International Electronic Journal of Pure and Applied Mathematics
Volume 4 No. 2 2012, 113-117

SOME DIFFERENTIAL GEOMETRIC INEQUALITIES FOR SURFACES IN EUCLIDEAN SPACE WITH NEGATIVE GAUSS CURVATURE

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Abstract: In this paper some differential geometric inequalities for surfaces in E^6 with negative Gauss curvature is derived. We compute some inequalities by means of the sum of betti numbers and Euler characteristics of surface.

- $\int_{M^4} GdV \leq -\frac{4}{3}\pi^2\beta(M^4)$;
- $\int_U \frac{\sqrt[4]{-\lambda_1\lambda_2}}{\sqrt{\lambda_1}+\sqrt{-\lambda_2}}[2\sqrt{-\lambda_1\lambda_2}+3(\lambda_1-\lambda_2)]dV \geq$
$\pi^3\beta(M^4)+6\pi^2\chi(M^4)-\frac{3}{2}\int_U |\alpha G(p)|dV$.

1. Introduction

Let M^4 be an oriented closed surface with an immersion $x : M^4 \to E^6$. Let $F(M^4)$ and $F(M^6)$ be the bundels of orthonormal frames of $M^4$and E^6 respectively. Let B be the set of elements $b = (p, e_1, e_2, e_3, e_4, e_5, e_6)$ such that $(p, e_1, e_2, e_3, e_4) \in F(M^4)$ and $b = (x(p), e_1, e_2, e_3, e_4, e_5, e_6) \in F(M^6)$ whose orientation is coherent with the one of E^6, identifying e_i with $dx(e_i), i = 1, 2, 3, 4$.

Received: February 29, 2012

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Define $\tilde{x} : B \to F(E^6)$ naturally by $b \to= (x(p), e_1, e_2, e_3, e_4, e_5, e_6)$. The structure equations of E^6 are given by

$$dx = \sum \tilde{w}_A e_A \qquad de_A = \sum \tilde{w}_{AB} e_B \qquad \tilde{w}_{AB} + \tilde{w}_{BA} = 0$$

$$d\tilde{w}_A = \sum \tilde{w}_B \wedge \tilde{w}_{BA} \qquad d\tilde{w}_{AB} = \sum \tilde{w}_{AC} \wedge \tilde{w}_{CB} \qquad A, B, C = 1, 2, 3, 4, 5, 6$$

where $\tilde{w}_A, \tilde{w}_{AB}$ are differential 1-forms on $F(E^6)$.

Let w_A, w_{AB} be the induced 1-forms on B from $\tilde{w}_A, \tilde{w}_{AB}$ by the mapping $\tilde{x}$. Then we have

$$w_r = 0 \qquad w_{ri} = \sum A_{rij} w_j \qquad A_{rij} = A_{rji} \quad for \quad r = 5, 6 \quad and \quad i, j = 1, 2, 3, 4$$

$$w_{5i} = A_{5i1} w_1 + A_{5i2} w_2 + A_{5i3} w_3 + A_{5i4} w_4$$

$$w_{6i} = A_{6i1} w_1 + A_{6i2} w_2 + A_{6i3} w_3 + A_{6i4} w_4$$

Let $(p, e_1, e_2, e_3, e_4, \bar{e}_5, \bar{e}_6)$ be a local cross-section of $B \to F(M^4)$. The restriction of A_{rij} onto the image of local cross-section is denoted by $\bar{A}_{rij}$. For a unit normal vector

$$e = e_6 = \cos\theta \bar{e}_5 + \sin\theta \bar{e}_6$$

$$A_{6ij} = \cos\theta \bar{A}_{5ij} + \sin\theta \bar{A}_{6ij}$$

The Lipschitz-Killing curvature $K(p, e)$ is determined by $K(p, e) \equiv \det(A_{6ij}) = \det(\cos\theta \bar{A}_{5ij} + \sin\theta \bar{A}_{6ij})$. It is a quadratic form of $\cos^2\theta$ and $\sin^2\theta$. It can be written as $K(p, e) = \lambda_1(p)\cos^4\theta + \lambda_2(p)\sin^4\theta$ by using an orthonormal frame where

$$\lambda_1(p) = \det(\bar{A}_{5ij}) \quad \lambda_2(p) = \det(\bar{A}_{6ij}) \quad and \quad \lambda_1(p) \geq \lambda_2(p)$$

$\lambda_1(p)$ and $\lambda_2(p)$ are continuous on M^4. The Gauss curvature $G(p)$ is given by

$$G(p) = \lambda_1(p) + \lambda_2(p) \quad as \quad in [1]$$

Theorem 1. *Let M^4 be an 4-dimensional oriented closed manifold with an immersion $x : M^4 \to E^6$ and $G(p) = \lambda_1(p) + \lambda_2(p)$ be negative Gauss curvature of M^4. $\beta(M^4)$ denotes the sum of betti numbers of M^4. If $\lambda_1(p) < 0$ and $\lambda_2(p) < 0$ then $\int_{M^4} G dV \leq -\frac{4}{3}\pi^2\beta(M^4)$.*

Proof. Let $\lambda_1(p) < 0$ and $\lambda_2(p) < 0$. The total absolute curvature is given by $K^*(p) = \int_0^{2\pi} |K(p, e)| d\theta$ where K(p,e) is the Lipschitz-Killing curvature

$$\begin{aligned} K^*(p) &= \int_0^{2\pi} |K(p, e)| d\theta \\ &= \int_0^{2\pi} |\lambda_1(p)\cos^4\theta + \lambda_2(p)\sin^4\theta| d\theta \end{aligned}$$

$$
\begin{aligned}
&= -\int_0^{2\pi} (\lambda_1(p)\cos^4\theta + \lambda_2(p)\sin^4\theta)d\theta \\
&= -\frac{3\pi}{4}(\lambda_1(p) + \lambda_2(p)) \\
&= -\frac{3\pi}{4}G(p)
\end{aligned}
$$

$$
\int_{M^4} K^* dV = \int_{M^4} -\frac{3\pi}{4}G(p)dV \quad and \quad \int_{M^4} K^* dV \geq c_5\beta(M^4) \quad as \quad in[2]
$$

$$
c_5 = \frac{2[\Gamma(\frac{1}{2})]^6}{\Gamma(3)} \quad c_5 = \pi^3
$$

$$
\int_{M^4} K^* dV \geq \pi^3\beta(M^4) \quad and \quad \int_{M^4} -\frac{3\pi}{4}GdV \geq \pi^3\beta(M^4) \quad then \quad we \quad have
$$

$\int_{M^4} GdV \leq -\frac{4\pi^2}{3}\beta(M^4)$.

Theorem 2. *Let M^4 be an 4-dimensional oriented closed manifold with an immersion $x : M^4 \to E^6$ and $G(p) = \lambda_1(p) + \lambda_2(p)$, $\lambda_1(p) \geq \lambda_2(p)$ be negative Gauss curvature of M^4. $\beta(M^4)$ denotes the sum of betti numbers of M^4. If $\lambda_1(p) > 0$ and $\lambda_2(p) < 0$ then*

$$
\int_U \frac{\sqrt[4]{-\lambda_1\lambda_2}}{\sqrt{\lambda_1} + \sqrt{-\lambda_2}}[2\sqrt{-\lambda_1\lambda_2} + 3(\lambda_1 - \lambda_2)]dV
$$

$$
\geq \pi^3\beta(M^4) + 6\pi^2\chi(M^4) - \frac{3}{2}\int_U |\alpha G(p)|dV
$$

where $\chi(M^4)$ is Euler characteristics of M^4 and $U = p \in M^4, \lambda_1(p) > 0$.

Proof. Define U and V as $U = p \in M^4, \lambda_1(p) > 0$ and $V = p \in M^4, \lambda_1(p) < 0$. Since $G(p) < 0$ we have $|\lambda_2(p)| \geq |\lambda_1(p)|$.

$$
\begin{aligned}
K^*(p) &= \int_0^{2\pi} |K(p,e)|d\theta \\
&= \int_0^{2\pi} |\lambda_1(p)\cos^4\theta + \lambda_2(p)\sin^4\theta|d\theta \\
&= \int_0^{2\pi} |\lambda_1(p)\cos^2\theta + \lambda_2(p)\sin^2\theta||\lambda_1(p)\cos^2\theta - \lambda_2(p)\sin^2\theta|d\theta
\end{aligned}
$$

where $a = \sqrt{\lambda_1}$, $b = \sqrt{-\lambda_2}$, $b > a > 0$. Since $a\cos^2\theta + b\sin^2\theta \geq 0$ for every θ we have

$$
K^*(p) = \int_0^{2\pi} (a\cos^2\theta + b\sin^2\theta)|a\cos^2\theta - b\sin^2\theta|d\theta
$$

International Electronic Journal of Pure and Applied Mathematics – IEJPAM, Volume 4, No. 2 (2012)

$$\begin{aligned} &= \frac{1}{2}\int_0^{2\pi}(a\cos^2\theta + b\sin^2\theta)|(a-b)+(a+b)\cos 2\theta|d\theta \\ &= \frac{1}{2}(a+b)\int_0^{2\pi}(a\cos^2\theta + b\sin^2\theta)|\frac{a-b}{a+b}+\cos 2\theta|d\theta \end{aligned}$$

Define an angle α such that $0<\alpha<\frac{\pi}{2}$, $\cos\alpha = -\frac{a-b}{a+b}$ so $\sin\alpha = \frac{2\sqrt{ab}}{a+b}$.

$$\begin{aligned} G(p) &= \lambda_1(p)+\lambda_2(p) \\ &= a^2-b^2 \\ K^*(p) &= \frac{1}{2}(a+b)\int_0^{2\pi}(a\cos^2\theta + b\sin^2\theta)|\cos 2\theta - \cos\alpha|d\theta \end{aligned}$$

$$2\theta = t \quad d\theta = \frac{1}{2}dt$$

$$\begin{aligned} K^*(p) &= \frac{1}{4}(a+b)\int_0^{4\pi}(a\cos^2\frac{t}{2}+b\sin^2\frac{t}{2})|\cos t-\cos\alpha|dt \\ &= (a+b)\int_0^{\pi}(a\cos^2\frac{t}{2}+b\sin^2\frac{t}{2})|\cos t-\cos\alpha|dt \\ &= \frac{1}{2}(a+b)\int_0^{\pi}[(a+b)+(a-b)\cos t]|\cos t-\cos\alpha|dt \\ &= \frac{a^2-b^2}{2}\int_0^{\pi}[\frac{a+b}{a-b}+\cos t]|\cos t-\cos\alpha|dt \\ &= \frac{a^2-b^2}{2}\int_0^{\pi}[-\frac{1}{\cos\alpha}+\cos t]|\cos t-\cos\alpha|dt \\ &= \frac{a^2-b^2}{2}\int_0^{\alpha}[-\frac{1}{\cos\alpha}+\cos t](\cos t-\cos\alpha)dt \\ &\quad -\frac{a^2-b^2}{2}\int_\alpha^{\pi}[-\frac{1}{\cos\alpha}+\cos t](\cos t-\cos\alpha)dt \\ &= \frac{a^2-b^2}{2}[\frac{1}{4}\sin 2t+\frac{t}{2}-\frac{\cos^2\alpha+1}{\cos^2\alpha}\sin t]\Big|_0^{\alpha} \\ &\quad -\frac{a^2-b^2}{2}[\frac{1}{4}\sin 2t+\frac{t}{2}-\frac{\cos^2\alpha+1}{\cos^2\alpha}\sin t]\Big|_0^{\pi} \\ &= G(p)[\frac{1}{4}\sin 2\alpha+\frac{3\alpha}{2}-\frac{\cos^2\alpha+1}{\cos^2\alpha}\sin\alpha]-\frac{3\pi}{4}G(p) \\ &= \frac{3\alpha}{2}G(p)-\frac{3\pi}{4}G(p)+G(p)[\frac{1}{4}\sin 2\alpha-\frac{\cos^2\alpha+1}{\cos^2\alpha}\sin\alpha] \\ &= \frac{3\alpha}{2}G(p)-\frac{3\pi}{4}G(p)+\frac{\sqrt{ab}}{a+b}[2ab+3(a^2+b^2)] \end{aligned}$$

$$K^*(p) = \int_{M^4} K^* dV \quad for \quad V = \{p \in M^4, \lambda_1(p) < 0\}$$

$$K^*(p) = -\frac{3\pi}{4} G(p)$$

$$\begin{aligned}
\int_{M^4} K^* dV &= \int_U K^* dV + \int_V K^* dV \\
&= \int_U \left[\frac{3\alpha}{2} G(p) - \frac{3\pi}{4} G(p) + \frac{\sqrt{ab}}{a+b}[2ab + 3(a^2+b^2)]\right] dV \\
&\quad + \int_V -\frac{3\pi}{4} G(p) dV \\
&= \int_U -\frac{3\pi}{4} G(p) dV + \int_V -\frac{3\pi}{4} G(p) dV \\
&\quad + \int_U \frac{\sqrt{ab}}{a+b}[2ab + 3(a^2+b^2)] dV + \frac{3}{2}\int_U \alpha G(p) dV \\
&= \int_{M^4} -\frac{3\pi}{4} G(p) dV + \frac{3}{2}\int_U \alpha G(p) dV \\
&\quad + \int_U \frac{\sqrt{ab}}{a+b}[2ab + 3(a^2+b^2)] dV
\end{aligned}$$

$$\int_U \frac{\sqrt{ab}}{a+b}[2ab + 3(a^2+b^2)] dV = \int_{M^4} K^* dV + \frac{3\pi}{4}\int_{M^4} G(p) dV - \frac{3}{2}\int_U \alpha G(p) dV$$

$\int_{M^4} G(p) dV = 8\pi^2 \chi(M^4)$ by Gauss-Bonnet formula

$$\int_U \frac{\sqrt{ab}}{a+b}[2ab + 3(a^2+b^2)] dV \quad \geq \quad c_5 \beta(M^4) + 6\pi^2 \chi(M^4) - \frac{3}{2}\int_U |\alpha G(p)| dV$$

which is

$$\begin{aligned}
&\int_U \frac{\sqrt[4]{-\lambda_1 \lambda_2}}{\sqrt{\lambda_1} + \sqrt{-\lambda_2}}[2\sqrt{-\lambda_1\lambda_2} + 3(\lambda_1 - \lambda_2)] dV \\
&\qquad \geq \pi^3 \beta(M^4) + 6\pi^2 \chi(M^4) - \frac{3}{2}\int_U |\alpha G(p)| dV.
\end{aligned}$$

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