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COMMON ALPHA-FIXED POINTS THEOREMS FOR MULTIVALUED MAPPINGS IN INTUITIONISTIC FUZZY METRIC SPACES

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Abstract: In this paper we prove a common α-fixed point theorem for multivalued mapping in intuitionistic fuzzy metric space satisfying a new rational inequality. in this paper we used the results of Srivastav and Pawar, see [9], 2006; S.L. Singh, see [7], 1979; and use the concept of α − fixed point theorems from the B. Singh and R.K. Sharma (see [6], 2001). The main objective of this paper is to investigate a common α-fixed point theorem in intuitionistic fuzzy metric space.

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Key Words: intuitionistic fuzzy metric space, t-norm, t-conorm, α-fixed points, common α-fixed point, fixed point, fuzzy metric space, metric space, inequality and complete intuitionistic fuzzy metric space

1. Introduction

Srivastav and Pawar (see [9], 2006) have proved various results related with the set of fixed point of Banach Contraction Mapping by introducing the notion of α-fixed point and some other concepts.

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Srivastav and Pawar ([9], 2006) obtained the new setting by mentioning α-fixed point through α-contraction mapping as well as mentioned relationship of contraction and α-contraction mappings by giving examples and remarks. Singh and Sharma ([6], 2001) proved common fixed point theorems for mappings satisfying rational inequality in intuitionistic fuzzy metric space.

Using the concept of α-fixed point we generalize the result of Singh and Sharma ([6], 2001) and proved common α-fixed point theorems for mappings satisfying rational inequalities in intuitionistic fuzzy metric space. In J. H. Park ([4], 2004) introduced the notion of intuitionistic fuzzy metric spaces with the help of continuous t-norms and continuous t-conorms as a generalization of fuzzy metric spaces due to A. George and P. Veeramani ([2], 1994).

2. Preliminaries

Basic terminology and properties of $\alpha-$fixed point and Intuitionistic fuzzy metric space are given in this paper.

Definition 2.1. A binary operation $* : [0, 1]\times[0, 1] \to [0, 1]$ is a continuous t-norm if $*$ satisfying the following conditions.

(a) $*$is commutative and associative,

(b) $*$is continuous,

(c) $a * 1 = a$for all $a \in [0, 1]$;

(d) $a * b \le c * d$whenever $a \le c$and $b \le d$, and $a, b, c, d \in [0, 1]$.

Definition 2.2. A binary operation $\diamondsuit : [0,1] \times [0,1] \to [0,1]$ is a continuous t-conorm if $\diamondsuit$ satisfies the following conditions:

(1) $\diamondsuit$ is associative and commutative,

(2) $\diamondsuit$ is continuous,

(3) $a\diamondsuit 0 = a$ for all $a \in [0, 1]$,

(4) $a\diamondsuit b \le c\diamondsuit d$ whenever $a \le c$ and $b \le d$, for each $a, b, c, d \in [0, 1]$.

Definition 2.3. A 5-tuple $(X, M, N, *, \diamondsuit)$ is called a intuitionistic fuzzy metric space if X is an arbitrary (non-empty) set, $*$ is a continuous t-norm, $\diamondsuit$is a continuous t-conorm and M, N are fuzzy sets on $X^2 \times (0, \infty)$, satisfying the following conditions (for each $x, y, z \in X$and $t, s > 0$):

(a) $M(x, y, t) + N(x, y, t) \le 1$,

(b) $M(x, y, t) > 0$,

(c) $M(x, y, t) = 1$ *if and only if* $x = y$,

(d) $M(x, y, t) = M(y, x, t)$;

(e) $M(x, y, t) * M(y, z, s) \le M(x, z, t + s)$,

(f) $M(x, y, .) : (0,\infty) \to [0, 1]$ *is continuous.*

(g) $N(x, y, t) > 0$,

(h) $N(x, y, t) = 0$ *if and only if* $x = y$,

(i) $N(x, y, t) = N(y, x, t)$,

(j) N(x, y, t) $\diamondsuit$ *N(y, z, s)* $\geq$ *N(x, z, t +s),*

(k) N(x, y, .) : (0,∞*)* $\to$ *[0, 1] is continuous.*

Then (M, N) is called an intuitionistic fuzzy metric on X. The functions *M(x, y, t)* and *N(x, y, t)* denote the degree of nearness and the degree of non nearness between x and $y$with respect to t, respectively.

Definition 2.4. A point $x \in X$ is said to be a α-fixed point of self map α, $T : X \to X$ if $(\alpha oT)x = x$.

Remark 2.5. A fixed point is not necessarily α-fixed point and α-fixed point is not necessarily a fixed point.

Remark 2.6. In intuitionistic fuzzy metric space *M(x, y, .)* is non-decreasing and

N(x, y, .) is non-increasing for all $x, y \in X$.

Definition 2.7. A sequence $\{x_n\}$ in a intuitionistic fuzzy metric space *(X, M, N, *,*$\diamondsuit$*)* converges to x if and only if $M(x_n, x, t) \to 1$and $N(x_n, x, t) \to 0$ as $n \to \infty$, for each $t > 0$.

It is called a Cauchy sequence if for each $0 < \in < 1$ and $t > 0$, there exits $n_0 \in N$such that $M(x_n, x_m, t) > 1 - \in$ and $N(x_n, x_m, t) < \in$for each $n, m \geq n_0$. The intuitionistic fuzzy metric space *(X, M, N, *,* $\diamondsuit$*)* is said to be complete if every Cauchy sequence is convergent.

Lemma 2.8. *If x, y are any two points in an intuitionistic fuzzy metric space X and k is a positive number with $k < 1$ and $M(x, y, kt) \geq M(x, y, t)$, $N(x, y, kt) \leq N(x, y, t)$,*

Then $x = y$.

Proof. It is immediate from remark 2.6 and Definition 2.3.

3. Main Results

Theorem 3.1. *Let (X, M, N, *,* $\diamondsuit$*) be a complete intuitionistic fuzzy metric space with $a * b = \min(a, b)$ and $a\diamondsuit b = \max(a, b)$.Let $\alpha : X \to X$ and $S, T : X \to CB(X)$ be single and multivalued mappings satisfying the following conditions:*

(i) $SX \subset TX$,

(ii) S and T are continuous,

(iv) there exists $q \in (0, 1)$ and $0<\beta<1$ such that for every $x, y \in X$ and $t > 0$,

$$M(\alpha oS, \alpha oT, qt) \geq \beta \frac{[M(x, (\alpha oS)x, t)]^2}{M(x, (\alpha oS)x, t) * M(x, (\alpha oT)y, 2t) * M(x, y, t)},$$

$$N(\alpha oS, \alpha oT, qt) \leq \beta \frac{[N(x, (\alpha oS)x, t)]^2}{N(x, (\alpha oS)x, t)\diamondsuit N(x, (\alpha oT)y, 2t)\diamondsuit N(x, y, t)}.$$

Here

$$M(x, (\alpha oS)x, t) * M(x, (\alpha oT)y, t) * M(x, y, t) \neq 0$$

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and

$$N(x,(\alpha oS)x,t)\Diamond N(x,(\alpha oT)y,t)\Diamond N(x,y,t)\neq 0$$

for all $x,y\in X$ and $0<\beta<1$.

Then S and$T$ have a unique common α-fixed point.

Proof. We define a sequence $\{x_n\}$in Xsuch that $x_n=(\alpha oS)x_{n-1}$ and $x_{n+1}=(\alpha oT)\ x_n$ for all a in X.

$$M(x_n,x_{n+1},qt)=M((\alpha oS)x_{n1},(\alpha oT)x_n,qt)$$

$$\geq\beta\frac{[M(x_{n-1},(\alpha oS)x_{n-1},t)]^2}{M(x_{n-1},(\alpha oS)x_{n-1},t)*M(x_{n-1},(\alpha oT)x_n,2t)*M(x_{n-1},x_n,t)},$$

$$M(x_n,x_{n+1},qt)\geq\quad\beta\frac{[M(x_{n-1},x_n,t)]^2}{M(x_{n-1},x_n,t)*M(x_{n-1},x_{n+1},2t)*M(x_{n-1},x_n,t)}\quad,$$

$$M(x_n,x_{n+1},qt\geq\beta\frac{[M(x_{n-1},x_n,t)]^2}{M(x_{n-1},x_n,t)*M(x_{n-1},x_{n+1},2t)},$$

$$M(x_n,x_{n+1},qt)\geq\beta\frac{[M(x_{n-1},x_n,t)]^2}{M(x_{n-1},x_n,t)*M(x_{n-1},x_n,t)*M(x_n,x_{n+1},t)}\quad,$$

$$M(x_n,x_{n+1},qt)\geq\beta\frac{[M(x_{n-1},x_n,t)]^2}{M(x_{n-1},x_n,t)*M(x_n,x_{n+1},t)}.$$

Using $a*b=\min\{a,b\}$

$$M(x_n,x_{n+1},qt)\geq\beta\frac{[M(x_{n-1},x_n,t)]^2}{\min[M(x_{n-1},x_n,t),M(x_n,x_{n+1},t)]}.$$

Case 1. If $M(x_n,x_{n+1},t)\geq M(x_{n-1},x_n,t)$ then we have

$$M(x_n,x_{n+1},qt)\leq\beta\frac{[M(x_{n-1},x_n,t)]^2}{M(x_n,x_{n+1},t)}.$$

Letting $q\to 1$:

$$M(x_n\,,x_{n+1},t)^2\leq\beta M(x_{n-1},x_n,t)^2.$$

Taking square both sides we get

$$M(x_n,x_{n+1},t)\leq\sqrt{\beta}\quad M(x_{n-1},x_n,t).$$

Case 2. If $M(x_n\,,x_{n+1},t)\leq M(x_{n-1},x_n,t)$, then we have

$$M(x_n,x_{n+1},qt)\geq\beta\frac{[M(x_{n-1},x_n,t)]^2}{M(x_{n-1},x_n,t)},$$

$$M(x_n,x_{n+1},qt)\geq\beta M(x_{n-1},x_n,t).$$

Now

$$N(x_n, x_{n+1}, qt) = N((\alpha oS)x_{n1}, (\alpha oT)x_n, qt)$$
$$\leq \beta \frac{[N(x_{n-1}, (\alpha oS)x_{n-1}, t)]^2}{N(x_{n-1}, (\alpha oS)x_{n-1}, t)\Diamond N(x_{n-1}, (\alpha oT)x_n, 2t)\Diamond N(x_{n-1}, x_n, t)},$$

$$N(x_n, x_{n+1}, qt) \leq \beta \frac{[N(x_{n-1}, x_n, t)]^2}{N(x_{n-1}, x_n, t)\Diamond N(x_{n-1}, x_{n+1}, 2t)\Diamond N(x_{n-1}, x_n, t)},$$

$$N(x_n, x_{n+1}, qt) \leq \beta \frac{[N(x_{n-1}, x_n, t)]^2}{N(x_{n-1}, x_n, t)\Diamond N(x_{n-1}, x_{n+1}, 2t)},$$

$$N(x_n, x_{n+1}, qt) \leq \beta \frac{[N(x_{n-1}, x_n, t)]^2}{N(x_{n-1}, x_n, t)\Diamond N(x_{n-1}, x_n, t)\Diamond N(x_n, x_{n+1}, t)},$$

$$N(x_n, x_{n+1}, qt) \leq \beta \frac{[N(x_{n-1}, x_n, t)]^2}{N(x_{n-1}, x_n, t)\Diamond N(x_n, x_{n+1}, t)},$$

Using $a\Diamond b = \max\{a, b\}$:

$$N(x_n, x_{n+1}, qt) \leq \beta \frac{[N(x_{n-1}, x_n, t)]^2}{\max[N(x_{n-1}, x_n, t), N(x_n, x_{n+1}, t)]}.$$

Now two cases arises:

Case 1. If $N(x_n, x_{n+1}, t) \geq N(x_{n-1}, x_n, t)$ then we have

$$N(x_n, x_{n+1}, qt) \leq \beta \frac{[N(x_{n-1}, x_n, t)]^2}{N(x_n, x_{n+1}, t)}.$$

Letting $q \to 1$:

$$N(x_n, x_{n+1}, t)^2 \leq \beta N(x_{n-1}, x_n, t)^2.$$

Taking square both sides we get

$$N(x_n, x_{n+1}, t) \leq \sqrt{\beta} N(x_{n-1}, x_n, t).$$

Case 2. If $N(x_n, x_{n+1}, t) \leq N(x_{n-1}, x_n, t)$, then we have

$$N(x_n, x_{n+1}, qt) \leq \beta \frac{[N(x_{n-1}, x_n, t)]^2}{N(x_{n-1}, x_n, t)}.$$

Letting $q \to 1$:

$$N(x_n, x_{n+1}, t) \leq \beta N(x_{n-1}, x_n, t).$$

Both the cases we have from Lemma 2.8 that

$$M(x_{n+1}, x_{n+2}, qt) \geq M(x_n, x_{n+1}, t). \qquad (3.1.1.)$$

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$$N(x_{n+1}, x_{n+2}, qt) \leq N(x_n, x_{n+1}, t).$$

Similarly, we have also

$$M(x_{n+2}, x_{n+3}, qt) \geq M(x_{n+1}, x_{n+2}, t). \tag{3.1.2}$$

$$N(x_{n+2}, x_{2n+3}, qt) \leq N(x_{n+1}, x_{n+2}, t).$$

From (3.1.1) and (3.1.2), we have

$$M(x_{n+1}, x_{n+2}, qt) \geq M(x_n, x_{n+1}, t). \tag{3.1.3}$$

$$N(x_{n+1}, x_{n+2}, qt) \leq N(x_n, x_{n+1}, t).$$

From (3.1.3)

$$M(x_n, x_{n+1}, t) \geq M(x_n, x_{n-1}, t/q) \geq M(x_{n-2}, x_{n-1}, t/q^2) \geq \cdots \geq M(x_1, x_2, t/q^n) \to 1$$

as $n \to \infty$.

So, $M(x_n, x_{n+1}, t) \to 1$ as $n \to \infty$ for any $t > 0$.

$$N(x_n, x_{n+1}, t) \leq N(x_n, x_{n-1}, t/q) \leq N(x_{n-2}, x_{n-1}, t/q^2) \leq \cdots \leq N(x_1, x_2, t/q^n) \to 0$$

as $n \to \infty$.

So, $N(x_n, x_{n+1}, t) \to 0$ as $n \to \infty$, for any $t > 0$.

For each $\varepsilon > 0$ and each $t > 0$, we can choose $n_0 \in N$ such that

$$M(x_n, x_{n+1}, t) > 1 - \varepsilon,$$

$$N(x_n, x_{n+1}, t) < \varepsilon,$$

for all $n > n_0$.

For $m, n \in N$, we suppose $m \geq n$. Then we have that

$$M(x_n, x_m, t) \geq M(x_n, x_{n+1}, \ t/m - n) \ M(x_{n+1}, x_{n+2}, t/m - n) \\ M(x_{m-1}, x_m, t/m - n) > \frac{m-n}{(1-\varepsilon) * (1-\varepsilon) ... (1-\varepsilon)} \geq (1-\varepsilon),$$

$$N(x_n, x_m, t) = N(x_n, x_{n+1}, t/m-n) \diamondsuit N(x_{n+1}, x_{n+2}, t/m-n) \diamondsuit ..... \diamondsuit_{N(xm-1, xm, t/m-n)} \\ < \frac{m-n}{\varepsilon \diamondsuit \varepsilon \diamondsuit ... \diamondsuit \varepsilon} \leq \varepsilon$$

and hence $\{x_n\}$ is a Cauchy sequence in X.

Here $0<\sqrt{\beta}<1$ then by lemma 2.8 $\{x_n\}$ converges to a point (say z) in x

Now

$$M(z,(\alpha oT)z,t) \geq M(z,x_n,\frac{t}{2}) * M(x_{n,}(\alpha oT)z,\frac{t}{2})$$
$$\geq M(z,x_n,\frac{t}{2}) * M((\alpha oS)x_{n-1,}(\alpha oT)z,\frac{t}{2})$$
$$\geq M(z,x_n,\frac{t}{2}) * \beta\frac{[M(x_{n-1},x_n,t)]^2}{M(x_{n-1},(\alpha oS)x_{n-1},t) * M(x_{n-1},(\alpha oT)z,t) * M(x_{n-1},x_n,t)}.$$

As $n \to \infty$:

$$\geq M(z,x_n,\frac{t}{2}) * \beta\frac{[M(x_{n-1},x_n,t)]^2}{M(x_{n-1},x_n,t) * M(x_{n-1},(\alpha oT)z,t) * M(x_{n-1},x_n,t)} \to 1.$$

Now,

$$N(z,(\alpha oT)z,t) \leq N(z,x_n,\frac{t}{2})\Diamond N(x_{n,}(\alpha oT)z,\frac{t}{2})$$
$$\leq N(z,x_n,\frac{t}{2})\Diamond N((\alpha oS)x_{n-1,}(\alpha oT)z,\frac{t}{2})$$
$$\leq N(z,x_n,\frac{t}{2})\Diamond\beta\frac{[N(x_{n-1},x_n,t)]^2}{N(x_{n-1},(\alpha oS)x_{n-1},t)\Diamond N(x_{n-1},(\alpha oT)z,t)\Diamond N(x_{n-1},x_n,t)}$$

As $n \to \infty$:

$$\leq N(z,x_n,\frac{t}{2})\Diamond\beta\frac{[N(x_{n-1},x_n,t)]^2}{N(x_{n-1},x_n,t)\Diamond N(x_{n-1},(\alpha oT)z,t)\Diamond N(x_{n-1},x_n,t)} \to 0.$$

As $n \to \infty$ implies that $(\alpha oT)z= z$ i.e z is the $\alpha-$fixed point of T.Similarly it can be shown that z is α-fixed point of S. Consequently z is the common $\alpha-$fixed point of $S\&T$. Now for uniqueness of z let z'be another common α- fixed point of S and T then:

$$M(z,z',t) = M((\alpha oS),(\alpha oT)z',t)$$
$$\geq \beta\frac{[M(z,(\alpha oS)z,t)]^2}{M(z,(\alpha oS)z,t) * M(z,(\alpha oT)z',t) * M(z,z',t)} = 1,$$

$$N(z,z',t) = N((\alpha oS),(\alpha oT)z',t)$$
$$\leq \beta\frac{[N(z,(\alpha oS)z,t)]^2}{N(z,(\alpha oS)z,t)\Diamond N(z,(\alpha oT)z',t)\Diamond N(z,z',t)} = 0.$$

Which is contradiction so $z = z'$. This completes the proof. □

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