International Electronic Journal of Pure and Applied Mathematics
Volume 3 No. 4 2011, 341-343

THE ASSUMPTIVE DEVELOPMENT OF A SUBSTITUTION FORMULA TO EVALUATE INTEGRALS OF THE FORM $\int c/(x^2 - a^2)^2 dx$, BY SOLUTION OBSERVATION

Rishan Singh
Westville Campus and Nelson R. Mandela School of Medicine
University of Kwa Zulu-Natal
SOUTH AFRICA
e-mail: rshnsingh1@webmail.co.za

*

In the year 2005, I had encountered an integration problem in the form $\int \frac{c}{(x^2-a^2)^2}dx$, which was given to a class of students by Dr. Aneshkumar Maharaj from the School of Mathematical Sciences at UKZN, Howard College as part of a tutorial exercise. Most of the students in the class battled to find the solution the conventional way as indicated in Neuhauser (2004) and at the end of the year, when examinations arrived, no one was able to figure out the steps involved to ascertain the solution provided in the prescribed textbook. Since integration is my field of expertise, I was very curious about finding a quick and reliable method to determine the solution so I made a number of observations along the way, while studying aspects of numerical integration, multivariate calculus and integration techniques. To date, there has been no published formula for integrals of this form, so I thought why not get creative and assumptive and develop one.

Now lets consider the integral $\int \frac{1}{x^2-2x-3}dx$.

In this integral, as most mathematicians may know, the denominator needs to be evaluated by completing the square. This is because the integral is in the form of a rational function and therefore it is not possible for it to be evaluated exactly. Therefore, one needs to achieve this through numerical integration.

Received: April 6, 2011

International Electronic Journal of Pure and Applied Mathematics – IEJPAM, Volume 3, No. 4 (2011)

By completing the square in the denominator, we get:

$$\begin{aligned}\frac{1}{x^2-2x-3} &= \frac{1}{x^2-2x+1-1-3}\\ &= \frac{1}{(x-1)^2-4}.\end{aligned}$$

One of the rational functions used to evaluate indefinite integrals of the form $c/(x^2-a^2)dx$ is, $1/2a \ln\left|\frac{x+a}{x-a}\right| + c$, where c is a number constant. We would now need to bring $\frac{1}{(x-1)^2} - 4$ to this form by first making $x-1=u$ with $du = dx$. By doing this we get,

$$\int \frac{dx}{(x-1)^2-4}dx = \int \frac{du}{u^2-4}du = -\int \frac{du}{4-u^2}.$$

Therefore

$$\int \frac{1}{x^2-2x-3}dx = -\int \frac{du}{4-u^2} = -\frac{1}{4}\ln\left|\frac{u-2}{u+1}\right| + c = -\frac{1}{4}\left|\ln\frac{x+1}{x-3}\right| + c.$$

Now back to our problem, the solution given to the integral $\int \frac{1}{(x^2-9^2)^2}dx$, which is in the form of $\int \frac{c}{(x^2-a^2)^2}dx$ in Neuhauser (2004) is:

$$\frac{1}{108}\ln\left|\frac{x+3}{x-3}\right| - \frac{1}{36}\left(\frac{1}{x+3}+\frac{1}{x-3}\right) + c.$$

When one looks at this solution it contains part of the rational function $\frac{1}{2a}\ln\left|\frac{x+a}{x-a}\right|$ $+c$. However, in this integral the entire part, (x^2-9^2), is squared. This squaring makes it difficult to obtain an exact solution and there is no defined formula that can be used. So here is my assumption to the problem based on the answer provided. From the integral it is evident that $a^2 = 9$ so then $a = 3$. Using the following substitution formula we get the answer provided by Neuhauser (2004):

$$\begin{aligned}&\frac{1}{2a}\left[\frac{1}{2a^2}\ln\left|\frac{x+a}{x-a}\right|\right] - \frac{1}{2a}\left[\frac{1}{x+a}+\frac{1}{x-a}\right] + c \quad \text{(see Rishan's Formula)}\\ &= \frac{1}{2(3)}\left[\frac{1}{2(3)^2}\ln\left|\frac{x+3}{x-3}\right|\right] - \frac{1}{2(3)}\left[\frac{1}{x+3}+\frac{1}{x-3}\right] + c\\ &= \frac{1}{6}\left[\frac{1}{18}\ln\left|\frac{x+3}{x-3}\right|\right] - \frac{1}{6\left[\frac{1}{x+3}+\frac{1}{x-3}\right]} + c\\ &= \frac{1}{108}\ln\left|\frac{x+3}{x-3}\right| - \frac{1}{36}\left(\frac{1}{x+3}+\frac{1}{x-3}\right) + c.\end{aligned}$$

International Electronic Journal of Pure and Applied Mathematics – IEJPAM, Volume 3, No. 4 (2011)

The core into the development of this formula was being assertive and observing the variety of indefinite rational function formulae as provided in Neuhauser (2004). However, in most developments, it is difficult to be completely certain about the outcome, especially when it is something novel. I have tested this formula using several randomly selected integrals (of this form of course), an in all cases the outcome was the expected answer. Based on this fact, I am absolutely sure that this formula is completely correct.

References

[1] C. Neuhauser, *Calculus for Biology and Medicine*, Second Edition, Pearson Education, Inc, Upper Saddle River, New Jersey (2004).