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WEYL'S THEOREM FOR A CLASS OF NON-PARANORMAL OPERATORS

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Abstract: For a bounded linear operator T on a separable complex infinite dimensional Hilbert space H, we say that T is class (p, k)operator if $T^{*k+2}T^{k+2} - 2\lambda T^{*k+1}T^{k+1} + \lambda^2 T^{*k}T^k \geq 0$ for all $\lambda \in \mathcal{R}$ and for a fixed integer $k \geq 1$. In this paper, we obtain some properties of this operator and prove that if T is a class (p, k) operator and f is an analytic function on an open neighbourhood of the spectrum of T, then $f(T)$ satisfies Weyl's theorem.

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Key Words: paranormal operators, class (p, k) operators, nilpotent operators, spactral picture, Weyl's theorem

1. Introduction

In [17] H. Weyl proved, for Hermitian operators on Hilbert space, his celebrated theorem on the structure of the spectrum viz the complement in the spectrum of the Weyl spectrum coincides with the isolated points of the spectrum which are eigen values of finite multiplicity given by the equation

$$\sigma(T)\backslash w(T) = \pi_{00}(T)$$

This result has been extended from Hermitian operators to hyponormal [4], several class of operators including semi - normal [2] [3], p - hyponormal [6], p - quasihyponormal[16], class A [15], quasiclass A [7], class A(k) [14] and paranormal operators [5]. The largest class for which Weyl's theorem holds is paranormal operators. We introduce a new class (p, k) larger than paranormal operator class and show that this class also satisfies Weyl's theorem.

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In this paper, let H be a separable complex infinite dimensional Hilbert space with inner product $\langle \cdot, \cdot \rangle$. Let $B(H)$ denote the C^*- algebra of all bounded linear operator on H. We say that $T \in B(H)$ is paranormal if $\|T^2x\|\|x\| \geq \|Tx\|^2$ for all x in H. In [1], T. Ando showed that T is paranormal if and only if $T^{*2}T^2 - 2\lambda T^*T + \lambda^2 \geq 0$ for all $\lambda \in \mathcal{R}$. Motivated by the work of In Hyoun Kim [9] in which class A is extended to quasi - class (A, k), we extend the paranormal class to class (p, k).

Definition 1.1. An operator $T \in B(H)$ belongs to class (p, k) if it satisfies the following operator inequality:

$$T^{*k+2}T^{k+2} - 2\lambda T^{*k+1}T^{k+1} + \lambda^2 T^{*k}T^k \geq 0$$

for all $\lambda \in \mathcal{R}$ and for a fixed integer $k \geq 1$.

Example 1.2. Consider the unilateral weighted shift operator W on l^2 given by

$$We_n = \alpha_n e_{n+1} \text{ for all } n \geq 0$$

where $\{e_n\}_0^\infty$ is the canonical orthonormal basis for l^2. It is easy to verify that W is paranormal if and only if W is hyponormal if and only if $|\alpha_0| \leq |\alpha_1| \leq |\alpha_2| \leq .....$ Further simple calculation show that W is of class (p, k) if and only if $|\alpha_k| \leq |\alpha_{k+1}| \leq |\alpha_{k+2}| \leq ....$ and $\alpha_0, \alpha_1, ...\alpha_{k-1}$ are arbitrary. So, if we take $\alpha_0 = \alpha_1 = ... = \alpha_{k-1} = 2$ and $\alpha_i = \frac{1}{2}$ for $i \geq 1$, we see that W is of class (p, k) but is not a normaloid because $\|W\| = 2 \neq 1 = r(W)$.

2. Weyl's Theorem for $f(T)$

If $T \in B(H)$, we write ker$T$ and ran$T$ for the null space and range of T, respectively. An operator $T \in B(H)$ is called upper semi - Fredholm if it has closed range and finite dimensional null space i.e., $\alpha(T) = \dim \ker T < \infty$. Further, we call $T \in B(H)$ as lower semi - Fredholm if it has closed range and finite Co - dimensional i.e., $\beta(T) = \dim \ker T^* < \infty$. If $T \in B(H)$ is both upper semi - Fredholm and lower semi - Fredholm, we call it Fredholm. If $T \in B(H)$ is semi - Fredholm, the index of T, denoted by ind$T$, is given by

$$\text{ind}T = \alpha(T) - \beta(T).$$

The index is an integer or $\{\pm\infty\}$.

The ascent of $T \in B(H)$, denoted by asc(T), is the least non - negative integer T such that kerT^n = kerT^{n+1} and the descent of T denoted by dsc(T) is the least non - negative integer n such that ranT^n = ranT^{n+1}. We say that $T \in B(H)$ is of finite ascent (resp., finite descent) if asc$(T - \lambda) < \infty$ (resp., dsc$(T - \lambda) < \infty$) for all $\lambda \in C$, the set of all complex numbers.

An operator $T \in B(H)$ is called Weyl if it is Fredholm of index zero. The spectrum of $T \in B(H)$ is defined by

$$\sigma(T)=\{\lambda \in C : T - \lambda \text{ is not invertible } \}.$$

The essential spectrum $\sigma_e(T)$ and the Weyl specrum $w(T)$ are defined by

$$\sigma_e(T)=\{\lambda \in C : T-\lambda \text{ is not Fredholm }\}$$

and

$$w(T)=\{\lambda \in C : T-\lambda \text{ is not Weyl }\}.$$

The set of isolated points and accumulation points of $\sigma(T)$ are denoted by iso$\sigma(T)$ and acc$\sigma(T)$ respectively.

Let $\pi_{00}(T)=\{\lambda \in C/\lambda \in \text{iso}\sigma(T) \text{ and } 0 < \alpha(T-\lambda) < \infty \}$ denote the set of isolated eigen values of finite multiplicity.

We say that $T \in B(H)$, satisfies Weyl's theorem if

$$\sigma(T)\backslash w(T) = \pi_{00}(T)$$

Let $H(\sigma(T))$ be the set of analytic function on an open neighbourhood of $\sigma(T)$. If $T \in B(H)$ and $f \in H(\sigma(T))$, then Weyl's theorem holds for $f(T)$ if T is hyponormal [12] or p - quasihyponormal [16] or class A [15] or quasi - class A operator [7] or class A(k) [14] or paranormal [5]. In this paper, we show that if $T \in B(H)$ is class (p, k) operator and $f \in H(\sigma(T))$, then Weyl's theorem holds for $f(T)$.

Lemma 2.1. *If $T \in B(H)$ is class (p, k) operator and T does not have dense range, then T has the following matrix representation*

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix} \text{ on } H = \overline{ranT^k} \oplus kerT^{*k}$$

where T_1 is paranormal on $\overline{ranT^k}$ and T_3 is nilpotent with nilpotency k. Further $\sigma(T) = \sigma(T_1) \cup \{0\}$.

Proof. Let $T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$ on $H = \overline{ranT^k} \oplus kerT^{*k}$ and P be the orthogonal projection of T onto $\overline{ranT^k}$. Then $\begin{pmatrix} T_1 & 0 \\ 0 & 0 \end{pmatrix} = TP = PTP$.

Since T is class (p, k) operator, we have

$$P(T^{*2}T^2 - 2\lambda T^*T + \lambda^2)P \geq 0 \text{ for all } \lambda \in \mathcal{R}.$$

Since $P^2 = P$ and $P \leq I$ we have

$$P(T^{*2}T^2)P - 2\lambda P(T^*T)P + \lambda^2 \geq 0$$

which shows $T_1^{*2}T_1^2 - 2\lambda T_1^*T_1 + \lambda^2 \geq 0$ for all $\lambda \in \mathcal{R}$ and so T_1 is paranormal on $\overline{ranT^k}$.

Further, for any $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in H$, we have

$$\begin{aligned} \langle T_3^k x_2, x_2 \rangle &= \langle T^k(I-P)x, (I-P)x \rangle \\ &= \langle (I-P)x, T^{*k}(I-P)x \rangle \\ &= 0 \end{aligned}$$

showing that $T_3^k = 0$.

By [8, Corollary 7], $\sigma(T_1) \cup \sigma(T_3) = \sigma(T) \cup G$ where G is the union of certain holes in $\sigma(T)$ which is a subsets of $\sigma(T_1) \cap \sigma(T_3)$. Further $\sigma(T_1) \cap \sigma(T_3)$ has no interior points. So we have

$$\sigma(T) = \sigma(T_1) \cup \sigma(T_3) = \sigma(T_1) \cup \{0\}.$$

□

Corollary 2.2. *If $T \in B(H)$ is class (p, k) operator and T_1 is invertible, then T is similar to a direct sum of a paranormal operator and a nilpotent operator.*

Proof. By assumption $0 \notin \sigma(T)$ and so $\sigma(T_1) \cap \sigma(T_3) = \phi$. Then, by Rosenblum's corollary there exists an operator S such that $T_1S - ST_3 = T_2$. So,

$$\begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix} = \begin{pmatrix} I & S \\ 0 & I \end{pmatrix}^{-1} \begin{pmatrix} T_1 & 0 \\ 0 & T_3 \end{pmatrix} \begin{pmatrix} I & S \\ 0 & I \end{pmatrix}$$

and hence the proof. □

Lemma 2.3. *If $T \in B(H)$ is class (p, k), then T is an isoloid.*

Proof. Let $T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$ on $H = \overline{ranT^k} \oplus kerT^{*k}$. Let $\lambda_0 \in \text{iso}\sigma(T)$. Since $\sigma(T) = \sigma(T_1) \cup \{0\}$, it follows that $\lambda_0 \in \text{iso}\sigma(T_1)$ or $\lambda_0 = 0$. If $\lambda_0 \in \text{iso}\sigma(T_1)$, then $\lambda_0 \in \sigma_p(T_1)$ because T_1 is a paranormal operator and every paranormal operator is an isoloid [5]. So, we assume that $\lambda_0 = 0$ and $\lambda_0 \notin \sigma(T_1)$. Then dim $\ker T_3 > 0$. Thus, if $x \in \ker T_3$ then $-T_1^{-1}T_2x \oplus x \in kerT$. Hence λ_0 is an eigen value of T and the proof is complete. □

Lemma 2.4. *If $T \in B(H)$ is class (p, k) and $(T - \mu)x = 0$ for $\mu \neq 0$ and $x \in H$, then $(T - \mu)^*x = 0$.*

Proof. Since T is class (p, k), we have

$$\langle (T^{*k+2}T^{k+2} - 2\lambda T^{*k+1}T^{k+1} + \lambda^2 T^{*k}T^k)x, x\rangle \geq 0$$

Since $Tx = \mu x$, this inequality reduces to $\langle (T^{*2}T^2 - 2\lambda T^*T + \lambda^2)x, x\rangle \geq 0$ and so T is paranormal. Then for each unit vector x, $\|Tx\|^2 \leq \|T^2x\|\|x\|$ and so $\ker T = \ker T^2$ which shows $\ker(T - \mu) \subset \ker(T - \mu)^*$. □

Lemma 2.5. *If $T \in B(H)$ is class (p, k), then T is of finite ascent.*

Proof. We show that $\ker(T-\mu)^{k+1} = \ker(T-\mu)^{k+2}$. By Lemma 2.4, $(T-\mu)x = 0$ implies $(T-\mu)^*x = 0$ for each $\mu \neq 0$. So, it follows that $\ker(T-\mu) = \ker(T-\mu)^2$ for $\mu \neq 0$. So, it sufficies to show that $\ker T^{k+1} = \ker T^{k+2}$.

Assume $\|T^{k+2}x\| = 0$. Since T is class (p, k), we have

$$\|T^{k+2}x\|^2 - 2\lambda\|T^{k+1}x\|^2 + \lambda^2\|T^kx\|^2 \geq 0 \text{ for all } \lambda \in \mathcal{R}$$

so that $\lambda^2\|T^kx\|^2 \geq 2\lambda\|T^{k+1}x\|^2$ for all $\lambda \in \mathcal{R}$ showing that $T^{k+1}x = 0$. Thus $\ker T^{k+2} \subset \ker T^{k+1}$. □

Lemma 2.6. *[11, Theorem 6] For A, B $C \in B(H)$, we have $w(A) \cup w(B) = w(M_c) \cup G$ where $M_c = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$ and G is the union of certain holes in $w(M_c)$ which is a subset of $w(A) \cup w(B)$.*

It is known that the spectral mapping theorem for Weyl spectrum holds for paranormal operators [5]. We now extend this result to class (p, k) operators.

Theorem 2.7. *If $T \in B(H)$ is class (p, k) operator, then $w(f(T)) = f(w(T))$ for every analytic function f on a neighborhood of $\sigma(T)$.*

Proof. For any polynomial p, let us prove $w(p(T)) = p(w(T))$.
Since spectral mapping theorem holds for paranormal operator and

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix},$$

where T_1 is paranormal and T_3 is nilpotent, we have

$$\begin{aligned} w(p(T)) &= w(p(T_1)) \cup w(p(T_3)) \\ &= p(w(T_1)) \cup p(w(T_3)) \\ &= p(w(T_1) \cup w(T_3)) \\ &= p(w(T)) \text{ as desired.} \end{aligned}$$

□

Definition 2.8. [13] The spectral picture of the operator $T \in B(H)$, denoted by $SP(T)$, consists of the set $\sigma_e(T)$, the collection of holes and pseudohole in $\sigma_e(T)$ and the indices associated with these holes and pseudoholes.

Lemma 2.9. *[11, Theorem 2.4] If either $SP(A)$ or $SP(B)$ has no pseudoholes and if A is an isoloid operator for which Weyl's theorem holds for every $C \in B(H)$, Weyl's theorem holds for $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \Rightarrow w\begin{pmatrix} A & C \\ 0 & B \end{pmatrix} = w(A) \cup w(B)$*

Corollary 2.10. *Weyl's theorem holds for class (p, k) operators.*

Proof. Let $T \in B(H)$ be of class (p, k). By Lemma 2.1, T has the matrix representation

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix} \text{ on } \overline{ranT^k} \oplus kerT^{*k}$$

where T_1 is paranormal on $\overline{ranT^k}$ and T_3 is nilpotent with nilpotency k.
Since Weyl's theorem holds for paranormal operators and nilpotent operators and both are isoloids, we find that Weyl's theorem holds for $\begin{pmatrix} T_1 & 0 \\ 0 & T_3 \end{pmatrix}$. Since

$SP(T_3)$ has no pseudoholes, it follows that Weyl's theorem holds for $\begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$ by Lemma 2.9. □

Theorem 2.11. *Let $T \in B(H)$ is class (p, k) then $f(T)$ satisfies Weyl's theorem for every analytic functions f on an neighborhood of $\sigma(T)$.*

Proof. By [10, Corollary 2.3.12], if
i) Weyl's theorem hold for T.
ii) T is isoloid.
iii) T satisfies the spectral mapping theorem for Weyl's spectrum, for $T \in B(H)$, then Weyl's theorem holds for $f(T)$, for every $f \in H(\sigma(T))$. So by corollary 2.10, Lemma 2.3 and Theorem 2.7 we get the result. □

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