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INTUITIONISTIC MENGER n-INNER PRODUCT SPACES, INTUITIONISTIC MENGER n-NORMED SPACES AND SOME RESULTS

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Abstract: In this paper first we introduce the definition of an intuitionistic Menger n-normed space, an intuitionistic Menger n-inner product space and then we prove some results in both mentioned spaces. Also we prove a fixed point theorem for generalized contraction maps in intuitionistic Menger n-normed spaces and we utilize it to prove the existence theorems of solutions to differential equations for intuitionistic Menger n-normed spaces.

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1. Introduction

The theory of 2-inner and n-inner product spaces has been effectively constructed by C. R. Dimmini, S. Gähler and A. White [4], [5]. It was developed by A. Misiak [14], [15]. Recent results in n-inner product spaces can be viewed in [2], [3]. There are many different viewpoints in introducing the definition of n- normed linear spaces [7], [8], [10], [11], [13]. In this paper we intend to introduce the definition of an intuitionistic Menger n-normed space and an intuitionistic Menger n-inner product space. Then we prove some interesting results. On the other hand, as the classical

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cases, fixed point theorems have many interesting applications in the realm of the intuitionistic Menger n-normed spaces. So generalization of theorems to the intuitionistic Menger n-normed spaces is a good instrument to solve the problems of this area. In the last part of this article we prove a fixed point theorem for generalized contraction maps in an intuitionistic Menger n-normed space and by applying this theorem we prove the existence theorems of solutions to differential equations for intuitionistic Menger n-normed spaces..

2. Preliminaries

Definition 2.1. [9] Let $n \geq 2$ be an integer and X be a real vector space of dimension greater than or equal n. A real valued function $< \bullet, \bullet|\bullet, ..., \bullet >$ on X^{n+1} satisfying the following properties, for all $x, y, x', z_1, ..., z_n \in X$ and $\alpha \in \mathbb{R}$,
i) $< z_1, z_1|z_2, ..., z_n > \geq 0$; $< z_1, z_1|z_2, ..., z_n >= 0$ if and only if $z_1, z_2, ..., z_n$ are linearly dependent
ii) $< z_1, z_1|z_2, ..., z_n >=< z_{i_1}, z_{i_1}|z_{i_2}, ..., z_{i_n} >$, for any permutation $(i_1, ..., i_n)$ of $(1, ..., n)$
iii) $< x, y|z_2, ..., z_n >=< y, x|z_2, ..., z_n >$
iv) $< \alpha x, y|z_2, ..., z_n >= \alpha < x, y|z_2, ..., z_n >$
v) $< x + x', y|z_2, ..., z_n >=< x, y|z_2, ..., z_n > + < x', y|z_2, ..., z_n >$,
is called an n-inner product on X and the pair $(X, < \bullet, \bullet|\bullet, ..., \bullet >)$ is called a n-inner product space.

Definition 2.2. [1] Let $n \in \mathbb{N}$ and X be a real linear space of dimension greater than or equal to n. A real valued function $\|\bullet, \bullet, ..., \bullet\|$ on $X \times ... \times X = X^n$ satisfying the following properties, for all $x_1, ..., x_n, y, z \in X$ and $\alpha \in \mathbb{R}$:
1) $\|x_1, ..., x_n\| = 0$ if and only if $x_1, ..., x_n$ are linearly dependent
2) $\|x_1, ..., x_n\|$ is invariant under any permutation
3) $\|x_1, ..., \alpha x_n\| = |\alpha|\|x_1, ..., x_n\|$
4) $\|x_1, ..., x_{n-1}, y + z\| \leq \|x_1, ..., x_{n-1}, y\| + \|x_1, ..., x_{n-1}, z\|$,
is called an n-norm on X and the pair $(X, \|\bullet, \bullet, ..., \bullet\|)$ is called a n-normed linear space.

Definition 2.3. [12] A triangular norm(t-norm) T on [0,1] is defined as increasing, commutative and associative mapping $T : [0, 1]^2 \to [0, 1]$ satisfying $T(1, x) = x$, for all $x \in [0, 1]$. If T_1 and T_2 be two t-norms on $[0, 1]$, we write $T_1 \leq T_2$ if $T_1(a, b) \leq T_2(a, b)$, for each $a, b \in [0, 1]$. A t-norm can also be defined recursively as an (n+1)-ary operation($n \in \mathbb{N}$) by $T^1 = T$ and $T^n(x_1, x_2, ..., x_{n+1}) = T(T^{n-1}(x_1, x_2, ..., x_n), x_{n+1})$, for $n \geq 2$ and $x_i \in [0, 1]$.

In what follows T_{mi} denotes a special continuous t-norm defined by:

$$T_{mi}(a, b) = \max \{a + b - 1, 0\}$$

Definition 2.4. [12] A binary operation $S : [0,1] \times [0,1] \longrightarrow [0,1]$ is a t-conorm if S is increasing, commutative, associative and $S(a,0) = a$, $\forall a \in [0,1]$. For any continuous t-norm T, a binary operation T^* on [0,1] which is related to T by $T^*(a,b) = 1 - T(1-a, 1-b)$, for all $a, b \in [0,1]$, is called the t-conorm of T. A t-conorm can also be defined recursively as an (n+1)-ary operation($n \in \mathbb{N}$) by $S^1 = S$ and $S^n(x_1, x_2, ..., x_{n+1}) = S(S^{n-1}(x_1, x_2, ..., x_n), x_{n+1})$, for $n \geq 2$ and $x_i \in [0,1]$.

In what follows S_{ma} denotes a special continuous t-conorm defined by:

$$S_{ma}(a,b) = \min\{a+b, 1\}$$

Throughout this paper we let, $\mathbb{R} = (-\infty, +\infty)$ and $\mathbb{R}^+ = [0, +\infty)$.

Definition 2.5. [12] A distance distribution function is a non-decreasing and left continuous mapping $F : \mathbb{R} \to \mathbb{R}^+$ with $\inf_{t\in\mathbb{R}} F(t) = 0$ and $\sup_{t\in\mathbb{R}} F(t) = 1$. We will denote by D the family of all distance distribution functions and by $\mathcal{H}$ the special element of D defined by

$$\mathcal{H}(t) = \begin{cases} 1 & t > 0 \\ 0 & t \leq 0. \end{cases}$$

If X is a nonempty set, $F : X \times X \longrightarrow D$ is called a probabilistic distance on X and $F(x,y)$ is usually denoted by F_{xy}.

Definition 2.6. [12] A non-distance distribution function is a non-increasing and right continuous mapping $L : \mathbb{R} \to \mathbb{R}^+$ with $\sup_{t\in\mathbb{R}} L(t) = 0$ and $\inf_{t\in\mathbb{R}} L(t) = 1$. We will denote by E the family of all non-distance distribution functions and by $\mathcal{G}$ the special element of E defined by

$$\mathcal{G}(t) = \begin{cases} 1 & t \leq 0 \\ 0 & t > 0. \end{cases}$$

If X is a nonempty set, $L : X \times X \longrightarrow E$ is called a probabilistic non-distance on X and $L(x,y)$ is usually denoted by L_{xy}.

3. Main Results

In this section first we introduce the definitions of intuitionistic Menger n-normed spaces, intuitionistic Menger n-inner product spaces and then we consider the notions such as Schwartz inequality, continuity of intuitionistic Menger n-inner product, two distinct sorts of convergency and Pythagorean theorem.

Definition 3.1. Let X be a real linear space over a field F, $n \in \mathbb{N}$, $dimX \geq n$, T and S be a continuous t-norm and a continuous t-conorm respectively, $F : X^n \longrightarrow D$ (probabilistic n-distance), $L : X^n \longrightarrow E$(probabilistic non n-distance) and the following conditions hold for all $x_1, x_2, ..., x_n \in X$ and $s, t \in \mathbb{R}$.
IMN-1) $F(x_1, x_2, ..., x_n)(t) + L(x_1, x_2, ..., x_n)(t) \leq 1$
IMN-2) $F(x_1, x_2, ..., x_n)(0) = 0$
IMN-3) $F(x_1, x_2, ..., x_n)(t) = \mathcal{H}(t)$ if and only if $x_1, x_2, ..., x_n$ are linearly dependent
IMN-4) $F(x_1, x_2, ..., x_n)(t)$ is invariant under any permutation of $x_1, x_2, ..., x_n$
IMN-5) $F(x_1, x_2, ..., cx_n, t) = F(x_1, x_2, ..., x_n, \frac{t}{|c|})$ if $0 \neq c \in F$
IMN-6) $F(x_1, x_2, ..., x_n + x_n^{'})(s + t) \geq T(F(x_1, x_2, ..., x_n)(s), F(x_1, x_2, ..., x_n^{'})(t))$
IMN-7) $L(x_1, x_2, ..., x_n)(0) = 1$
IMN-8) $L(x_1, x_2, ..., x_n)(t) = \mathcal{G}(t)$ if and only if $x_1, x_2, ..., x_n$ are linearly dependent
IMN-9) $L(x_1, x_2, ..., x_n)(t)$ is invariant under any permutation of $x_1, x_2, ..., x_n$
IMN-10) $L(x_1, x_2, ..., cx_n, t) = L(x_1, x_2, ..., x_n, \frac{t}{|c|})$ if $0 \neq c \in F$
IMN-11) $L(x_1, x_2, ..., x_n + x_n^{'})(s + t) \leq S(L(x_1, x_2, ..., x_n)(s), L(x_1, x_2, ..., x_n^{'})(t))$
In this case (X, F, L, T, S) is called an intuitionistin Menger n-normed space or in short IMNS.

Definition 3.2. An intuitionistic Menger n-inner product space($IMNI$) is a 5-tuple (X, F, L, T, S), where X is a real vector space, T and S are continuous t-norm and continuous t-conorm respectively, F is a probabilistic (n+1)-distance and L is a probabilistic non (n+1)-distance such that the following conditions hold for every $x_1, x_2, ..., x_n, y, z \in X$ and $s, t, r \in \mathbb{R}$.
IMI-1) $F(x, x|x_2, ..., x_n)(t) + L(x, x|x_2, ..., x_n)(t) \leq 1$
IMI-2) $F(x, x|x_2, ..., x_n)(0) = 0$
IMI-3) $F(x, x|x_2, ..., x_n)(t) = \mathcal{H}(t)$, for all $t \in \mathbb{R}$ if and only if $x, x_2, ..., x_n$ are linearly dependent
IMI-4) $F(x, x|x_2, ..., x_n)(t)$ is invariant under any permutation of $x_2, ..., x_n$
IMI-5) $F(x, y|x_2, ..., x_n)(t) = F(y, x|x_2, ..., x_n)(t)$
IMI-6) $F(x, x|x_2, ..., x_n)(t) = F(x_2, x_2|x, ..., x_n)(t)$
IMI-7) For any real number α,

$$F(\alpha x, y|x_2, ..., x_n)(t) = \begin{cases} F(x, y|x_2, ..., x_n)(t/\alpha) & \alpha > 0 \\ \mathcal{H}(t) & \alpha = 0 \\ 1 - F(x, y|x_2, ..., x_n)(-t/\alpha) & \alpha < 0 \end{cases}$$

IMI-8) $T(F(x, x|x_2, ..., x_n)(t), F(y, y|x_2, ..., x_n)(s)) \leq F(x + y, x + y|x_2, ..., x_n)(t + s)$
IMI-9) $\sup_{s+r=t} T(F(x, y|x_2, ..., x_n)(s), F(z, y|x_2, ..., x_n)(r)) = F(x+z, y|x_2, ..., x_n)(t)$
IMI-10) $L(x, x|x_2, ..., x_n)(0) = 0$
IMI-11) $L(x, x|x_2, ..., x_n)(t) = \mathcal{G}(t)$, for all $t \in \mathbb{R}$ if and only if $x, x_2, ..., x_n$ are linearly dependent
IMI-12) $L(x, x|x_2, ..., x_n)(t)$ is invariant under any permutation of $x_2, ..., x_n$

IMI-13) $L(x,y|x_2,...,x_n)(t) = L(y,x|x_2,...,x_n)(t)$
IMI-14) $L(x,x|x_2,...,x_n)(t) = L(x_2,x_2|x,...,x_n)(t)$
IMI-15) For any real number α,

$$L(\alpha x,y|x_2,...,x_n)(t) = \begin{cases} L(x,y|x_2,...,x_n)(t/\alpha) & \alpha > 0 \\ \mathcal{G}(t) & \alpha = 0 \\ L(x,y|x_2,...,x_n)(-t/\alpha) & \alpha < 0 \end{cases}$$

IMI-16) $S(L(x,x|x_2,...,x_n)(t), L(y,y|x_2,...,x_n)(s)) \geq L(x+y,x+y|x_2,...,x_n)(t+s)$
IMI-17) $\inf_{s+r=t} S((L(x,y|x_2,...,x_n)(s), L(z,y|x_2,...,x_n)(r))) = L(x+z,y|x_2,...,x_n)(t)$

Example 3.3. Let $(X, < \bullet,\bullet|\bullet,..., \bullet >)$ be an ordinary n-inner product space. We define the mappings $F : X^{n+1} \to D$ and $L : X^{n+1} \to E$ as follows:

$$F(x,y|x,...,x_n,t) = \mathcal{H}(t- < x,x|x_2,...,x_n >)$$

$$L(x,y|x,...,x_n,t) = \mathcal{G}(t- < x,x|x_2,...,x_n >)$$

If we define $T(a,b) = \min\{a,b\}$ and $S(a,b) = \max\{a,b\}$ then (X, F, L, T, S) is a IMNI, which is called the standard intuitionistic Menger n-inner product induced by the n-inner product $< \bullet, \bullet|\bullet,..., \bullet >$.

From now on, we apply T_s and S_s to show any continuous t-norm T and any continuous t-conorm S on $[0,1]$ with the property $T(a,a) \geq a$ and $S(a,a) \leq a$, for each $a \in [0,1]$, respectively. There are many t-norms and t-conorms having this property.

Example 3.4. Let $T(a,b) = \min\{a,b\}$ and $S(a,b) = \max\{a,b\}$ on [0,1]. Then T and S have the above property.

Example 3.5. For $0 < p < +\infty$, the Aczél-Alsina t-norms are defined by

$$T_p^{AA}(x,y) = e^{-(|logx|^p+|logy|^p)^{\frac{1}{p}}},$$

for any $x,y \in [0,1]$. If we consider S_p^{AA} as the t-conorm of T_p^{AA} then for any even number p, we can show that T_p^{AA} and S_p^{AA} have the above properties.

Also the standard intuitionistic Menger n-inner product space is an example of an intuitionistic Menger n-inner product space such that its t-norm and its t-conorm has the above condition.

In the following we prove the Schwartz inequality in a IMNI which has a key role in the theory of IMNS.

Theorem 3.6. *Let (X, F, L, T_s, S_s) be a IMNI. Then for any $x,y,x_2,...,x_n \in X$ and $t,s > 0$ we have*

$$F(x,y|x_2,...,x_n)(ts) \geq T_s(F(x,x|x_2....,x_n)(t^2), F(y,y|x_2,...,x_n)(s^2)). \tag{3.1}$$

$$L(x,y|x_2,...,x_n)(ts) \leq S_s(L(x,x|x_2....,x_n)(t^2), L(y,y|x_2,...,x_n)(s^2)).$$

Proof. Let $\alpha = -s/t$ *i.e.* $\alpha t + s = 0$. Putting $a = F(x,x|x_2,...,x_n)(s^2)$, $b = F(\alpha y, x|x_2,...,x_n)(\alpha ts)$ and $c = F(\alpha y, \alpha y|x_2,...,x_n)(\alpha^2 t^2)$. By IMI-2) and IMI-9) we have

$$\begin{aligned} 0 = F(x+\alpha y, x+\alpha y, y|x_2,...,x_n)((\alpha t+s)^2) &\geq T_s(T_s(a,b), T_s(b,c)) \\ &= T_s(a, T_s(T_s(b,b),c)) \\ &\geq T_s(a, T_s(b,c)). \end{aligned}$$

It is easy to deduce that $T_s \geq T_{mi}$. Besides

$$c = F(\alpha y, \alpha y|x_2,...,x_n)(\alpha^2 t^2) = 1 - F(y, \alpha y|x_2,...,x_n)(\alpha t^2) = F(y,y|x_2,...,x_n)(t^2)$$
$$b = F(\alpha y, x|x_2,...,x_n)(\alpha ts) = 1 - F(y,x|x_2,...,x_n)(ts).$$

Substituting c and b into the above inequality, we have,

$$\begin{aligned} 0 &\geq T_s(F(x,x|x_2,...,x_n)(s^2), T_s(1 - F(y,x|x_2,...,x_n)(ts), F(y,y|x_2,...,x_n)(t^2))) \\ &= T_s(F(x,x|x_2,...,x_n)(s^2), T_s(F(y,y|x_2,...,x_n)(t^2), 1 - F(y,x|x_2,...,x_n)(ts))) \\ &\geq T_s(F(x,x|x_2,...,x_n)(s^2), F(y,y|x_2,...,x_n)(t^2)) + 1 - F(y,x|x_2,...,x_n)(ts) - 1. \end{aligned}$$

On the other hand we can deduce that $S_s \leq S_{ma}$ and the similar argument proves the second inequality in (3.1). So the proof is complete. □

Lemma 3.7. *Let* (X, F, L, T, S) *be a IMNI. Then it is a IMNS.*

Proof. Define:

$$N(x_1,x_2,...,x_n)(t) = \begin{cases} F(x_1,x_1|x_2,...,x_n)(t^2), & t > 0, \\ 0, & t \leq 0. \end{cases}$$

and

$$M(x_1,x_2,...,x_n)(t) = \begin{cases} L(x_1,x_1|x_2,...,x_n)(t^2), & t < 0, \\ 0, & t \geq 0. \end{cases}$$

We will show that N and M satisfy the conditions of definition 3.1. In fact conditions IMN-2)-IMN-5) are obtained from IMI-2)-$IMI-7)\backslash\{IMI-6)\}$ and the properties of distance distribution functions. To show IMN-6) If s or t is equal zero the inequality holds trivially. So we may assume that s and t are not zero. In this case by IMN-4) and IMI-4)-IMI-9) we have:

$$\begin{aligned} N(x_1,x_2,...,x_n+x_n^{'})(t+s) &= N(x_n+x_n^{'},x_2,...,x_{n-1},x_1)(t+s) \\ &= F(x_n+x_n^{'},x_n+x_n^{'},x_2,...,x_{n-1},x_1)((t+s)^2) \\ &\geq T(F(x_n+x_n^{'},x_n+x_n^{'},x_2,...,x_{n-1},x_1)(t^2+s^2) \\ &\quad ,F(0,x_n+x_n^{'},x_2,...,x_{n-1},x_1)(2ts)) \\ &= T(F(x_n+x_n^{'},x_n+x_n^{'},x_2,...,x_{n-1},x_1,)(t^2+s^2),1) \\ &\geq T(F(x_n,x_n,x_2,...,x_{n-1},x_1)(t^2) \\ &\quad ,F(x_n^{'},x_n^{'},x_2,...,x_{n-1},x_1)(s^2)) \\ &= T(N(x_1,x_2,...,x_n)(t), N(x_1,x_2,...,x_n^{'})(s)). \end{aligned}$$

To show the properties of M we use the properties of L and the similar argument can help us. □

For any $p \in X$, $\varepsilon > 0$ and $0 < \lambda < 1$, define an intuitionistic Menger n-neighborhood in an intuitionistin Menger n-normed space (X, N, M, T, S) by:

$$U_p^n(\varepsilon,\lambda) = \{x \in X : N(x-p, x_2, ..., x_n)(\varepsilon) > 1-\lambda \\ \textit{and } M(x-p, x_2, ..., x_n)(\varepsilon) < \lambda \ \forall x_2, ..., x_n \in X\}.$$

Let T_I^N be the family of neighborhoods $\{U_p^n(\varepsilon,\lambda) : \varepsilon > 0,\ 0 < \lambda < 1\}$. Then T_I^N induced a topology on X. So by lemma 3.7 each IMNI is a topological space and hence the following definition will be reasonable.

Definition 3.8. Let (X, F, L, T, S) be a IMNI. The sequence $\{x_k\} \subset X$ is called T_I^N-convergent to $x \in X$ (we write $x_k \xrightarrow{T_I^N} x$) if for any given $\varepsilon > 0$ and $0 < \lambda < 1$, there exists a positive integer $N_0 = N_0(\varepsilon, \lambda)$ such that

$$N(x_k - x, z_2, ..., z_n)(\varepsilon) > 1 - \lambda$$

and

$$M(x_k - x, z_2, ..., z_n)(\varepsilon) < \lambda,$$

whenever $k \geq N_0$. In this case, we say that $\{x_k\}$ converges to x strongly. Meanwhile $\{x_k\}$ is said to converge weakly to x(we write $x_k \xrightarrow{W_N} x$) whenever $F(x_k - x, y|z_2, ..., z_n)(\varepsilon) > 1 - \lambda$ and $L(x_k - x, y|z_2, ..., z_n)(\varepsilon) < \lambda$, for any $y, z_2, ..., z_n \in X$, whenever $k \geq N_0$.

Theorem 3.9. *Let (X, F, L, T_s, S_s) be a IMNI. Let $\{u_k\}$ be a sequence in X such that $u_k \xrightarrow{T_I^N} 0$. Then for any $v \in X$, $\varepsilon > 0$, $z_2, ..., z_n \in X$ and $0 < \lambda < 1$, there exists $N = N(\varepsilon, \lambda)$ such that $F(u_k, v|z_2, ..., z_n)(\varepsilon) > 1-\lambda$ and $L(u_k, v|z_2, ..., z_n)(\varepsilon) < \lambda$, whenever $k \geq N$.*

Proof. As a natural way it is enough to show the proof for F. Since $T_s(a, 1) = a$, for each $a \in [0, 1]$ and T_s is continuous, then for any $\varepsilon > 0$ and $0 < \lambda < 1$ there exists $a, b \in (0, 1)$ such that

$$T_s((1-a), (1-b)) > 1 - \lambda. \tag{3.2}$$

By definition of IMNI and the properties of distance distribution functions there exists $t_0 > 0$ such that

$$F(v, v|z_2, ..., z_n)(t_0^2) > 1 - b.$$

Since $u_k \xrightarrow{T_I^N} 0$, therefore, for the given number a, there exists n_0 such that

$$F(u_k, u_k|z_2, ..., z_n)((\varepsilon/t_0)^2) > 1 - a,$$

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whenever $k \geq n_0$. It follows from Schwartz inequality and (3.2) that

$$\begin{aligned} F(u_k, v|z_2, ..., z_n)(\varepsilon) &= F(u_k, v|z_2, ..., z_n)((\varepsilon/t_0)t_0) \\ &\geq T_s(F(u_k, u_k|z_2, ..., z_n)((\varepsilon/t_0)^2), F(v, v|z_2, ..., z_n)(t_0^2)) \\ &\geq T_s((1-a), (1-b)) \\ &> 1 - \lambda. \end{aligned}$$

Theorem 3.10. *Let $(X,\ F,\ L,\ T_s,\ S_s)$ be a IMNI and:*

$$F(x+y, z|z_2, ..., z_n)(t) \leq \Delta_{T_s}(F(x, z|z_2, ..., z_n)(r), F(y, z|z_2, ..., z_n)(s))$$

and

$$L(x+y, z|z_2, ..., z_n)(t) \geq \Delta_{S_s}(L(x, z|z_2, ..., z_n)(r), L(y, z|z_2, ..., z_n)(s)), \tag{3.3}$$

where $t \in \mathbb{R}$, $t = r + s$, $x, y, z, z_2, ..., z_n \in X$,

$$\Delta_{T_s}(a, b) = 1 - T_s((1-a), (1-b))$$

and

$$\Delta_{S_s}(a, b) = 1 - S_s((1-a), (1-b))$$

, for all $a, b \in [0,1]$. Suppose further $\{u_k\}$ be a sequence in X such that $u_k \xrightarrow{T_I^N} u_0 \in X$. Then for any $v \in X$,

$$\lim_{k\to\infty} F(u_k, v|z_2, ..., z_n)(t) = F(u_0, v|z_2, ..., z_n)(t),$$

and

$$\lim_{k\to\infty} L(u_k, v|z_2, ..., z_n)(t) = L(u_0, v|z_2, ..., z_n)(t).$$

Proof. Since T_s is continuous and $T_s(a,1) = a,\ for\ all\ a \in [0,1]$, then for any $\varepsilon > 0$ there exists $0 < \lambda < 1$ such that

$$T_s((1-\lambda), F(u_0, v|z_2, ..., z_n)(t-\varepsilon)) > F(u_0, v|z_2, ..., z_n)(t-\varepsilon) - \varepsilon,$$

and as $\varepsilon \searrow 0$ we have $\lambda \searrow 0$. Now since $(u_0 - u_k) \xrightarrow{T_I^N} 0$, by theorem 3.9 there exists n_0 such that

$$F(u_0 - u_k, v|z_2, ..., z_n)(\varepsilon) > 1 - \lambda \quad (\forall k \geq n_0). \tag{3.4}$$

Therefore we have

$$\begin{aligned} F(u_k, v|z_2, ..., z_n)(t) &\geq T_s(F(u_k - u_0, v|z_2, ..., z_n)(\varepsilon), F(u_0, v|z_2, ..., z_n)(t-\varepsilon)) \\ &\geq T_s((1-\lambda), F(u_0, v|z_2, ..., z_n)(t-\varepsilon)) \end{aligned}$$

$$> F(u_0, v|z_2, ..., z_n)(t-\varepsilon) - \varepsilon \quad (\forall\ k \geq n_0). \qquad (3.5)$$

By virtue of condition (3.3) we have

$$\begin{aligned} F(u_k, v|z_2, ..., z_n, t) &= F(u_k - u_0 + u_0, v|z_2, ..., z_n)(t) \\ &\leq 1 - T_s((1 - F(u_k - u_0, v|z_2, ..., z_n)(-\varepsilon)), \\ &\quad (1 - F(u_0, v|z_2, ..., z_n)(t+\varepsilon))) \\ &= 1 - T_s(F(u_0 - u_k, v|z_2, ..., z_n)(\varepsilon) \\ &\quad , (1 - F(u_0, v|z_2, ..., z_n)(t+\varepsilon))). \quad (3.6) \end{aligned}$$

Since $T_s \geq T_{min}$, it follows from (3.6) that

$$\begin{aligned} F(u_k, v|z_2, ..., z_n)(t) &\leq 1 - (F(u_0 - u_k, v|z_2, ..., z_n)(\varepsilon) + \\ &\quad 1 - F(u_0, v|z_2, ..., z_n)(t) + \varepsilon) - 1) \\ &= 1 - F(u_0 - u_k, v|z_2, ..., z_n)(\varepsilon) + \\ &\quad F(u_0, v|z_2, ..., z_n)(t+\varepsilon). \end{aligned}$$

Noting (3.4) and (3.5) hence we have

$$F(u_0, v|z_2, ..., z_n)(t-\varepsilon) - \varepsilon < F(u_k, v|z_2, ..., z_n)(t) < \lambda + F(u_0, v|z_2, ..., z_n)(t+\varepsilon).$$

This implies that

$$\begin{aligned} \liminf_{k\to\infty} F(u_k, v|z_2, ..., z_n)(t) &\geq F(u_0, v|z_2, ..., z_n)(t-\varepsilon) - \varepsilon, \\ \limsup_{k\to\infty} F(u_k, v|z_2, ..., z_n)(t) &\leq F(u_0, v|z_2, ..., z_n)(t+\varepsilon) + \lambda. \end{aligned}$$

Letting $\varepsilon \searrow 0$ (hence $\lambda \searrow 0$) we have

$$\begin{aligned} \liminf_{k\to\infty} F(u_k, v|z_2, ..., z_n)(t) &\geq F(u_0, v|z_2, ..., z_n)(t), \\ \limsup_{k\to\infty} F(u_k, v|z_2, ..., z_n)(t) &\leq F(u_0, v|z_2, ..., z_n)(t), \end{aligned}$$

if $t \neq 0$. From the above two inequalities we have the first conclusion for F and the second conclusion for L can be proved by the similar argument and hence the proof is complete. □

Note. The above theorem along with the condition IMI-5) and IMI-13) show that the n-inner product as a function is continuous, under some restrictions, in terms of the first two variables and with respect to the strong topology.

Lemma 3.11. *Suppose that* $(X,\ F,\ L,\ T,\ S)$ *be a IMNI and* N *and* M *be their induced intuitionistic Menger n-norm, respectively. If* $\{x_k\}$ *and* $\{y_k\}$ *converge strongly/ weakly to* x *and* y *respectively, then for each* $\alpha, \beta \in \mathbb{R}$, $\{\alpha x_k + \beta y_k\}$ *converges strongly/weakly to* $\alpha x + \beta y$.

Proof. It is easy to show that the definition strongly and weakly convergent, introduced in 3.8 are equivalent to the conditions

$$N(x_k - x, z_2, ..., z_n)(t) \longrightarrow 1$$

$$M(x_k - x, z_2, ..., z_n)(t) \longrightarrow 0,$$

and

$$F(x_k - x, , y|z_2, ..., z_n)(t) \longrightarrow 1$$

$$L(x_k - x, , y|z_2, ..., z_n)(t) \longrightarrow 0,$$

for all $y, z_2, ..., z_n \in X$ and $t > 0$, as $k \longrightarrow \infty$, respectively. Suppose that $x_k \xrightarrow{W_N} x$ and $y_k \xrightarrow{W_N} y$. Then by IMI-8) and IMI-16) for each $t \in \mathbb{R}$, we have

$$F((x_k+y_k)-(x+y), z_2, ..., z_n)(t) \geq T(F(x_k-x, z_2, ..., z_n)(r), F(y_k-y, z_2, ..., z_n)(s))$$

$$L((x_k+y_k)-(x+y), z_2, ..., z_n)(t) \leq S(L(x_k-x, z_2, ..., z_n)(r), L(y_k-y, z_2, ..., z_n)(s)),$$

where $r+s=t$. Also for each $\alpha \in \mathbb{R}$, from conditions IMI-7) and IMI-15), we imply that

$$F(\alpha x_k - \alpha x, z_2, ..., z_n)(t) = F(x_k - x, z_2, ..., z_n)(\frac{t}{\alpha}),\ if\ \alpha > 0,$$

$$L(\alpha x_k - \alpha x, z_2, ..., z_n)(t) = L(x_k - x, z_2, ..., z_n)(\frac{t}{\alpha}),\ if\ \alpha > 0,$$

and

$$F(\alpha x_k - \alpha x, z_2, ..., z_n)(t) = 1 - F(x_k - x, z_2, ..., z_n)(\frac{-t}{\alpha}),\ if\ \alpha < 0.$$

$$L(\alpha x_k - \alpha x, z_2, ..., z_n)(t) = L(x_k - x, z_2, ..., z_n)(\frac{-t}{\alpha}),\ if\ \alpha < 0.$$

These prove the convergency in the weak topology. The proof in strong convergency is the same as above. □

Theorem 3.12. *Suppose that $(X,\ F,\ L,\ T,\ S)$ be a IMNI for which F and L are continuous in the first two variables and $(X,\ N,\ M,\ L,\ S)$ be its induced intuitionistic Menger n-normed space. If $\{x_k\}$ converges weakly to x and x' simultaneously, then $x = x'$.*

Proof. It is enough to check the conditions for F. By hypothesis, for each $t > 0$, we have:

$$F(x_k, y|z_2, ..., z_n)(t) \xrightarrow{W_N} F(x, y|z_2, ..., z_n)(t),$$

and

$$F(x_k, y|z_2, ..., z_n)(t) \xrightarrow{W_N} F(x', y|z_2, ..., z_n)(t),$$

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for all $y, z_2, ..., z_n \in X$. According to the uniqueness of the limit of a sequence of real numbers, we must have

$$F(x, y|z_2, ..., z_n)(t) = F(x^{'}, y|z_2, ..., z_n)(t).$$

But by IMI-7)

$$F(x - x^{'}, y|z_2, ..., z_n)(t) = \sup_{s+r=t} T(F(x, y|z_2, ..., z_n)(s), (1 - F(x^{'}, y|z_2, ..., z_n)(-r)))$$

for each $y, z_2, ..., z_n \in X$. According to IMI-3), a simple calculation shows that the right hand side of the above equality is equal to $F(x, y|z_2, ..., z_n)(t)$. So for each $n \in \mathbb{N}$, we have

$$\begin{aligned} F(x, y|z_2, ..., z_n)(t) &= \sup_{s+r=t} T(F(x - x^{'}, y|z_2, ..., z_n)(s), F(x^{'}, y|z_2, ..., z_n)(r)) \\ &= \sup_{s+r=t} T(F(x, y|z_2, ..., z_n)(s), F(x, y|z_2, ..., z_n)(r)) \\ &= F(2x, y|z_2, ..., z_n)(t) \\ &= .... = F(nx, y|z_2, ..., z_n)(t) \\ &= F(x, y|z_2, ..., z_n)(\tfrac{t}{n}). \end{aligned}$$

Then we imply that $F(x - x^{'}, y|z_2, ..., z_n)(t) = \mathcal{H}(t)$, for each $y, z_2, ..., z_n \in X$ and $t \in \mathbb{R}$. In particular if we let $y = x - x^{'}$, we obtain

$$N(x - x^{'}, z_2, ..., z_n)(t) = 1,$$

for each $y, z_2, ..., z_n \in X$ and $t > 0$. But by IMN-3) and the elementary facts from linear algebra, this can only happens if $x - x^{'} = 0$ or $x = x^{'}$. □

Theorem 3.13. *Let (X, F, L, T_s, S_s) be a IMNI. If $\{x_k\}$ converges strongly to x, then it converges weakly to x.*

Proof. By Schwartz inequality, we have:

$$F(x - x_k, y|z_2, ..., z_n)(t^2) \geq T_s(N(x - x_k, z_2...., z_n)(t), N(y, z_2, ..., z_n)(t)).$$

$$L(x - x_k, y|z_2, ..., z_n)(t^2) \leq S_s(M(x - x_k, z_2...., z_n)(t), M(y, z_2, ..., z_n)(t)).$$

These equalities hold for each $y \in X$. if we take $y = 0$, the above equalities yield the following equalities.

$$F(x - x_k, y|z_2, ..., z_n)(t^2) \geq N(x - x_k, z_2...., x_n)(t).$$

$$L(x - x_k, y|z_2, ..., z_n)(t^2) \leq M(x - x_k, z_2...., x_n)(t).$$

By the hypothesis, the right hand side of the above equalities tend to 1 and 0 respectively, for each $y, z_2, ..., z_n \in X$ and $t > 0$. So the proof is complete. □

Corollary 3.14. *The limit of a sequence in strong topology is unique.*

Now we turn to introduce the notion of n-orthogonality in these spaces.

4. Intuitionistic Menger n-Orthogonality

Definition 4.1. Let $(X,\ F,\ L,\ T,\ S)$ be a IMNI. $u, v \in X$ is said to be intuitionistic Menger n-orthogonal if $F(u, v|z_2, ..., z_n)(t) = \mathcal{H}(t)$ and $L(u, v|z_2, ..., z_n)(t) = \mathcal{G}(t)$, $(\forall t \in \mathbb{R}, z_2, ..., z_n \in X)$. It is denoted by $u \perp v$.

Theorem 4.2. *Let $(X,\ F,\ L,\ T,\ S)$ be a IMNI. The intuitionistic Menger n-orthogonality has the following properties:*
i) $0 \perp u \quad \forall u \in X$
ii) If $u \perp v$ then $v \perp u$
iii)If $u \perp u$ then $u = 0$
iv) If $u \perp v$ then for any $a \in \mathbb{R}$, $u \perp av$
v) Let $T = T_s$, $S = S_s$ and the condition (3.3) holds. If

$$u_k \xrightarrow{T_I^N} u,\ v \perp u_k\ (n = 1, 2, ...),$$

then $v \perp u$.

Proof. i)-iii) follows immediately from conditions IMI-7), IMI-5) and IMI-3) respectively. Moreover, suppose $u \perp v$. Then for any $a > 0$ we have

$$F(u, av|z_2, ..., z_n)(t) = F(u, v|z_2, ..., z_n)(t/a) = \mathcal{H}(t/a) = \mathcal{H}(t) \qquad (\forall t \in \mathbb{R})$$

and if $a = 0$ by IMI-7)

$$F(u, av|z_2, ..., z_n)(t) = \mathcal{H}(t). \qquad (\forall\ t \in \mathbb{R})$$

For any $a < 0$, again by IMI-7) we have

$$\begin{aligned} F(u, av|z_2, ..., z_n)(t) &= 1 - F(u, v|z_2, ..., z_n)(t/a) \\ &= \begin{cases} 0 & t < 0 \\ 1 & t > 0. \end{cases} \end{aligned}$$

Then $F(u, av|z_2, ..., z_n)(t) = \mathcal{H}(t),\ \forall t \in \mathbb{R}$. the same argument shows that

$$L(u, av|z_2, ..., z_n)(t) = \mathcal{G}(t)\ \forall t \in \mathbb{R}$$

and these prove (iv). Finally to prove (v), since $v \perp u_k$ $(k = 1, 2, ...)$, by theorem 3.10 we have:

$$F(u, v|z_2, ..., z_n)(t) = \mathcal{H}(t),\ \forall t \in \mathbb{R},$$
$$L(u, v|z_2, ..., z_n)(t) = \mathcal{G}(t),\ \forall t \in \mathbb{R}.$$

Then $v \perp u$ and so the proof is complete. □

Definition 4.3. Let $(X,\ F,\ L,\ T,\ S)$ be a IMNI and $E \subset X$. $E^{\perp}$ is the set of all $v \in X$ that are intuitionistic Menger n-orthogonal to every $u \in E$.

Theorem 4.4. *Let $(X,\ F,\ L,\ T,\ S)$ be a IMNI with a continuous t-norm and a continuous t-conorm T and S, satisfying the conditions of theorem 3.10 and M is a subset of X. Then $M^{\perp}$ is a T_I^N-closed subspace of X and $M \cap M^{\perp} = \{0\}$.*

Proof. By 4.2(i), $0 \in M^{\perp}$ and hence M is nonempty. Let $u, v \in M^{\perp}$ and $\alpha \in \mathbb{R}$. For each $w \in M$, if $\alpha > 0$ then

$$\begin{aligned} F(\alpha u + v, w|z_2, ..., z_n)(t) &= \sup_{s+r=t} T(F(u, w|z_2, ..., z_n)(r/\alpha) \\ &\qquad , F(v, w|z_2, ..., z_n)(s)) \\ &= \sup_{s+r=t} T(\mathcal{H}(r/\alpha), \mathcal{H}(s)) \\ &= \sup_{s+r=t} T(\mathcal{H}(r), \mathcal{H}(s)) \\ &= \mathcal{H}(t). \end{aligned}$$

Because, if $t < 0$, then at least one of s or r is negative, and so $T(\mathcal{H}(r), \mathcal{H}(s)) = 0$. Also if $t > 0$, we can choose $r, s \in \mathbb{R}$ such that both r and s be positive and $T(\mathcal{H}(r), \mathcal{H}(s)) = 1$. If $\alpha < 0$, the similar argument shows that

$$F(\alpha u + v, w|z_2, ..., z_n)(t) = \mathcal{H}(t),\ \forall t \in \mathbb{R},\ w \in M,$$

and also we can show that:

$$L(\alpha u + v, w|z_2, ..., z_n)(t) = \mathcal{G}(t),\ \forall t \in \mathbb{R},\ w \in M.$$

The case $\alpha = 0$ is obvious. So $M^{\perp}$ is a subspace of X. Now we show that M is T_I^N-closed. Let $\{u_k\} \subset M^{\perp}$ be a sequence which $u_k \xrightarrow{T_I^N} u$. Definition 4.3 and theorem 3.10 imply that

$$F(u, w|z_2, ..., z_n)(t) = \lim_{k\to\infty} F(u_k, w|z_z, ..., z_n)(t) = \mathcal{H}(t),\ \forall t \in \mathbb{R},\ z_2, ..., z_n \in X,$$

and

$$L(u, w|z_2, ..., z_n)(t) = \lim_{k\to\infty} L(u_k, w|z_z, ..., z_n)(t) = \mathcal{G}(t),\ \forall t \in \mathbb{R},\ z_2, ..., z_n \in X.$$

This shows that $u \in M^{\perp}$. If $u \in M \cap M^{\perp}$, then by definition 4.3

$$F(u, u|z_2, ..., z_n)(t) = \mathcal{H}(t),\ \forall t \in \mathbb{R}\ and\ z_2, ..., z_n \in X,$$

and

$$L(u, u|z_2, ..., z_n)(t) = \mathcal{G}(t),\ \forall t \in \mathbb{R}\ and\ z_2, ..., z_n \in X.$$

So by 4.2(iii), $u = 0$. □

The following theorem shows a case in which the equalities in IMN-6) and IMN-11) hold.

Theorem 4.5. (The Pythagorean Theorem) *Let* $(X,\ F,\ L,\ T,\ S)$ *be a IMNI with induced intuitionistic Menger norms N and M, respectively. If* $u \perp v$ *then*

$$N(u+v, z_2, ..., z_n)(t) = T(N(u, z_2, ..., z_n)(t), N(v, z_2, ..., z_n)(t)),$$

and

$$M(u+v, z_2, ..., z_n)(t) = S(M(u, z_2, ..., z_n)(t), M(v, z_2, ..., z_n)(t)).$$

Proof. We show the first part of theorem.

$$\begin{aligned} N(u+v, z_2, ..., z_n)(t) &= F(u+v, u+v|z_2, ..., z_n)(t^2), \\ &= \sup_{t_1+t_2=t^2} T(F(u, u+v|z_2, ..., z_n)(t_1) \\ &\quad , F(v, u+v|z_2, ..., z_n)(t_2)) \\ &= \sup_{t_1+t_2=t^2} T(\sup_{t_1'+t_2'=t_1} T((F(u, u|z_2, ..., z_n)(t_1') \\ &\quad , F(u, v|z_2, ..., z_n)(t_2'))) \\ &\quad , \sup_{t_1''+t_2''=t_2} T((F(v, u|z_2, ..., z_n)(t_1'') \\ &\quad , F(v, v|z_2, ..., z_n)(t_2'')))). \end{aligned}$$

Now suppose $u \perp v$. Since $t^2 > 0$, we can choose t_2' and t_2'' such that $t_2' > 0$ and $t_2'' > 0$. Therefore

$$\begin{aligned} N(u+v, z_2, ..., z_n)(t) &= \sup_{t_1+t_2=t^2}[\sup_{t_1'+t_2'=t_1} T(F(u, u|z_2, ..., z_n)(t_1') \\ &\quad , \sup_{t_1''+t_2''=t_2} F(v, v|z_2, ..., z_n)(t_1'')) \\ &\le \sup_{t_1+t_2=t^2} T(F(u, u|z_2, ...z_n)(t_1), F(v, v|z_2, ..., z_n)(t_2))], \end{aligned}$$

where the last inequality holds by the definition of distribution functions. Now by definition 2.4(iv):

$$N(u+v, z_2, ..., z_n)(t) = T(N(u, z_2, ..., z_n)(t), N(v, z_2, ..., z_n)(t)),$$

for all $t \in \mathbb{R}$. □

5. Fixed point Theorem in Intuitionistic Menger n-Normed Spaces and Application to Differential Equations in Intuitionistic Menger n-Normed Spaces

Our main purpose in the first section of this section is to prove the fixed point theory for generalized contraction maps in an intuitionistic Menger n-normed space. The proof will be deduced after proving some results. At first, for a given intuitionistic Menger n-normed space, we offer an algorithm to derive an intuitionistic Menger (n-1)-norm from an intuitionistic Menger n-norm and hence we will show that any intuitionistic Menger n-normed space is an intuitionistic Menger $(n-1)$-normed space. Then we show that, in certain conditions, the intuitionistic Menger (n-1)-norm can be derived from the intuitionistic Menger n-norm in such a way that the convergence and completeness in the intuitionistic Menger n-norm is equivalent to those in deduced intuitionistic Menger (n-1)-norm. Finally, this fact aims us to generalize the fixed point theorem for generalized contraction maps in intuitionistic Menger n-normed spaces. in the last part we utilize the results to prove the existence theorems of solutions to differential equations for intuitionistic Menger n-normed spaces.

Theorem 5.1. *Let $n \geq 2$, (X, N, M, T, S) be a IMNS of dimension $d \geq n$. Take a linear independent set $\{a_1, a_2, ..., a_n\}$ in X and for each $t > 0$ define:*

$$N_i(x_1, x_2, ..., x_{n-1})(t) = \min\{N(x_1, x_2, ..., x_{n-1}, a_k)(t) : k = 1, 2, ..., n\}.$$

$$M_i(x_1, x_2, ..., x_{n-1})(t) = \max\{M(x_1, x_2, ..., x_{n-1}, a_k)(t) : k = 1, 2, ..., n\}.$$

The functions N_i and M_i define an intuitionistic Menger (n-1)-norm on X.

Proof. Suppose that $x_1, x_2, ..., x_{n-1}$ be linearly dependent. Then by IMN-3) for each $k = 1, 2, ..., n$ and $t > 0$, $N(x_1, x_2, ..., x_{n-1}, a_k)(t) = 1$. This means that $N_i(x_1, x_2, ..., x_{n-1})(t) = 1$. Conversely suppose that $N_i(x_1, x_2, ..., x_{n-1})(t) = 1$. Then for each $k = 1, 2, ..., n$, $N(x_1, x_2, ..., x_{n-1}, a_k)(t) = 1$ and accordingly $x_1, x_2, ..., x_{n-1}, a_k$ are linearly independent, for each $k = 1, 2, ..., n$. But this can only happen when $x_1, x_2, ..., x_{n-1}$ are linearly dependent. Since for each k, $N(x_1, x_2, ..., x_{n-1}, a_k)(t)$ is invariant under any permutation of $\{x_1, x_2, ..., x_{n-1}\}$ it implies that $N_i(x_1, x_2, ..., x_{n-1})(t)$ is also invariant under any permutation. We observe that

$$\begin{aligned} N_i(x_1, x_2, ..., x_{n-2}, \alpha x_{n-1}, t) &= \min\{N(x_1, x_2, ..., \alpha x_{n-1}, a_k)(t) : k = 1, 2, ..., n\} \\ &= \min\{N(x_1, x_2, ..., x_{n-1}, a_k)(\tfrac{t}{|\alpha|}) : k = 1, 2, ..., n\} \\ &= N_i(x_1, x_2, ..., x_{n-2}, x_{n-1})(\tfrac{t}{|\alpha|}), \end{aligned}$$

for each $0 \neq \alpha \in \mathbb{R}$. Also for each $y, z \in X$, we can see that

$$\begin{aligned} N_i(x_1, x_2, ..., x_{n-2}, y + z, t) &= \min\{N(x_1, x_2, ..., x_{n-2}, y + z, a_k)(t) : k = 1, 2, ..., n\} \\ &\geq T(\min\{N(x_1, x_2, ..., x_{n-2}, y, a_k)(t) : k = 1, 2, ..., n\} \\ &\quad , \min\{N(x_1, x_2, ..., x_{n-2}, z, a_k)(t) : k = 1, 2, ..., n\}) \\ &= T(N_i(x_1, x_2, ..., x_{n-2}, y, t), N_i(x_1, x_2, ..., x_{n-2}, z)(t)). \end{aligned}$$

The other conditions of definition 3.1 are trivially hold for M_i and by applying the similar proof we can check the conditions for M_i. So we imply that (X, N, M, T, S) is a IMNS. ∎

Corollary 5.2. *Every intuitionistic Menger n-normed space is an intuitionistic Menger normed space.*

For a finite dimensional intuitionistic Menger n-normed space $(X,\ N,\ M,\ T,\ S)$, we can apply the previous theorem to drive an intuitionistic Menger (n-1)-norm from an intuitionistic Menger n-norm i.e. Take a linearly independent set $\{a_1, a_2, ..., a_m\}$ in X with $n \le m \le d$. With respect to $\{a_1, a_2, ..., a_m\}$, define the following functions.

$$N_i(x_1, x_2, ..., x_{n-1})(t) = \min\{N(x_1, x_2, ..., x_{n-1}, a_k)(t) : k = 1, 2, ..., m\},\ \forall t \in \mathbb{R}.$$

$$M_i(x_1, x_2, ..., x_{n-1})(t) = \max\{M(x_1, x_2, ..., x_{n-1}, a_k)(t) : k = 1, 2, ..., m\},\ \forall t \in \mathbb{R}.$$

Recall that a sequence $\{x_k\}$ in an intuitionistic Menger n-normed space (X, N, M, T, S) is said to converge to $x \in X$ if

$$\lim_{k\to\infty} N(x_1, x_2..., x_{n-1}, x_k - x)(t) = 1,$$

and

$$\lim_{k\to\infty} M(x_1, x_2..., x_{n-1}, x_k - x)(t) = 0,$$

for each $x_1, x_2, ..., x_{n-1} \in X$ and $t > 0$. Also $\{x_k\}$ is called a Cauchy sequence if

$$\lim_{k,l\to\infty} N(x_1, x_2..., x_{n-1}, x_k - x_l)(t) = 1,$$

and

$$\lim_{k,l\to\infty} M(x_1, x_2..., x_{n-1}, x_k - x_l)(t) = 0,$$

for each $x_1, x_2, ..., x_{n-1} \in X$ and $t > 0$.

Lemma 5.3. *Let $(X,\ N,\ M,\ T,\ S)$ be an intuitionistic Menger n-normed space and $\{x_k\} \subseteq X$ converges to $x \in X$ in the intuitionistic Menger n-norm. Then $\{x_k\}$ converges to x in the derived intuitionistic Menger (n-1)-norm N_i and M_i, that is*

$$\lim_{k\to\infty} N_i(x_1, x_2..., x_{n-2}, x_k - x)(t) = 1,$$
$$\lim_{k\to\infty} M_i(x_1, x_2..., x_{n-2}, x_k - x)(t) = 0,$$

for each $x_1, x_2, ..., x_{n-2} \in X$ and $t > 0$.

Proof. Suppose that $\{x_k\}$ converges to x in the intuitionistic Menger n-norm. Then

$$\lim_{k\to\infty} N(x_1, x_2..., x_{n-2}, x_k - x, a_k)(t) = 1,$$

and

$$\lim_{k\to\infty} M(x_1, x_2..., x_{n-2}, x_k - x, a_k)(t) = 0,$$

for each $x_1, x_2, ..., x_{n-2} \in X$, $t > 0$ and $k = 1, 2, ..., n$. Since min and max are continuous functions then

$$\lim_{k\to\infty} N_i(x_1, x_2..., x_{n-2}, x_k - x)(t) = 1,$$

and

$$\lim_{k\to\infty} M_i(x_1, x_2..., x_{n-2}, x_k - x)(t) = 0,$$

for each $x_1, x_2, ..., x_{n-2} \in X$ and $t > 0$. This means that $\{x_k\}$ converges to x in the derived intuitionistic Menger (n-1)-norm N_i and M_i. □

Theorem 5.4. *A sequence in the finite dimensional intuitionistic Menger n-normed space $(X,\ N,\ M,\ T,\ S)$ is convergent in the intuitionistic Menger n-norm if and only if it is convergent in the induced intuitionistic Menger (n-1)-norm N_i and M_i.*

Proof. Let $\{x_k\}$ be a sequence in X. According to previous lemma it is enough to show that $x_k \xrightarrow{N_i} x$ implies that $x_k \xrightarrow{M} x$ and $x_k \xrightarrow{M_i} x$ implies that $x_k \xrightarrow{M} x$. We show the first part. Suppose that $x_k \xrightarrow{N_i} x$ and $\{b_1, b_2, ..., b_d\}$ be a basis for X. Take $x_1, x_2, ...x_{n-1} \in X$. Writing $x_{n-1} = \alpha_1 b_1 + \alpha_2 b_2 + ... + \alpha_n b_n$, we get

$$\begin{aligned} N(x_1, x_2..., x_{n-2}, x_{n-1}, x_k - x)(t) \geq{}& T^d(N(x_1, x_2..., x_{n-2}, b_1, x_k - x)(\tfrac{t}{|\alpha_1|}) \\ &, ..., N(x_1, x_2..., x_{n-2}, b_d, x_k - x)(\tfrac{t}{|\alpha_d|})) \\ \geq{}& N_i(x_1, x_2, ..., x_{n-2}, x_k - x)(\tfrac{t}{|\alpha_1|+|\alpha_2|,...+|\alpha_d|}). \end{aligned}$$

But

$$\lim_{k\to\infty} N_i(x_1, x_2..., x_{n-2}, x_k - x, a_i)(t) = 1,$$

for each $t > 0$. So we obtain

$$\lim_{k\to\infty} N(x_1, x_2..., x_{n-1}, x_k - x)(t) = 1,$$

and this means that $x_k \xrightarrow{N} x$. □

Corollary 5.5. *A sequence in the finite dimensional intuitionistic Menger n-normed space $(X,\ N,\ M,\ T,\ S)$ is Cauchy in the intuitionistic Menger n-norm if and only if it is Cauchy in the intuitionistic Menger derived n-norm N_i and M_i.*

The theorem of generalized contraction for intuitionistic Menger spaces has been proved in [12](theorem 6), for a complete intuitionistic Menger space and by some conditions. On the other hand, if X is a finite dimensional complete intuitionistic Menger n-normed space, by corollary 5.5, X is a complete intuitionistic Menger normed space. So we have proved the following theorem.

Theorem 5.6. (Fixed Point Theorem for generalized contraction maps in intuitionistic Menger n-normed spaces) *Let $(X,\ N,\ M,\ T_s,\ S_s)$ be a finite dimensional complete intuitionistic Menger n-normed space and A a generalized contractive mapping of X into itself, that is,*

$$N(x_1, x_2, ..., x_{n-1}, Ay - Az)(t) \geq N(x_1, x_2, ..., x_{n-1}, y - z)(\frac{t}{k(\alpha, \beta)}),$$

and

$$M(x_1, x_2, ..., x_{n-1}, Ay - Az)(t) \leq M(x_1, x_2, ..., x_{n-1}, y - z)(\frac{t}{k(\alpha, \beta)}),$$

for all $x_1, x_2, ..., x_{n-1}, y, z \in X$ *and* $t \geq 0$. *Also suppose that*

$$N(x_1, x_2, ..., x_n)(\alpha) > 0 \text{ and } N(x_1, x_2, ..., x_n)(\beta) < 1,$$

and

$$M(x_1, x_2, ..., x_n)(\alpha) < 1 \text{ and } N(x_1, x_2, ..., x_n)(\beta) > 0,$$

for each $\alpha, \beta \in (0, \infty)$ *and a function* $k(\alpha, \beta) : (0, \infty)^2 \longrightarrow (0, 1)$. *Then A has a unique fixed point in X.*

Let $(X,\ F,\ L,\ T_s,\ S_s)$ be a complete intuitionistic Menger n-normed space and $C([0, K], X)$ be a set of mappings $x(.) : [0, K] \longrightarrow X$ which are continuous on $(X,\ F,\ L,\ T_s,\ S_s)$. For any $\varepsilon > 0$, $\lambda \in (0, 1)$ and $x_0 \in X$, we denote,

$$N(x_0, \varepsilon, \lambda) = \{x : F(x - x_0, x_2, ..., x_n)(\varepsilon) > 1 - \lambda \text{ and } L(x - x_0, x_2, ..., x_n)(\varepsilon) < \lambda\}$$

for all $x_2, ..., x_n \in X$.

Let f and g be two continuous mappings from $\mathbb{R} \times \overline{N(x_0, \varepsilon, \lambda)}$ into X satisfying the followings.

i) $F(f(t, x) - f(t, y), x_2, ..., x_n)(s) \geq F(x, y, ..., x_n)(\frac{s}{K})$ for $s \geq 0$, $x_2, ..., x_n \in X$, $x, y \in \overline{N(x_0, \varepsilon, \lambda)}$ and where $K > 0$ is a constant.

ii) $g(t, \overline{N(x_0, \varepsilon, \lambda)})$ is relatively compact subset in X for each t.

Also by applying the results of this section and theorem 8 in [12] we have the following result.

Theorem 5.7. *Let* $(X,\ F,\ L,\ T_s,\ S_s)$ *be a complete intuitionistic Menger* n*-normed space having the above conditions i) and ii). Then there exists* $\delta_0 > 0$ *small enough such that the differential equation:*

$$\begin{cases} x'(t) = f(t, x) + g(t, x), \\ x(0) = x_0 \end{cases} .$$

has a solution in $C([0, \delta], X)$.

6. Conclusion

In this paper first we introduced the definition of an intuitionistic Menger n-normed space, an intuitionistic Menger n-inner product space and then we proved some results in both mentioned spaces. Also we proved a fixed point theorem for generalized contraction maps in intuitionistic Menger n-normed spaces and we utilized it to prove the existence theorems of solutions to differential equations for intuitionistic Menger n-normed spaces and we investigated some interesting their properties.

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