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EXISTENCE OF SOLUTIONS FOR NONCONVEX n-TH ORDER DIFFERENTIAL INCLUSIONS

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Abstract: We prove an existence result for the nth order differential inclusion

$$x^{(n)} \in F\left(x, x', x'', \ldots, x^{(n-1)}\right) + f\left(t, x, x', \ldots, x^{(n-1)}\right),$$

with initial conditions

$$x(0) = a_0,\ \ x'(0) = a_1, \ldots, x^{(n-1)}(0) = a_{n-1},$$

where f is a Carathéodory function and where F is a compact valued upper semicontinuous multifunction such that $F(x_0, x_1, ..., x_{n-1}) \subset \partial V(x_{n-1})$ for some lower semicontinuous proper convex function V.

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1. Introduction

Lupulescu [7] showed that if Ω is an open set in $\mathbb{R}^{2m}$ and $F : \Omega \to \mathcal{P}(\mathbb{R}^m)$ is a compact valued upper semicontinuous multifunction with the property that there exists a proper convex and lower semicontinuous function $V : \mathbb{R}^m \to \mathbb{R}$ with

$$F(x, y) \subset \partial V(y),\ \ (x, y) \in \Omega,$$

and if f is a Carathéodory function, then there exists a solution to the differential inclusion

$$x'' \in F(x, x') + f(t, x, x'),\ \ x(0) = x_0,\ \ x'(0) = y_0.$$

By a solution it is meant that there is a $T > 0$ and an absolutely continuous function

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$x : [0,T] \to \mathbb{R}^m$ with absolutely continuous derivative x' such that $x(0) = x_0$, $x'(0) = y_0$, and

$$x''(t) \in F\left(x(t), x'(t)\right) + f\left(t, x(t), x'(t)\right), \text{ a.e. on } [0,T].$$

Hopkins [5] extended a variation of this result (without the Carathéodory function) to 3rd order differential inclusions.

In this paper we extend Lupulescu's result to a class of nth order differential inclusions. We define a sequence of approximate solutions on a certain interval and show that a subsequence converges to a solution.

2. Preliminaries

For topological spaces X and Y, a multifunction $F : X \to \mathcal{P}(Y)$, where $\mathcal{P}(Y)$ is the power set of Y, is said to be *compact valued* if $F(x)$ is a compact subset of Y for every $x \in X$. It is called *upper semicontinuous* if for every $x \in X$ and for every open set $N \supset F(x)$ there exists an open neighborhood M of x such that $F(M) \subset N$. The *graph* of F is:

$$\text{Graph}(F) = \{(x,y) \in X \times Y : y \in F(x)\}.$$

If X is compact, and if $F : X \to \mathcal{P}(Y)$ is a compact valued upper semicontinuous multifunction, then it is straightforward to check that

$$F(X) = \bigcup_{x \in X} F(x)$$

is compact in Y (see Proposition 1.1.3 in [2]).

For a lower semicontinuous (in the sense of ordinary functions), proper convex function $V : \mathbb{R}^m \to \mathbb{R}$, we define a multifunction $\partial V : \mathbb{R}^m \to \mathcal{P}(\mathbb{R}^m)$, called the *subdifferential* of V, as follows:

$$\partial V(x) = \{z \in \mathbb{R}^m : V(y) - V(x) \geq \langle z, y - x\rangle, \forall y \in \mathbb{R}^m\}.$$

It may be verified that $\partial V(x)$ is always a nonempty convex compact subset of $\mathbb{R}^m$.

A *Carathéodory function* is a function of the form $f : \mathbb{R} \times \prod_{j=0}^{n-1} \mathbb{R}^m \to \mathbb{R}^m$ that has the following properties:

(i) $t \mapsto f(t, x_0, x_1, \ldots, x_{n-1})$ is measurable for every $x_0, x_1, \ldots, x_{n-1} \in \mathbb{R}^m$;

(ii) $(x_0, x_1, \ldots, x_{n-1}) \mapsto f(t, x_0, x_1, \ldots, x_{n-1})$ is continuous for every $t \in \mathbb{R}$;

(iii) there exists $m \in L^2(\mathbb{R})$ so that for every $x_0, x_1, \ldots, x_{n-1} \in \mathbb{R}^m$ and for almost every $t \in \mathbb{R}$,

$$\|f(t, x_0, x_1, \ldots, x_{n-1})\| \leq m(t).$$

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3. The Main Result

Theorem 1. *Let Ω be an open set in $\mathbb{R}^{nm}$ where $n \in \mathbb{N}\backslash\{1\}$. Assume the following:*

(H1) $F : \Omega \to \mathcal{P}(\mathbb{R}^m)$ *is a compact valued upper semicontinuous multifunction.*

(H2) There exists a lower semicontinuous proper convex function $V : \mathbb{R}^m \to \mathbb{R}$ *such that* $F(x_0, x_1, ..., x_{n-1}) \subset \partial V(x_{n-1})$ *for every* $(x_0, x_1, ..., x_{n-1}) \in \Omega$.

(H3) $f : \mathbb{R} \times \prod_{j=0}^{n-1} \mathbb{R}^m \to \mathbb{R}^m$ *is a Carathéodory function.*

Then, for every $(a_0, a_1, \ldots, a_{n-1}) \in \Omega$, *there exists a solution to the differential inclusion*

$$x^{(n)}(t) \in F\left(x(t), x'(t), ..., x^{(n-1)}(t)\right) + f\left(t, x(t), x'(t), \ldots, x^{(n-1)}(t)\right) \quad (1)$$

with initial conditions

$$x(0) = a_0, x'(0) = a_1, ..., x^{(n-1)}(0) = a_{n-1}. \quad (2)$$

Note that by a solution we mean an absolutely continuous function $x : [0, T] \to \mathbb{R}^m$ having $n-1$ absolutely continuous derivatives that satisfies (1) for almost every $t \in [0, T]$ along with the initial conditions (2).

Proof. Suppose $\bar{a} = (a_0, a_1, ..., a_{n-1}) \in \Omega$. Since Ω is open, there exists $r > 0$ such that $\overline{B}_r(\bar{a}) \subset \Omega$. Since $\overline{B}_r(\bar{a})$ is compact and F is compact valued, the set

$$F(\overline{B}_r(\bar{a})) = \bigcup_{\bar{x} \in \overline{B}_r(\bar{a})} F(\bar{x})$$

is compact. Hence there exists $M^* > r$ such that

$$\sup\{\|v\| : v \in F(\overline{B}_r(\bar{a}))\} \leq M^*.$$

Let

$$M = \max\{\|a_0\|, \|a_1\|, ..., \|a_{n-1}\|, M^*\}$$

and

$$T_1 = \frac{r}{e\sqrt{n}M}.$$

By (H3), f is a Carathéodory function, so there exists $m \in L^2(\mathbb{R})$ so that for every $x_0, x_1, \ldots, x_{n-1} \in \mathbb{R}^m$ and for almost every $t \in \mathbb{R}$,

$$\|f(t, x_0, x_1, \ldots, x_{n-1})\| \leq m(t).$$

Since $m \in L^2(\mathbb{R})$, then $m \in L^1_{loc}(\mathbb{R})$, so there exists $T_2 > 0$ such that

$$\int_0^{T_2} (m(s) + M)ds < r. \tag{3}$$

We shall prove the existence of a solution of (1) defined on the interval $[0, T]$, where $T = \min\{T_1, T_2\} < 1$.

Step 1: Construction of a sequence of approximate solutions.

For integers k and q with $1 \leq q \leq n$, set

$$x_{0,k}^{q-1} = a_{q-1} \tag{4}$$

and let $v_{0,k} \in F(x_{0,k}^0, x_{0,k}^1, ..., x_{0,k}^{n-1})$. For $0 \leq j \leq k-1$ recursively define

$$x_{j+1,k}^{n-q} = \sum_{i=0}^{q-1} \frac{1}{i!}\left(\frac{T}{k}\right)^i x_{j,k}^{n-(q-i)} + \frac{1}{q!}\left(\frac{T}{k}\right)^q v_{j,k}, \tag{5}$$

where each $v_{j,k}$ is chosen so that $v_{j,k} \in F(x_{j,k}^0, x_{j,k}^1, ..., x_{j,k}^{n-1})$. Let $t_{j,k} = \frac{Tj}{k}$, and for $t \in [t_{j,k}, t_{j+1,k}]$, define

$$\begin{aligned} x_k(t) &= \sum_{i=0}^{n-1} \frac{1}{i!}(t - t_{j,k})^i x_{j,k}^i + \frac{1}{n!}(t - t_{j,k})^n v_{j,k} \\ &+ \int_{t_{j,k}}^t \frac{1}{(n-1)!}(t-s)^{n-1} f(s, x_{j,k}^0, x_{j,k}^1, \ldots, x_{j,k}^{n-1})ds. \end{aligned} \tag{6}$$

Set $\bar{x}_{j,k} = (x_{j,k}^0, x_{j,k}^1, \ldots, x_{j,k}^{n-1})$. Our goal is to show that $\bar{x}_{j,k} \in \bar{B}_r(\bar{a})$. We begin with the following claim.

Claim: $\bar{x}_{1,k} \in \overline{B}_r(\bar{a})$.

To prove our claim, we use (5), the definition of M, and our choice of T, and note that,

$$\begin{aligned} \|x_{1,k}^{n-q} - a_{n-q}\| &= \left\|\sum_{i=0}^{q-1} \frac{1}{i!}\left(\frac{T}{k}\right)^i x_{0,k}^{n-(q-i)} + \frac{1}{q!}\left(\frac{T}{k}\right)^q v_{0,k} - a_{n-q}\right\| \\ &= \left\|x_{0,k}^{n-q} + \sum_{i=1}^{q-1} \frac{1}{i!}\left(\frac{T}{k}\right)^i x_{0,k}^{n-(q-i)} + \frac{1}{q!}\left(\frac{T}{k}\right)^q v_{0,k} - a_{n-q}\right\| \end{aligned}$$

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$$
\begin{aligned}
&= \left\| \sum_{i=1}^{q-1} \frac{1}{i!} \left(\frac{T}{k}\right)^i x_{0,k}^{n-(q-i)} + \frac{1}{q!}\left(\frac{T}{k}\right)^q v_{0,k} \right\| \\
&\leq \sum_{i=1}^{q-1} \frac{1}{i!}\left(\frac{T}{k}\right)^i M + \frac{1}{q!}\left(\frac{T}{k}\right)^q M^* \\
&\leq M \sum_{i=1}^{q} \frac{1}{i!}\left(\frac{T}{k}\right)^i \\
&\leq M \sum_{i=0}^{q} \frac{1}{i!} T \\
&\leq \frac{r}{e\sqrt{n}} \sum_{i=0}^{q} \frac{1}{i!} \\
&< \frac{r}{\sqrt{n}}.
\end{aligned}
$$

We then have

$$
\|\bar{x}_{1,k} - \bar{a}\|_F^2 = \sum_{q=1}^{n} \|x_{1,k}^{n-q} - a_{n-q}\|^2 < n\left(\frac{r^2}{n}\right) = r^2
$$

where $\|\cdot\|_F$ denotes the Frobenius norm (i.e. the Euclidean norm on $\mathbb{R}^{nm}$). Hence $\|\bar{x}_{1,k} - \bar{a}\|_F < r$, which gives us that $\bar{x}_{1,k} \in \overline{B}_r(\bar{a})$.

Claim: Sequence (5) may be rewritten as

$$
x_{j,k}^{n-q} = \sum_{\alpha=0}^{q-1} \frac{1}{\alpha!}\left(\frac{Tj}{k}\right)^\alpha x_{0,k}^{n-(q-\alpha)} + \frac{1}{q!}\left(\frac{T}{k}\right)^q \sum_{\gamma=0}^{j-1} c_{\gamma,q} v_{j-1-\gamma,k}, \tag{7}
$$

where $c_{\gamma,q} = (\gamma+1)^q - \gamma^q$.

We verify the claim using induction. For $j = 1$,

$$
\begin{aligned}
x_{1,k}^{n-q} &= \sum_{i=0}^{q-1} \frac{1}{i!}\left(\frac{T}{k}\right)^i x_{0,k}^{n-(q-i)} + \frac{1}{q!}\left(\frac{T}{k}\right)^q v_{0,k} \\
&= \sum_{\alpha=0}^{q-1} \frac{1}{\alpha!}\left(\frac{Tj}{k}\right)^\alpha x_{0,k}^{n-(q-\alpha)} + \frac{1}{q!}\left(\frac{T}{k}\right)^q \sum_{\gamma=0}^{j-1} c_{\gamma,q} v_{j-1-\gamma,k}.
\end{aligned}
$$

Next, note that

$$
\sum_{i=0}^{q-1} \frac{1}{i!}\left(\frac{T}{k}\right)^i \left[\sum_{\alpha=0}^{q-i-1} \frac{1}{\alpha!}\left(\frac{Tj}{k}\right)^\alpha x_{0,k}^{n-(q-i-\alpha)}\right]
$$

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$$
\begin{aligned}
&= \sum_{i=0}^{q-1}\sum_{\beta=1}^{q-i} \frac{1}{i!(q-i-\beta)!} j^{q-i-\beta} \left(\frac{T}{k}\right)^{q-\beta} x_{0,k}^{n-\beta} \\
&= \sum_{\beta=1}^{q} \frac{1}{(q-\beta)!} \left[\sum_{i=0}^{q-\beta} \binom{q-\beta}{i} j^{q-\beta-i}\right] \left(\frac{T}{k}\right)^{q-\beta} x_{0,k}^{n-\beta} \\
&= \sum_{\beta=1}^{q} \frac{1}{(q-\beta)!} (j+1)^{q-\beta} \left(\frac{T}{k}\right)^{q-\beta} x_{0,k}^{n-\beta} \\
&= \sum_{\alpha=0}^{q-1} \frac{1}{\alpha!} \left[\frac{T(j+1)}{k}\right]^{\alpha} x_{0,k}^{n-(q-\alpha)} \qquad (8)
\end{aligned}
$$

and

$$
\begin{aligned}
&\sum_{i=0}^{q-1} \frac{1}{i!}\left(\frac{T}{k}\right)^{i} \left[\frac{1}{(q-i)!}\left(\frac{T}{k}\right)^{q-i} \sum_{\gamma=0}^{j-1} c_{\gamma,q-i} v_{j-1-\gamma,k}\right] + \frac{1}{q!}\left(\frac{T}{k}\right)^{q} v_{j,k} \\
&\quad= \frac{1}{q!}\left(\frac{T}{k}\right)^{q} \sum_{i=0}^{q-1}\sum_{\gamma=0}^{j-1} \binom{q}{i} c_{\gamma,q-i} v_{j-1-\gamma,k} + \frac{1}{q!}\left(\frac{T}{k}\right)^{q} v_{j,k} \\
&\quad= \frac{1}{q!}\left(\frac{T}{k}\right)^{q} \sum_{\gamma=0}^{j-1} \left(\sum_{i=0}^{q-1} \binom{q}{i} \left[(\gamma+1)^{q-i} - \gamma^{q-i}\right]\right) v_{j-1-\gamma,k} + \frac{1}{q!}\left(\frac{T}{k}\right)^{q} v_{j,k} \\
&\quad= \frac{1}{q!}\left(\frac{T}{k}\right)^{q} \sum_{\gamma=0}^{j-1} \left(\sum_{i=0}^{q} \binom{q}{i} \left[(\gamma+1)^{q-i} - \gamma^{q-i}\right]\right) v_{j-1-\gamma,k} + \frac{1}{q!}\left(\frac{T}{k}\right)^{q} v_{j,k} \\
&\quad= \frac{1}{q!}\left(\frac{T}{k}\right)^{q} \sum_{\gamma=0}^{j-1} c_{\gamma+1,q} v_{j-1-\gamma,k} + \frac{1}{q!}\left(\frac{T}{k}\right)^{q} v_{j,k} \\
&\quad= \frac{1}{q!}\left(\frac{T}{k}\right)^{q} \sum_{\gamma=0}^{j} c_{\gamma,q} v_{j-\gamma,k}. \qquad (9)
\end{aligned}
$$

Now assuming equality for j and using (5) and (8) and (9), we see

$$
\begin{aligned}
x_{j+1,k}^{n-q} &= \sum_{i=0}^{q-1} \frac{1}{i!}\left(\frac{T}{k}\right)^{i} x_{j,k}^{n-(q-i)} + \frac{1}{q!}\left(\frac{T}{k}\right)^{q} v_{j,k} \\
&= \sum_{i=0}^{q-1} \frac{1}{i!}\left(\frac{T}{k}\right)^{i} \left[\sum_{\alpha=0}^{q-i-1} \frac{1}{\alpha!} j^{\alpha} \left(\frac{T}{k}\right)^{\alpha} x_{0,k}^{n-(q-i-\alpha)}\right] \\
&\quad + \sum_{i=0}^{q-1} \frac{1}{i!}\left(\frac{T}{k}\right)^{i} \left[\frac{1}{(q-i)!}\left(\frac{T}{k}\right)^{q-i} \sum_{\gamma=0}^{j-1} c_{\gamma,q-i} v_{j-1-\gamma,k}\right] + \frac{1}{q!}\left(\frac{T}{k}\right)^{q} v_{j,k}
\end{aligned}
$$

$$=\sum_{\alpha=0}^{q-1}\frac{1}{\alpha!}\left[\frac{T(j+1)}{k}\right]^{\alpha}x_{0,k}^{n-(q-\alpha)}+\frac{1}{q!}\left(\frac{T}{k}\right)^{q}\sum_{\gamma=0}^{j}c_{\gamma,q}v_{j-\gamma,k},$$

thus verifying (7).

□

Claim: $\bar{x}_{j,k}\in\overline{B}_r(\bar{a})$ for each j.

Using a process similar to the first claim (where $j=1$), we have:

$$\begin{aligned}\left\|x_{j,k}^{n-q}-a_{n-q}\right\| &= \left\|\sum_{\alpha=0}^{q-1}\frac{1}{\alpha!}\left(\frac{Tj}{k}\right)^{\alpha}x_{0,k}^{n-(q-\alpha)}+\frac{1}{q!}\left(\frac{T}{k}\right)^{q}\left[\sum_{\gamma=0}^{j-1}c_{\gamma,q}v_{j-1-\gamma,k}\right]-a_{n-q}\right\|\\ &\le \sum_{\alpha=1}^{q-1}\frac{1}{\alpha!}\left(\frac{Tj}{k}\right)^{\alpha}\left\|x_{0,k}^{n-(q-\alpha)}\right\|+\frac{M^*}{q!}\left(\frac{Tj}{k}\right)^{q}\\ &\le M\sum_{\alpha=1}^{q}\frac{1}{\alpha!}T\\ &< \frac{r}{\sqrt{n}}\end{aligned}$$

As before, this gives that $\|\bar{x}_{j,k}-\bar{a}\|_F<r$, and thus $\bar{x}_{j,k}\in\overline{B}_r(\bar{a})$.

Step 2: Weak Convergence of $\{x_k^{(n)}\}$.

For $t\in(t_{j,k},t_{j+1,k})$ and $j=0,\dots,k-1$ define $f_k(t)=f(t,x_{j,k}^0,x_{j,k}^1,\dots,x_{j,k}^{n-1})$. Let $d=n-q$. By (6), we have for $d=0,\dots,n-1$ and $t\in(t_{j,k},t_{j+1,k})$,

$$x_k^{(d)}(t)=\sum_{i=0}^{n-(d+1)}\frac{1}{i!}(t-t_{j,k})^i x_{j,k}^{i+d} +\frac{1}{(n-d)!}(t-t_{j,k})^{n-d}v_{j,k}+\int_{t_{j,k}}^{t}\frac{(t-s)^{n-(d+1)}}{(n-(d+1))!}f_k(s)ds. \quad (10)$$

Furthermore,

$$x_k^{(n)}(t)=v_{j,k}+f_k(t),\quad t\in(t_{j,k},t_{j+1,k}). \quad (11)$$

Then $\|x_k^{(n)}(t)\|\le\|v_{j,k}\|+\|f_k(t)\|\le M+m(t)$, for $t\in[0,T]$.

Claim: $\{x_k^{(d)}(t)\}$ is uniformly bounded for each $d=0,\dots,n-1$ and $t\in(t_{j,k},t_{j+1,k})$.

Note that for $i=0,1,\dots,n-(d+1)$,

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$$\begin{aligned}
\left\|x_{j,k}^{i+d}\right\| &= \left\|x_{j,k}^{n-(q-i)}\right\| \\
&= \left\|\sum_{\alpha=0}^{q-i-1} \frac{1}{\alpha!}\left(\frac{Tj}{k}\right)^{\alpha} x_{0,k}^{n-(q-i-\alpha)} + \frac{1}{(q-i)!}\left(\frac{T}{k}\right)^{q-i}\left[\sum_{\gamma=0}^{j-1} c_{\gamma,q-i} v_{j-1-\gamma,k}\right]\right\| \\
&\leq \sum_{\alpha=0}^{q-i-1} \frac{1}{\alpha!}\left(\frac{Tj}{k}\right)^{\alpha}\left\|x_{0,k}^{n-(q-i-\alpha)}\right\| + \frac{M^*}{(q-i)!}\left(\frac{T}{k}\right)^{q-i}\sum_{\gamma=0}^{j-1} c_{\gamma,q-i} \\
&= \sum_{\alpha=0}^{q-i-1} \frac{1}{\alpha!}\left(\frac{Tj}{k}\right)^{\alpha}\left\|x_{0,k}^{n-(q-i-\alpha)}\right\| + \frac{M^*}{(q-i)!}\left(\frac{Tj}{k}\right)^{q-i} \\
&< TMe + M.
\end{aligned}$$

Let $d = 0, \ldots, n-2$, then using the above and the fact that $T < 1$, we have

$$\begin{aligned}
\left\|x_k^{(d)}(t)\right\| \leq& \sum_{i=0}^{n-(d+1)} \frac{1}{i!}(t-t_{j,k})^i \left\|x_{j,k}^{i+d}\right\| + \frac{1}{(n-d)!}(t-t_{j,k})^{n-d}\left\|v_{j,k}\right\| \\
&+ \left\|\int_{t_{j,k}}^{t} \frac{(t-s)^{n-(d+1)}}{(n-(d+1))!} f_k(s)ds\right\| \\
\leq& \sum_{i=0}^{n-(d+1)} \frac{1}{i!}\left(\frac{T}{k}\right)^i \left\|x_{j,k}^{i+d}\right\| + \frac{1}{(n-d)!}\left(\frac{T}{k}\right)^{n-d} M^* \\
&+ \left(\frac{T}{k}\right)^{n-1-d} \frac{1}{(n-d-1)!}\int_{t_{j,k}}^{t_{j+1,k}} \|f_k(s)\| ds \\
\leq&(TMe+M)\sum_{i=0}^{n-(d+1)} \frac{1}{i!}\left(\frac{T}{k}\right)^i \\
&+ \frac{1}{(n-d-1)!}\left(\frac{T}{k}\right)^{n-1-d}\left[\frac{T}{k}M + \int_{t_{j,k}}^{t_{j+1,k}} \|f_k(s)\| ds\right] \\
\leq& M(Te+1) + \left[1 + \sum_{i=1}^{n-(d+1)} \frac{1}{i!}\left(\frac{T}{k}\right)^i\right] \\
&+ \frac{1}{(n-d-1)!}\left(\frac{T}{k}\right)^{n-1-d}\left[\int_{t_{j,k}}^{t_{j+1,k}} M + \|f_k(s)\| ds\right] \\
\leq& M(e+1)[1+Te] + \frac{T}{(n-d-1)!}\int_0^T (M+m(s))ds \\
\leq& M(e+1)[1+Te] + \frac{Tr}{(n-d-1)!}. \qquad (12)
\end{aligned}$$

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This verifies the claim.

Using similar arguments to the above inequalities, it may be shown that for $d = 0, \ldots, n-2$ we have

$$\left\| x_k^{(d)}(t) - x_{j,k}^d \right\| \leq \frac{T}{k} \left[TMe^2 + Me + \frac{r}{(n-d-1)!} \right].$$

Also, for $d = n-1$,

$$\begin{aligned} \left\| x_k^{(d)}(t) - x_{j,k}^d \right\| &= \left\| (t - t_{j,k}) v_{j,k} + \int_{t_{j,k}}^{t} f_k(s) ds \right\| \\ &\leq \frac{T}{k} M + \int_{t_{j,k}}^{t} \| f_k(s) \| \, ds \\ &\leq \frac{T}{k} M + \int_{t_{j,k}}^{t} m(s) ds. \end{aligned}$$

Thus

$$\begin{aligned} & dist\left(\text{Graph}(F), \left(x_k(t), x_k'(t), \ldots, x_k^{(n-1)}(t), x_k^{(n)}(t) - f_k(t) \right) \right) \\ & \quad \leq \sum_{d=0}^{n-1} \left\| x_k^{(d)}(t) - x_{j,k}^d \right\| \\ & \quad \leq (n-1) \frac{T}{k} \left[TMe^2 + Me + \frac{r}{(n-d-1)!} \right] + \frac{T}{k} M + \int_{t_{j,k}}^{t} m(s) ds. \end{aligned} \tag{13}$$

Also, by (11), we have

$$\int_0^T \left\| x_k^{(n)}(t) \right\|^2 dt \leq \int_0^T (M + m(t))^2 dt.$$

Thus, the sequence $\left\{ x_k^{(n)} \right\}_{k=1}^{\infty}$ is bounded in $L^2([0,T], \mathbb{R}^m)$.

If $\tau, t \in [0,T]$, then

$$\left\| x_k^{(n-1)}(t) - x_k^{(n-1)}(\tau) \right\| \leq \left| \int_\tau^t \left\| x_k^{(n)}(s) \right\| ds \right| \leq \left| \int_\tau^t (M^* + m(s)) \, ds \right|.$$

Thus $\{x_k^{(n-1)}\}_{k=1}^{\infty}$ is equicontinuous. Also, for $d = 0, \ldots, n-2$ we have by (12) that $\{x_k^{(d)}\}_{k=1}^{\infty}$ is equi-Lipschitzian, thus equicontinuous. As a consequence of the Arzela-Ascoli Theorem (see Theorem 0.3.4 in [2]) there exists a subsequence, still denoted by $\{x_k\}_{k=1}^{\infty}$, and an absolutely continuous function $x : [0,T] :\to \mathbb{R}^m$ such that

(i) $\left\{x_k^{(d)}\right\}_{k=1}^{\infty}$ converges uniformly to $x^{(d)}$ for each $d = 0, 1, \ldots, n-1$;

(ii) $\left\{x_k^{(n)}\right\}_{k=1}^{\infty}$ converges weakly in $L^2([0,T],\mathbb{R}^m)$ to $x^{(n)}$.

Step 3: Strong Convergence of $\left\{x_k^{(n)}(t)\right\}$

Note that by (6) we have have that for each fixed $t \in [t_{j,k}, t_{j+1,k}]$,

$$\lim_{k\to\infty} x_k(t) - x_{j,k}^0 = 0.$$

Thus

$$\lim_{k\to\infty} x_{j,k}^0 = x(t).$$

Similarly, by (10), we have for each $d = 0, \ldots, n-1$ that

$$\lim_{k\to\infty} x_{j,k}^d = x^{(d)}(t).$$

Thus $\{f_k(\cdot)\}_{k=1}^{\infty}$ converges to $f(\cdot, x(\cdot), \ldots, x^{(n-1)}(\cdot))$ in $L^2([0,T],\mathbb{R}^m)$. So by (H2), (13), and Theorem 1.4.1 in [2] we have for almost every $t \in [0,T]$ that

$$x^{(n)}(t) - f\left(t, x(t), x'(t), \ldots, x^{(n-1)}(t)\right) \in \overline{co}F\left(x(t), x'(t), \ldots, x^{(n-1)}(t)\right) \subset \partial V\left(x^{(n-1)}(t)\right) \quad (14)$$

where $\overline{co}$ denotes the closed convex hull. So for almost every $t \in [0,T]$ we have by Lemma 3.3 in [3] that

$$\frac{d}{dt}V(x^{(n-1)}(t)) = \left\langle x^{(n)}(t), x^{(n)}(t) - f\left(t, x(t), x'(t), \ldots, x^{(n-1)}(t)\right)\right\rangle.$$

Thus

$$V(x^{(n-1)}(T)) - V(x^{(n-1)}(0)) = \int_0^T ||x^{(n)}(t)||^2 dt - \int_0^T \langle x^{(n)}(t), f(t, x(t), \ldots, x^{(n-1)}(t))\rangle dt. \quad (15)$$

Recall that by (11) we have, for almost every $t \in (t_{j,k}, t_{j+1,k})$, that

$$x_k^{(n)}(t) - f_k(t) = v_{j,k} \in F(x_{j,k}^0, \ldots, x_{j,k}^{n-1}) \subset \partial V(x_k^{(n-1)}(t_{j,k})).$$

So using the definition of subdifferential we have

$$V(x_k^{(n-1)}(t_{j+1,k})) - V(x_k^{(n-1)}(t_{j,k})) \geq \langle x_k^{(n)}(t) - f_k(t), x_k^{(n-1)}(t_{j+1,k}) - x_k^{(n-1)}(t_{j,k})\rangle$$

$$\begin{aligned} &= \langle v_{j,k}, \int_{t_{j,k}}^{t_{j+1,k}} x_k^{(n)}(t)dt\rangle \\ &= \int_{t_{j,k}}^{t_{j+1,k}} \langle v_{j,k}, x_k^{(n)}(t)\rangle dt \\ &= \int_{t_{j,k}}^{t_{j+1,k}} \langle x_k^{(n)}(t) - f_k(t), x_k^{(n)}(t)\rangle dt \\ &= \int_{t_{j,k}}^{t_{j+1,k}} ||x_k^{(n)}(t)||^2 dt - \int_{t_{j,k}}^{t_{j+1,k}} \langle f_k(t), x_k^{(n)}(t)\rangle dt. \end{aligned}$$

Adding the k inequalities above, we have

$$V(x_k^{(n-1)}(T)) - V(a_{n-1}) \geq \int_0^T ||x_k^{(n)}(t)||dt - \int_0^T \langle f_k(t), x_k^{(n)}(t)\rangle dt. \quad (16)$$

By the convergence of $\{f_k(\cdot)\}_{k=1}^{\infty}$ in the L^2-norm and of $\{x_k^{(n)}(\cdot)\}_{k=1}^{\infty}$ in the weak topology of L^2, we have

$$\lim_{k\to\infty} \int_0^T \langle f_k(t), x_k^{(n)}(t)\rangle dt = \int_0^T \langle f(t, x(t), x'(t), \dots, x^{(n-1)}(t)), x^{(n)}(t)\rangle dt.$$

Taking the limit as $k \to \infty$ in (16) we have by the continuity of V that

$$\begin{aligned} &V(x^{(n-1)}(T)) - V(a_{n-1}) \\ &\qquad \geq \limsup_{k\to+\infty} \int_0^T ||x_k^{(n)}(t)||^2 dt - \int_0^T \langle f(t, x(t), \dots, x^{(n-1)}(t)), x^{(n)}(t)\rangle dt. \quad (17) \end{aligned}$$

So by (15) and (17) we have

$$\|x^{(n)}(t)\|_{L^2}^2 \geq \limsup_{k\to\infty} \|x_k^{(n)}(t)\|_{L^2}^2.$$

But also, by the weak lower semicontinuity of the norm (see [4], Theorem 14.28),

$$\|x^{(n)}(t)\|_{L^2}^2 \leq \liminf_{k\to\infty} \|x_k^{(n)}(t)\|_{L^2}^2.$$

Thus,

$$\lim_{k\to\infty} \|x_k^{(n)}(t)\|_{L^2} = \|x^{(n)}(t)\|_{L^2}.$$

Since we also have that $\{x_k^{(n)}\}_{k=1}^{\infty}$ converges weakly in $L^2([0,T],\mathbb{R}^m)$ to $x^{(n)}$, then (see [4], Theorem 14.29) $\{x_k^{(n)}\}_{k=1}^{\infty}$ converges strongly in $L^2([0,T],\mathbb{R}^m)$ in $x^{(n)}$ (i.e. $\lim_{k\to\infty} \|x_k^{(n)} - x^{(n)}\|_{L^2} = 0$). So there is a subsequence, again denoted by $\{x_k^{(n)}\}_{k=1}^{\infty}$, that converges pointwise almost everywhere to $x^{(n)}$.

By (H1), F is both upper-semicontinuous and closed valued, so the graph of F is closed (see [2], Proposition 1.1.2). By (13) we have:

$$\lim_{k\to\infty} dist(\text{Graph}(F), (x_k(t), x'_k(t), \ldots, x_k^{(n-1)}(t), x_k^{(n)}(t) - f_k(t))) = 0.$$

Thus, for almost every $t \in [0, T]$,

$$x^{(n)}(t) \in F(x(t), x'(t), ..., x^{(n-1)}(t)) + f(t, x(t), x'(t), \ldots, x^{(n-1)}(t)).$$

Setting $t = 0$ in (6), letting $k \to \infty$, and using (4) yields that $x(0) = a_0$. Similarly, setting $t = 0$ in (10), letting $k \to \infty$, and using (4) gives that $x^{(d)}(0) = a_d$ for each $d = 1, \ldots, n-1$. Thus x satisfies the initial conditions (2).

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