

**UPPER AND LOWER BOUNDS FOR PRODUCTS OF
MULTIPLICATION OPERATOR AND HAUSDORFF
MATRIX ON BLOCK WEIGHTED SEQUENCE SPACES**

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Abstract: In this paper, we compute upper and lower bounds for products of multiplication operator and Hausdorff matrix on block weighted sequence spaces.

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1. Introduction and Preliminaries

Let l_p ($0 < p < \infty$) denote the space of all sequences for which $\sum_{n=1}^{\infty} |x_n|^p < \infty$. For $p \geq 1$, l_p is a Banach space with the norm

$$\|x\|_p = \left(\sum_{n=1}^{\infty} |x_n|^p \right)^{\frac{1}{p}}.$$

If $0 < p < 1$, then $\|\cdot\|_p$ does not satisfies the triangle inequality and therefore, it is not a norm. Abusing notation, we will write

$$\|x\|_p = \left(\sum_{n=1}^{\infty} |x_n|^p \right)^{\frac{1}{p}}$$

even if $0 < p < 1$. The space l_p for $0 < p < 1$ is an F space. Let $u : \mathbb{N} \rightarrow \mathbb{C}$ be a measurable function. Then a bounded linear transformation, $M_u : L^p(\mathbb{N}, \mathbb{C}) \rightarrow L^p(\mathbb{N}, \mathbb{C})$ defined by $(M_u f)(k) = (u \cdot f)(k) = u(k)(f(k))$ is called a multiplication operator induced by u .

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For a decreasing non-negative sequence $w = \{w_n\}$ and $0 < p < \infty$. We define the weighted sequence space $l_p(w)$ as follows:

$$l_p(w) = \left\{ \{x_n\} : \|x\|_{w,p} = \left(\sum_{n=1}^{\infty} w_n |x_n|^p \right)^{\frac{1}{p}} < \infty \right\}.$$

Let F be a partition of positive integers. If $F = (F_n)$, where each F_n is a finite interval of positive integers and

$$\max F_n < \min F_{n+1} \quad (n = 1, 2, \dots),$$

then we define the block weighted sequence space $l_p(w, F)$ as

$$l_p(w, F) = \left\{ x = \{x_n\} : \sum_{n=1}^{\infty} w_n |\langle x, F_n \rangle|^p \text{ is finite} \right\},$$

where $\langle x, F_n \rangle = \sum_{j \in F_n} x_j$. The norm on $l_p(w, F)$ denoted by $\|\cdot\|_{p,w,F}$ is defined as follows

$$\|x\|_{p,w,F} = \left(\sum_{n=1}^{\infty} w_n |\langle x, F_n \rangle|^p \right)^{\frac{1}{p}}.$$

For a certain I_n such as $I_n = \{n\}$, $I = (I_n)$ is a partition of positive integers, $l_p(w, I) = l_p(w)$ and also $\|x\|_{p,w,I} = \|x\|_{p,w}$.

Let $A : l_p(w, I) \rightarrow l_p(w, F)$ be a linear operator. The problem of the norm of matrix operators on $l_p(w)$ and $l_p(w, F)$ is considered in [4] and [9]. We consider upper and lower bounds U and L of the form

$$L \|x\|_{p,w,I} \leq \|Ax\|_{p,w,F} \leq U \|x\|_{p,w,I}$$

for all decreasing non-negative sequences x .

We consider the Hausdorff matrix operator $H(\mu) = (h_{j,k})$, such that

$$h_{j,k} = \begin{cases} \binom{j-1}{k-1} \Delta^{j-k} a_k, & \text{if } 1 \leq k \leq j \\ 0, & \text{if } k > j, \end{cases}$$

where Δ is the difference operator, that is,

$$\Delta a_k = a_k - a_{k+1},$$

and $\{a_k\}$ is a sequence of real numbers, normalized so that $a_1 = 1$.

If

$$a_k = \int_0^1 \theta^{k-1} d\mu(\theta), \quad (k = 1, 2, \dots),$$

where μ is a probability measure on $[0, 1]$, then for all $j, k = 1, 2, \dots$, we have

$$h_{j,k} = \begin{cases} \binom{j-1}{k-1} \int_0^1 \theta^{k-1}(1-\theta)^{j-k} d\mu(\theta), & \text{if } 1 \leq k \leq j \\ 0, & \text{if } k > j \end{cases}$$

The Hausdorff matrix contains following classes of matrices:

1. If $d\mu(\theta) = \alpha(1-\theta)^{\alpha-1}d\theta$, then $H(\mu)$ reduces to the Cesaro matrix of order α , which we denote by $T(\alpha)$;
2. If $d\mu(\theta) =$ point evaluation at $\theta = \alpha$, then $H(\mu)$ reduces to the Euler matrix of order α , which we denote by $E(\alpha)$;
3. If $d\mu(\theta) = \frac{|\log(\theta)|^{\alpha-1}}{\Gamma(\alpha)}d(\theta)$, then $H(\mu)$ reduces to Holder matrix of order α , which we denote by $S(\alpha)$;
4. If $d\mu(\theta) = \alpha\theta^{\alpha-1}d\theta$, then $H(\mu)$ reduces to Gamma matrix of order α , which we denote by $\Gamma(\alpha)$;

The Cesaro matrix, Holder matrix and Gamma matrix have non-negative entries, whenever $\alpha > 0$. Also the Euler matrix is non-negative, whenever $0 \leq \alpha \leq 1$.

Recently, R.Lashkaripour and D.Foroutannia have computed lower bounds for matrices on block weighted sequence space I see [7], whereas the study of the multiplication operator M_u studied by many authors see (see [5], [6]).

In this paper we compute lower bounds of the operator $M_uH(\mu)$. Throughout this paper, constants are denoted by c , they are positive and may differ from one occurrence to the other. The notation $a \sim b$ means that there is a positive constant c such that $\frac{1}{c}a \leq b \leq ca$.

2. Lower Bounds of $M_uH(\mu)$

In this section, we are computing the lower bound problem of $M_uH(\mu)$, where M_u be a multiplication operator and $H(\mu)$ is the Hausdorff matrix operator.

Theorem 1. *Let $0 < p \leq 1$ and $u : \mathbb{N} \rightarrow \mathbb{C}$ be a measurable function. Then*

$$\begin{aligned} \inf |u(x)| \left(\int_0^1 \theta^{\frac{1-p}{p}} d\mu(\theta) \right) &\leq \|M_uH^t(\mu)\|_{p,w,I} \\ &\leq \sup |u(x)| \left(\int_0^1 \theta^{\frac{1-p}{p}} d\mu(\theta) \right), \end{aligned}$$

for every non negative sequence x .

Proof. If $u(k) = 0$ for some $k \in \mathbb{N}$ then the inequality is obvious. Otherwise, let $E(\alpha)$ be the Euler matrix of order α . Since the row sums of $E^t(\alpha)$ are all $\frac{1}{\alpha}$ and column sums are all 1, applying the proposition 2.2 of [8], we have

$$L_{p,w,I}(E^t(\alpha)) \geq \alpha^{1-p/p}.$$

we now applying the Minkowski's inequality, we have

$$\begin{aligned} \|M_u H^t(\mu)x\|_{p,w,I} &= \left(\sum_{n=1}^{\infty} w_n |\langle M_u H^t(\mu)x, I_n \rangle|^p \right)^{\frac{1}{p}} \\ &= \left(\sum_{n=1}^{\infty} w_n \left| \sum_{j \in I_n} (M_u H^t(\mu)x)_j \right|^p \right)^{\frac{1}{p}} \\ &\geq \inf_{k \in \mathbb{N}} |u(k)| \left(\sum_{n=1}^{\infty} w_n \left(\sum_{k=1}^{\infty} h_{k,n} x_k \right)^p \right)^{\frac{1}{p}} \\ &= \inf_{k \in \mathbb{N}} |u(k)| \left(\sum_{n=1}^{\infty} w_n \left(\int_0^1 \sum_{k=1}^{\infty} e_{k,n} x_k d\mu(\alpha) \right)^p \right)^{\frac{1}{p}} \\ &\geq \inf_{k \in \mathbb{N}} |u(k)| \int_0^1 \left(\sum_{n=1}^{\infty} w_n \left(\sum_{k=1}^{\infty} e_{k,n} x_k \right)^p \right)^{\frac{1}{p}} d\mu(\alpha) \\ &= \inf_{k \in \mathbb{N}} |u(k)| \int_0^1 \|E^t(\alpha)x\|_{p,w,I} d\mu(\alpha) \\ &\geq \inf_{k \in \mathbb{N}} |u(k)| \left(\int_0^1 \alpha^{\frac{1-p}{p}} d\mu(\alpha) \right) \|x\|_{p,w,I}. \end{aligned}$$

Therefore, for any real number $\alpha > 0$, we have

$$\|M_u H^t(\mu)x\|_{p,w+\alpha,I} \geq \inf_{k \in \mathbb{N}} |u(k)| \left(\int_0^1 \theta^{\frac{1-p}{p}} d\mu(\theta) \right) \|x\|_{p,w+\alpha,I} \quad \dots(1)$$

for all non-negative sequence x in $l_p(w, I)$. Thus we have

$$\|M_u H^t(\mu)\|_{p,w+\alpha,I} \geq \inf_{k \in \mathbb{N}} |u(k)| \left(\int_0^1 \theta^{\frac{1-p}{p}} d\mu(\theta) \right)$$

Let $\rho > \frac{1}{p}$ and n be a fixed integer such that $n \geq \rho$. We define x by

$$x_k = \begin{cases} 0, & \text{if } k < n \\ \frac{\binom{k-\rho}{k-n}}{\binom{k}{n}}, & \text{if } k \geq n \end{cases}$$

Since

$$x_k = \frac{(k - \rho) \cdots (n + 1 - \rho)}{k \cdots (n + 1)} \sim k^{-\rho},$$

where $x \sim y$ means that there exist a constant $c > 0$ such that $\frac{1}{c}x \leq y \leq cx$. When $k \rightarrow \infty$, it follows that $\|x\|_p < \infty$ and $\|x\|_p \rightarrow \infty$ when $\rho \rightarrow \frac{1}{p}$. Since w is decreasing and also for all k , $w_{k+\alpha} \geq \alpha$ then we have,

$$\alpha^{1/p} \|x\|_p \leq \|x\|_{p,w+\alpha,I} \leq (w_1 + \alpha)^{1/p} \|x\|_p.$$

So $\|x\|_{p,w+\alpha,I} < \infty$ and $\|x\|_{p,w+\alpha,I} \rightarrow \infty$ when $\rho \rightarrow \frac{1}{p}$.

Moreover, for all $m > n$, we have

$$(M_u H^t(\mu)x)_m = u(m) \int_0^1 \theta^{p-1} d\mu(\theta).$$

Hence

$$\begin{aligned} \|M_u H^t(\mu)x\|_{p,w+\alpha,I}^p &\leq \sup_k |u(k)| \left(\sum_{m=1}^n (w_m + \alpha) \left(|M_u \sum_{k=1}^{\infty} h_{k,m}(x_k)| \right)^p \right. \\ &\quad \left. + \sum_{m=n+1}^{\infty} (w_m + \alpha) (|H^t(\mu)M_u x|_m^p) \right) \\ &\leq \sup_k |u(k)| \left(n(w_1 + \alpha) \sup_{k,m} |h_{k,m}|^p \|x\|_1^p \right. \\ &\quad \left. + \left(\int_0^1 \theta^{p-1} d\mu(\theta) \right)^p \|x\|_{p,w+\alpha,I}^p \right) \end{aligned}$$

and also

$$\begin{aligned} \|M_u H^t(\mu)\|_p^p &\leq \frac{\sup_k |u(k)| n(w_1 + \alpha) \sup_{k,m} |h_{k,m}|^p \|x\|_1^p}{\|x\|_{p,w+\alpha,I}^p} \\ &\quad + \sup_k |u(k)| \left(\int_0^1 \theta^{p-1} d\mu(\theta) \right)^p. \end{aligned}$$

If $\rho \rightarrow \frac{1}{p}$, then

$$L_{p,w+\alpha,I}(H^t(\mu)M_u) \leq \sup_k |u(k)| \left(\int_0^1 \theta^{\frac{1}{p}-1} d\mu(\theta) \right)^p.$$

□

Corollary 2. Let $p > 0$ and $u : \mathbb{N} \rightarrow \mathbb{C}$ be a measurable function. Then

$$\|H^t(\mu)M_u(x)\|_{p,w,I} \geq \inf |u(x)| \left(\int_0^1 \theta^{\frac{1-p}{p}} d\mu(\theta) \right) \|x\|_{p,w,I}$$

for every non-negative sequence x . The constant is the best possible.

Proof. Since $0 < p^* < 1$, by applying Theorem 1, we establish the statement. \square

Corollary 3. Suppose $p > 0$ and let $H(\mu)$ be the Hausdorff matrix operator. Then

$$\|H^t(\mu)(x)\|_{p,w,I} \geq \left(\int_0^1 \theta^{\frac{1-p}{p}} d\mu(\theta) \right) \|x\|_{p,w,I}$$

for every non-negative sequence x . The constant is the best possible.

Proof. Follows by taking $|u(x)| = x$ in Corollary 2. \square

Corollary 4. Let $0 < p \leq 1$ and $u : \mathbb{N} \rightarrow \mathbb{C}$ be a measurable function. Then

$$\|H^t(\mu)M_u\|_p \geq \inf |u(x)| \left(\int_0^1 \theta^{\frac{1-p}{p}} d\mu(\theta) \right) \|x\|_p$$

for every non - negative sequence x . The constant is the best possible.

Proof. By taking $w_n = 1$ for all n in Corollary 3, we have the above result. \square

Corollary 5. Suppose $0 < p \leq 1$ and $H(\mu)$ is the Hausdorff matrix operator. Then

$$\|H^t(\mu)(x)\|_p \geq \int_0^1 \theta^{\frac{1-p}{p}} d\mu(\theta) \|x\|_p$$

for every non -negative sequence x . The constant is the best possible.

Proof. Again proof follows by taking $|u(x)| = x$ in Corollary 4. \square

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