpha})$ for every $n \geq 1$. □

Remark 4.2. If $y_n : [0,T]\times\Omega \to \mathbb{R}$ are Carathéodory positive functions and M, b are positive constants such that $y_0(t,\omega) \leq M$ and

$$y_n(t,\omega) \leq b\int_0^t g_{\alpha}(t-s)y_{n-1}(t,\omega)ds, \quad t\in[0,T], \quad n=1,2,\dots,$$

for all $\omega\in\Omega$, then it easy to check by mathematical induction that for every $n \geq 1$

$$y_n(t,\omega) \leq \frac{M\,(bT^{\alpha})^n}{\Gamma(n\alpha+1)},\ t\in[0,T], \quad \omega\in\Omega.$$

Theorem 4.3. *Let $f : [0,T]\times\mathbb{R}\times\Omega \to \mathbb{R}$ be a given function, and let x_0 be a random variable. If (H1)-(H4) hold, then the S-problem (3.3) has a unique solution on $[0,T]$.*

Proof. To prove the theorem, we shall use the method of successive approximations. For this, we define a sequence of functions $x_n(\cdot,\cdot) : [0,T]\times\Omega \to \mathbb{R}$, $n\in\mathbb{N}$, as follows:

$$\begin{cases} x_0(t,\omega) = x_0(\omega), \\ x_n(t,\omega) = x_0(\omega) + \int_0^t g_{\alpha}(t-s)f(s,x_{n-1}(s,\omega),\omega)ds, \ \ n\geq 1, \end{cases} \tag{4.1}$$

for every $t\in[0,T]$ and every $\omega\in\Omega$. First, we observe that

$$\begin{aligned} |x_1(t,\omega)-x_0(t,\omega)| &\leq \int_0^t g_{\alpha}(t-s)\,|f(s,x_0(\omega),\omega)|\,ds \\ &\leq \frac{M}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}ds \leq \frac{Mt^{\alpha}}{\Gamma(\alpha+1)} \leq \frac{MT^{\alpha}}{\Gamma(\alpha+1)}, \end{aligned}$$

and for every $n \geq 2$

$$|x_n(t,\omega)-x_{n-1}(t,\omega)| \leq \int_0^t g_{\alpha}(t-s)\,|f(s,x_{n-1}(s,\omega),\omega)-f(s,x_{n-2}(s,\omega),\omega)|\,ds$$

$$\leq \quad K(\omega)\int_0^t g_\alpha(t-s)\left|x_{n-1}(s,\omega)-x_{n-2}(s,\omega)\right|ds,$$

where $K(\omega)=\sup_{t\in[0,T]} k(t,\omega)$. If we take $y_n(t,\omega)=|x_n(t,\omega)-x_{n-1}(t,\omega)|$ for $n\geq 1$ and $y_0(t,\omega)=|f(t,x_0(\omega),\omega)|\leq M$ then, by Remark 4.2, for $n\geq 1$ we obtain that

$$|x_n(t,\omega)-x_{n-1}(t,\omega)|\leq \frac{M\,[K(\omega)T^\alpha]^n}{\Gamma(n\alpha+1)},\quad t\in[0,T],\quad \omega\in\Omega. \tag{4.2}$$

Now, we show that the functions $x_n(\cdot,\omega):[0,T]\to\mathbb{R}$, $n\in\mathbb{N}$, are continuous on $[0,T]$ for every $\omega\in\Omega$. Obviously, $x_0(\cdot,\omega)$ is continuous on $[0,T]$ for every $\omega\in\Omega$. Next, we suppose that $x_k(\cdot,\omega)$, $0\leq k\leq n-1$, are continuous on $[0,T]$ for every $\omega\in\Omega$. Then for $t\in[0,T)$ and $h>0$ such that $t+h\in(0,T]$ we have

$$\begin{aligned}|x_n(t+h,\omega)-x_n(t,\omega)| \quad &\leq \quad \int_0^t |g_\alpha(t+h-s)-g_\alpha(t-s)|\,|f(s,x_{n-1}(s,\omega),\omega)|\,ds\\ &\quad + \int_t^{t+h} g_\alpha(t+h-s)\,|f(s,x_{n-1}(s,\omega),\omega)|\,ds. \end{aligned}\tag{4.3}$$

By (H3) and (H4) we have that

$$\begin{aligned}|f(t,x_{n-1}(t,\omega),\omega)| \quad &\leq \quad |f(t,x_{n-1}(t,\omega),\omega)-f(t,x_0(\omega),\omega)|+|f(t,x_0(\omega),\omega)|\\ &\leq \quad M+k(t,\omega)\,|x_{n-1}(t,\omega)-x_0(\omega)|\\ &\leq \quad M+K(\omega)\int_0^t g_\alpha(t-s)\,|f(s,x_{n-2}(s,\omega),\omega)|\,ds.\end{aligned}$$

If we put $y_n(t,\omega)=|f(t,x_{n-1}(t,\omega),\omega)|$, then for every $n\geq 1$

$$y_n(t,\omega)\leq M+K(\omega)\int_0^t g_\alpha(t-s)y_{n-1}(t,\omega)ds,$$

and, by Lemma 4.1, it follows that

$$y_n(t,\omega)\leq ME_\alpha\left(K(\omega)T^\alpha\right);$$

that is,

$$|f(t,x_{n-1}(t,\omega),\omega)|\leq ME_\alpha\left(K(\omega)T^\alpha\right),\quad \text{for every}\ \ n\geq 1.$$

Using the last estimation and (4.3), we have

$$|x_n(t+h,\omega)-x_n(t,\omega)|$$

$$\begin{aligned}&\le ME_\alpha\left(K(\omega)T^\alpha\right)\left[\int\limits_0^t |g_\alpha(t+h-s)-g_\alpha(t-s)|\,ds+\int\limits_t^{t+h} g_\alpha(t+h-s)ds\right]\\&\le \frac{ME_\alpha\left(K(\omega)T^\alpha\right)}{\Gamma(\alpha+1)}\left[2h^\alpha+t^\alpha-(t+h)^\alpha\right],\end{aligned}$$

and therefore, $x_n(t+h,\omega)-x_n(t,\omega)\to 0$ as $h\to 0^+$. Similarly for $t\in(0,T]$ and $h>0$ such that $t-h\in[0,T)$, we obtain that $x_n(t-h,\omega)-x_n(t,\omega)\to 0$ as $h\to 0^+$. Hence the functions $x_n(\cdot,\omega):[0,T]\to\mathbb{R}$, $n\in\mathbb{N}$, are continuous on $[0,T]$ for every $\omega\in\Omega$. Since $\omega\mapsto x_n(t,\omega)$ are obviously measurable, then by (H1), (H2) and Proposition 2.3 it follows that $x_n(\cdot,\cdot)$ are Carathéodory functions. In the sequel we shall show that for the sequence $x_n(t,\omega)$ the Cauchy convergence condition is satisfied uniformly on the variable t, and as a consequence $x_n(t,\omega)$ is uniformly convergent for all $\omega\in\Omega$. For $n>m\ge 0$ using (4.2) one obtains

$$\begin{aligned}\sup_{t\in[0,T]}|x_n(t,\omega)-x_m(t,\omega)| &\le \sum_{k=m+1}^{n}\sup_{t\in[0,T]}|x_k(t,\omega)-x_{k-1}(t,\omega)|\\&\le M\sum_{k=m+1}^{n}\frac{[K(\omega)T^\alpha]^k}{\Gamma(k\alpha+1)}.\end{aligned}$$

On the other hand, using the inequality (see [3])

$$\Gamma(k\alpha)\ge(k-1)!\alpha^{2k-2}\left[\Gamma(\alpha)\right]^k,\quad k\ge 1$$

it follows that

$$\frac{[K(\omega)T^\alpha]^k}{\Gamma(k\alpha+1)}\le\frac{\alpha\,[K(\omega)T^\alpha]^k}{k!\alpha^{2k}\left[\Gamma(\alpha)\right]^k}=\alpha\frac{1}{k!}\left[\frac{K(\omega)T^\alpha}{\alpha\Gamma(\alpha+1)}\right]^k,$$

and so

$$\sup_{t\in[0,T]}|x_n(t,\omega)-x_m(t,\omega)|\le\alpha M\sum_{k=m+1}^{n}\frac{1}{k!}\left[\frac{K(\omega)T^\alpha}{\alpha\Gamma(\alpha+1)}\right]^k.$$

The convergence of the series $\sum_{n=0}^{n}\frac{1}{n!}\left[\frac{K(\omega)T^\alpha}{\alpha\Gamma(\alpha+1)}\right]^n$ implies that for any $\varepsilon>0$ we can find $n_0\in\mathbb{N}$ such that for $n,m\ge n_0$

$$\sup_{t\in[0,T]}|x_n(t,\omega)-x_m(t,\omega)|\le\varepsilon. \tag{4.4}$$

Therefore, $x_n(t,\omega)$ is uniformly convergent on $[0,T]$ for all $\omega\in\Omega$. For $\omega\in\Omega$, let $x(t,\omega)=\lim_{n\to\infty}x_n(t,\omega)$, $t\in[0,T]$. We shall show that $x(\cdot,\cdot)$ is a solution of the integral equation (3.4). Since for $n,m\ge 0$

$$\sup_{t\in[0,T]}|f(t,x_n(t,\omega),\omega)-f(t,x_m(t,\omega),\omega)|\le K(\omega)\sup_{t\in[0,T]}|x_n(t,\omega)-x_m(t,\omega)|,$$

then it follows that the sequence $f(t, x_n(t,\omega), \omega)$ is uniformly convergent on $[0, T]$ for all $\omega \in \Omega$. Moreover, for any $\varepsilon > 0$ we can find $n_0 \in \mathbb{N}$ such that for $n \geq n_0$

$$\sup_{t\in[0,T]} |f(t, x_n(t,\omega), \omega) - f(t, x(t,\omega), \omega)| \leq K(\omega) \sup_{t\in[0,T]} |x_n(t,\omega) - x(t,\omega)| \leq \varepsilon,$$

and so $\lim_{n\to\infty} f(t, x_n(t,\omega), \omega) = f(t, x(t,\omega), \omega)$ uniformly on $[0, T]$ for all $\omega \in \Omega$. Further, since

$$\begin{aligned}
&\sup_{t\in[0,T]} \left| \int_0^t g_\alpha(t-s) f(s, x_n(s,\omega), \omega) ds - \int_0^t g_\alpha(t-s) f(s, x(s,\omega), \omega) ds \right| \\
\leq\ & \sup_{t\in[0,T]} \int_0^t g_\alpha(t-s) \left| f(s, x_n(s,\omega), \omega) - f(s, x(s,\omega), \omega) \right| ds \\
\leq\ & \frac{K(\omega) T^\alpha}{\Gamma(\alpha+1)} \sup_{t\in[0,T]} |x_n(t,\omega) - x(t,\omega)| \, ds,
\end{aligned}$$

thus, by Lebesgue dominated convergence theorem, we infer that

$$\lim_{n\to\infty} \int_0^t g_\alpha(t-s) f(s, x_n(s,\omega), \omega) ds = \int_0^t g_\alpha(t-s) f(s, x(s,\omega), \omega) ds,$$

for all $t \in [0, T]$ and $\omega \in \Omega$. Next we have that

$$\begin{aligned}
&\left| x(s,\omega) - x_0(\omega) - \int_0^t g_\alpha(t-s) f(s, x(s,\omega), \omega) ds \right| \\
\leq\ & |x_n(t,\omega) - x(t,\omega| + \left| x_n(t,\omega) - x_0(\omega) - \int_0^t g_\alpha(t-s) f(s, x_n(s,\omega), \omega) ds \right| \\
&+ \left| \int_0^t g_\alpha(t-s) f(s, x_n(s,\omega), \omega) ds - \int_0^t g_\alpha(t-s) f(s, x(s,\omega), \omega) ds \right|,
\end{aligned}$$

hence

$$\begin{aligned}
&\sup_{t\in[0,T]} \left| x(t,\omega) - x_0(\omega) - \int_0^t g_\alpha(t-s) f(s, x(s,\omega), \omega) ds \right| \\
\leq\ & \sup_{t\in[0,T]} |x_n(t,\omega) - x(t,\omega|
\end{aligned}$$

$$+\sup_{t\in[0,T]}\int_0^t g_\alpha(t-s)\left|f(s,x_n(s,\omega),\omega)-f(s,x(s,\omega),\omega)\right|ds.$$

Thus, in view of the two previous convergences, we infer that

$$x(t,\omega)=x_0(\omega)+\int_0^t g_\alpha(t-s)f(s,x(s,\omega),\omega)ds.$$

From the Remark 3.4 it follows that $x(\cdot,\cdot)$ is a solution for (3.3). Now we show that this solution $x(\cdot,\cdot)$ is unique. Let us assume that $x(\cdot,\cdot)$, $y(\cdot,\cdot):[0,T]\times\Omega\to\mathbb{R}$ are two Carathéodory functions which are solutions of the integral equation (3.4). Then we have that

$$\begin{aligned}|x(t,\omega)-y(t,\omega| &\le \int_0^t g_\alpha(t-s)\left|f(s,x(s,\omega),\omega)-f(s,y(s,\omega),\omega)\right|ds\\ &\le K(\omega)\int_0^t g_\alpha(t-s)\left|x(s,\omega)-y(s,\omega)\right|ds.\end{aligned}$$

Therefore, applying the Gronwall inequality with singularity (see [16]) we obtain that $|x(t,\omega)-y(t,\omega)|=0$, which completes the proof. □

Now we consider the analogous result for the L^p-problem (3.1).

Theorem 4.4. *Assume that:*

(h1) The function $F:[0,T]\times L^p(\Omega)\to L^p(\Omega)$ is continuous and satisfies the Lipschitz condition

$$\|F(t,X)-F(t,Y)\|_{L^p(\Omega)}\le K\|X-Y\|_{L^p(\Omega)}$$

for all $X,Y\in L^p(\Omega)$, where K is a positive constant.

(h2) For any L^p-continuous function

$$X:[0,T]\to L^p(\Omega),$$

the function $t\mapsto F(t,X(t))$ is Bochner integrable on $[0,T]$.

Then the L^p-problem (3.1) has a unique L^p-solution on $[0,T]$.

Proof. Since $t \mapsto \|F(t, X_0)\|_{L^p(\Omega)}$ is continuous on $[0, T]$ then, by the properties of convolution product (see [1]), it follows that the function

$$t \mapsto \int_0^t g_\alpha(t-s)\, \|F(s, X_0)\|_{L^p(\Omega)}\, ds$$

is a continuous function from $[0, T]$ into $\mathbb{R}_+$. Let

$$M = \sup_{t\in[0,T]} \int_0^t g_\alpha(t-s)\, \|F(s, X_0)\|_{L^p(\Omega)}\, ds < \infty.$$

Next, we define a sequence of functions $X_n : [0, T] \to L^p(\Omega)$ as follows:

$$X_0(t) = X_0,$$

$$X_n(t) = X_0 + \textstyle\int_0^t g_\alpha(t-s) F(s, X_{n-1}(s)) ds, \quad n \geq 1.$$

Then we have that $\|X_1(t) - X_0\|_{L^p(\Omega)} \leq M$, $t \in [0, T]$. For $n \geq 1$ let us put

$$y_n(t) = \|X_n(t) - X_{n-1}(t)\|_{L^p(\Omega)}, \quad t \in [0, T].$$

Using (h1) it follows that

$$y_n(t) \leq K \int_0^t g_\alpha(t-s) y_{n-1}(s) ds$$

for $t \in [0, T]$ and $n \geq 1$. For $n \geq 1$ the last inequality implies that $y_n(t) \leq \frac{M[KT^\alpha]^{n-1}}{\Gamma((n-1)\alpha+1)}$, and therefore

$$\|X_n(t) - X_{n-1}(t)\|_{L^p(\Omega)} \leq \frac{M(KT^\alpha)^{n-1}}{\Gamma((n-1)\alpha+1)}$$

for $t \in [0, T]$ and $n \geq 1$. The remainder of the existence proof is analogous to that of Theorem 4.3, and is, therefore, omitted. Similarly, we omit the proof of uniqueness, since it follows from the Gronwall inequality with singularity (see [16]). □

References

[1] W. Arendt, C.J.K. Batty, M. Hiber, F. Neubrander, *Vector-Valued Laplace Transforms and Cauchy Problems*, Monographs in Mathematics, Volume 96, Birkhäuser, Basel (2011).

[2] A.T. Bharucha-Reid, *Random Integral Equations*, Academic Press, New York (1972).

[3] S.S. Dragomir, R.P. Agarwal, N.S. Barnett, Inequalities for Beta and Gamma functions via some classical and new integral inequalities, *J. Inequal. Appl.*, **5** (2000), 103-165.

[4] N. Dunford, J.T. Schwartz, *Linear Operators I*, Interscience, New York (1958).

[5] R. Edsinger, *Random Ordinary Differential Equations*, Ph.D. Thesis, Univ. of California, Berkeley (1968).

[6] E. Hille, R. S. Phillips, *Functional Analysis and Semi-Groups*, AMS, Colloquium Publications, Volume XXXI, New York (1957).

[7] C.J. Himmelberg, Measurable relations, *Fund. Math.*, **87** (1975), 53-72.

[8] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, *Theory and Applications of Fractional Differential Equations*, North-Holland Mathematics Studies, **204**, Elsevier Science B.V., Amsterdam (2006).

[9] G.S. Ladde, V. Lakshmikantham, *Random Differential Inequalities*, Academic Press, New York (1980).

[10] H.A.H. Salem, A.M.A. El-Sayed, O.L. Moustafa, A note on the fractional calculus in Banach spaces, *Studia Sci. Math. Hungar.*, **42** (2005), 115-130.

[11] S.G. Samko, A.A. Kilbas, O.I. Marichev, *Fractional Integrals and Derivatives. Theory and Applications*, Gordon and Breach, Yverdon (1993).

[12] T.T. Soong, *Random Differential Equations in Science and Engineering*, Academic Press, New York (1973).

[13] J.L. Strand, Random Ordinary Differential Equations, *J. Differential Equations*, **7** (1970), 538-553.

[14] J.L. Strand, *Stochastic Ordinary Differential Eqnations*, Ph.D. Thesis, Univ. of California, Berkeley (1968).

[15] C.P. Tsokos, W.J. Padgett, *Random Integral Equations with Applications to Life Sciences and Engineering*, Academic Press, New York (1974).

[16] H. Ye, J. Gao, Y. Ding, A generalized Gronwall inequality and its application to a fractional differential equation, *J. Math. Anal. Appl.*, **328** (2007), 1075-1081.