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A NEW FAMILY OF NONCONFORMING MIXED FINITE ELEMENTS FOR SECOND ORDER ELLIPTIC PROBLEMS IN $\mathbb{R}^3$

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Abstract: In this paper, we introduce a new family of nonconforming mixed finite elements on parallelepiped to approximate the solutions of second order elliptic problems. We suggest a degrees of freedom and prove that new space is uniquely determined by the degrees of freedom. Using new elements, we obtain a priori error estimates. The results of numerical experiments are presented to illustrate the convergence of the new elements.

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1. Introduction

In this paper, we consider the following second order elliptic boundary value problem:

$$\begin{cases} -\mathrm{div}(\kappa\nabla p) &= f, \quad \text{in } \Omega, \\ p &= 0, \quad \text{on } \partial\Omega, \end{cases} \tag{1}$$

over a bounded Lipschitz domain Ω in $\mathbb{R}^3$. Where f is a given function in $L^2(\Omega)$ and $\kappa = \kappa(\mathbf{x})$ is a symmetric and uniformly positive definite matrix, i.e., there exists two positive constants c_1 and c_2 such that

$$c_1\xi^T\xi \le \xi^T\kappa(\mathbf{x})\xi \le c_2\xi^T\xi, \quad \forall\xi \in \mathbb{R}^3,\ \mathbf{x} \in \overline{\Omega}.$$

We are interested in finite element discretization of (1) over unstructured meshes. The finite element method has achieved great success in many fields of science and technology since it was first suggested in elasticity. So it has become a powerful tool for solving partial differential equation today. However, there are some situations ordinary finite element method can barely cope with. Among these are the presence

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of large jumps in the diffusion coefficient κ. Then it is vital for the discretization to preserve certain conservation properties of continuous problem. Sometimes we take an interest in other quantity such as velocity beside the solution p. In this case, mixed finite element method should be preferred.

To explain mixed finite element methods, let us introduce a vector variable $\mathbf{u} = -\kappa\nabla p$. Then the problem (1) can be factored to give the following first order system:

$$\begin{cases} \mathbf{u} + \kappa\nabla p &= 0, \quad \text{in } \Omega, \\ \operatorname{div} \mathbf{u} &= f, \quad \text{in } \Omega, \\ p &= 0, \quad \text{on } \partial\Omega. \end{cases} \tag{2}$$

The first equation of (2), which relates $\mathbf{u}$ and p, is called Darcy's law, and the second equation represents the conservation of mass. Now we introduce the function spaces

$$\begin{aligned} &\mathbf{V} = H(\operatorname{div}, \Omega) = \{\mathbf{v} \in (L^2(\Omega))^3 : \operatorname{div} \mathbf{v} \in L^2(\Omega)\}, \\ &W = L^2(\Omega). \end{aligned}$$

The weak form of (2) leads to the following saddle point problem: find $(\mathbf{u}, p) \in \mathbf{V} \times W$ such that

$$\begin{aligned} (\kappa^{-1}\mathbf{u}, \mathbf{v}) - (p, \operatorname{div} \mathbf{v}) &= 0, & \forall \mathbf{v} \in \mathbf{V}, \\ (\operatorname{div} \mathbf{u}, q) &= (f, q), & \forall q \in W, \end{aligned} \tag{3}$$

where $(\cdot, \cdot)$ indicates the inner product in $L^2(\Omega)$ or $(L^2(\Omega))^3$. For simplicity of notation, we let

$$a(\mathbf{u}, \mathbf{v}) = \int_\Omega \kappa^{-1}\mathbf{u}\mathbf{v}d\mathbf{x}, \tag{4}$$

$$b(p, \mathbf{v}) = -\int_\Omega p \operatorname{div} \mathbf{v}d\mathbf{x}. \tag{5}$$

Then the weak formulation becomes like this: find $(\mathbf{u}, p) \in \mathbf{V} \times W$ such that

$$\begin{aligned} a(\mathbf{u}, \mathbf{v}) + b(p, \mathbf{v}) &= 0, & \forall \mathbf{v} \in \mathbf{V}, \\ b(\mathbf{u}, q) &= -(f, q), & \forall q \in W. \end{aligned} \tag{6}$$

The variational problem (6) is the starting point for mixed finite element methods. The mixed finite element method has been investigated for the simulation of porous media problems or semiconductor device equations([1], [2], [3], [4], [5]). In the latter application, the major drawback of the common mixed finite element method is vulnerability for ill shaped elements. This is due to poor local approximation of the velocity variable. A solution is to enhance the discrete space for velocity using a nonconforming method. Nonconforming methods rely on approximations not contained in the space the exact solution lives in. This method have been widely researched for ordinary finite element method but not for the mixed case. In this paper, we consider the nonconforming mixed finite elements methods.

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The remainder of the paper is organized as follows: In Section 2, we introduce a new family of nonconforming mixed finite elements on parallelepiped and prove the unisolvence of degrees of freedom. The next section is devoted to a priori estimates of the discretization error. Finally, we present the results of numerical experiments.

2. Construction of New Nonconforming Mixed Finite Elements

In this section, we introduce new nonconforming mixed finite elements on parallelepiped K. First, we introduce some notations for simplicity.

$$\begin{aligned} Q_{\ell,m}(x,y) &= \{\text{polynomials of degree up to } \ell, \textit{ and } m \text{ in } x \text{ and } y, \text{ respectively}\}, \\ Q'_{\ell,m,n}(K) &= \text{Span}\{x^py^qz^r \mid 0 \le p \le \ell,\ 0 \le q \le m,\ 0 \le r \le n,\ (p,q,r) \ne (\ell,m,i), \\ &\qquad i = 1, \dots n\}, \\ Q''_{\ell,m,n}(K) &= \text{Span}\{x^py^qz^r \mid\ x^py^qz^r \in Q'_{\ell,m,n}(K),\ (p,q) = (l+1,j) \text{ for} \\ &\qquad j = 0, \dots m, \text{ and } (p,q) = (k, m+1) \text{ for } k = 0, \dots \ell\}. \end{aligned}$$

Definition 1. We define

$$\mathbf{V}_h(K) = Q'_{k+1,k,k}(K) \times Q'_{k,k+1,k}(K) \times Q''_{k,k,k+1}(K),$$

where the elements $(x^{k+1}y^k,\ 0,\ 0)$ and $(0,\ x^ky^{k+1},\ 0)$ are replaced by the single element $(x^{k+1}y^k,\ -x^ky^{k+1},\ 0)$. And

$$W_h(K) = Q'_{k,k,k}(K) \backslash \{x^ky^k\}.$$

Then, we see that

$$\begin{aligned} \dim \mathbf{V}_h(K) &= [\ 2((k+1)^2(k+2) - k) + ((k+1)^2(k+2) - (k+1) + 2(k+1))\ ] - 1 \\ &= 3k^3 + 12k^2 + 14k + 6 \end{aligned}$$

and

$$\dim W_h(K) = [\ (k+1)^3 - k\ ] - 1 = k(k+1)(k+2).$$

From the above definition, we give an following example for $k = 1$. The space $\mathbf{V}_h(K)$ consists of all vector polynomials (u_1, u_2, u_3) where

$$\begin{aligned} u_1 &= P_1(x,y,z) + a_1xy + a_2yz + a_3zx + a_4xyz + a_5x^2 + a_6x^2z + dx^2y, \\ u_2 &= P_1(x,y,z) + b_1xy + b_2yz + b_3zx + b_4xyz + b_5y^2 + b_6y^2\hat{z} - dxy^2, \\ u_3 &= P_1(x,y,z) + c_1xy + c_2yz + c_3zx + c_4z^2 + c_5xz^2 + c_6yz^2 \\ &\quad + c_7x^2 + c_8y^2 + c_9x^2y + c_{10}xy^2, \end{aligned}$$

and $W_h(K)$ is a space containing polynomial of the following form:

$$p = r_1 + r_2x + r_3y + r_4z + r_5yz + r_6zx.$$

Lemma 2. *Let $\mathbf{V}_h(K)$ and $W_h(K)$ are given spaces by Definition 1. Then we have*

$$\operatorname{div} \mathbf{V}_h(K) = W_h(K).$$

Proof. Let $\mathbf{v} = (v_1, v_2, v_3)$ be any element in $Q'_{k+1,k,k}(K) \times Q'_{k,k+1,k}(K) \times Q''_{k,k,k+1}(K)$. Then

$$\frac{\partial v_1}{\partial x} = q_1, \quad \text{for some} \quad q_1 \in Q'_{k,k,k}(K)$$

and

$$\frac{\partial v_2}{\partial y} = q_2, \quad \text{for some} \quad q_2 \in Q'_{k,k,k}(K).$$

Since $Q'_{k,k,k}(K)\backslash\{x^k y^k\} = \text{Span}\{x^p y^q z^r \mid 0 \le p,\ q,\ r \le k,\ (p,q,r) \ne (k,k,i)\ for\ i = 0, \dots k\}$,

$$\frac{\partial v_3}{\partial z} = q_3, \quad \text{for some} \quad q_3 \in Q'_{k,k,k}(K)\backslash\{x^k y^k\}.$$

From the construction of $\mathbf{V}_h(K)$, we know that the elements $(x^{k+1}y^k,\ 0,\ 0)$ and $(0,\ x^k y^{k+1},\ 0)$ are replaced by the single element $(x^{k+1}y^k,\ -x^k y^{k+1},\ 0)$. Hence for $\mathbf{v} \in \mathbf{V}_h(K)$, we see that $\operatorname{div} \mathbf{v} = q$, for some $q \in Q'_{k,k,k}(K)\backslash\{x^k y^k\}$. Therefore we see that $\operatorname{div} \mathbf{V}_h(K) \subseteq W_h(K)$. The reverse inclusion $\operatorname{div} \mathbf{V}_h(K) \supseteq W_h(K)$ is clear, thus the proof is complete. □

Now we need to define the degrees of freedom. For this purpose, we define

$$\Psi_k(K) = Q'_{k-1,k,k}(K) \times Q'_{k,k-1,k}(K),$$

where the elements $(x^{k-1}y^k,\ 0)$ and $(0,\ x^k y^{k-1})$ are replaced by the single element $(x^{k-1}y^k,\ -x^k y^{k-1})$.

For any $\mathbf{u} = (u_1, u_2, u_3) \in \mathbf{V}_h(K)$, we consider the following degrees of freedom for new velocity spaces.

$$\int_{f_H} \mathbf{u} \cdot \mathbf{n}\, q\, dA,\ \forall q \in Q_{k,k}(f_H)\backslash\{x^k y^k\},\ \text{for each horizontal faces } f_H, \quad (7)$$

$$\int_{f_V} \mathbf{u} \cdot \mathbf{n}\, q\, dA,\ \forall q \in Q_{k,k}(f_V),\ \text{for each vertical faces } f_V, \quad (8)$$

$$\int_K u_3 \phi_3\, d\mathbf{x}, \qquad \forall \phi_3 \in Q''_{k,k,k-1}, \quad (9)$$

$$\int_K (u_1\phi_1 + u_2\phi_2)\, d\mathbf{x}, \quad \forall \boldsymbol{\phi} = (\phi_1, \phi_2) \in \Psi_k(K). \quad (10)$$

The number of conditions is $2[\ (k+1)^2 - 1\ ] + 4(k+1)^2 + [\ k(k+1)^2 - (k-1) + 2(k+1)\ ] + 2[\ k(k+1)^2 - k\ ] - 1$ which is also the dimension of $\mathbf{V}_h(K)$. See Figure 1 for an illustration of degrees of freedom when $k = 1$.

We need to show that the degrees of freedom are independent.

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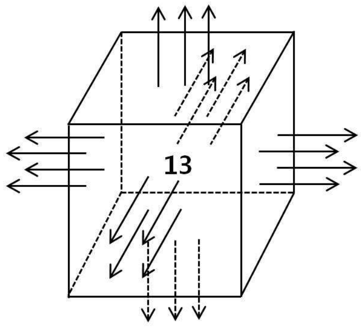

Figure 1: For simplicity, we only show the degrees of freedom for $k = 1$. There are three normal component degrees of freedom per upper and bottom faces. Also, there are four normal component degrees of freedom per side faces and thirteen interior degrees of freedom.

Theorem 3. (Unisolvence) *A vector function* $\mathbf{u} = (u_1, u_2, u_3) \in \mathbf{V}_h(K)$ *in the Definition* 1 *is uniquely determined by the degrees of freedom* $(7) - (10)$.

Proof. To prove unisolvence, we first note that the dimension of $\mathbf{V}_h(K)$ is also the number of degrees of freedom. Hence it suffices to show that if all the quantities $(7)-(10)$ vanish, then $\mathbf{u} = 0$. Since $\mathbf{u}\cdot\mathbf{n} \in Q_{k,k}(f_V)$ on the vertical faces, (8) implies $\mathbf{u} \cdot \mathbf{n} = 0$ for each vertical faces. Then we have

$$u_1 = x(1-x)v_1, \quad u_2 = y(1-y)v_2, \quad \text{where } \mathbf{v} = (v_1, v_2) \in \Psi_k(K).$$

Hence choosing $\boldsymbol{\phi} = \mathbf{v}$ in (10) proves that $\mathbf{v} = 0$. Therefore $\tilde{\mathbf{u}} = (u_1, u_2) = 0$. Now consider the $u_3 \in Q''_{k,k,k+1}(K)$. We note that

$$u_3 = s + rz^k + r'z^{k+1}, \quad \forall s \in Q''_{k,k,k-1}(K), \quad \forall r,\ r' \in Q_{k,k}(f_H)\backslash\{x^k y^k\}.$$

Since $Q_{k,k}(f_H)\backslash\{x^k y^k\}$ is a subspace of $Q''_{k,k,k-1}(K)$, the degrees of freedom (9) implies

$$u_3 = r_1 z^k + r'_1 z^{k+1}, \quad \forall r_1,\ r'_1 \in Q_{k,k}(f_H)\backslash\{x^k y^k\}.$$

From the degrees of freedom (7), we know that $u_3 = 0$. This completes the proof. □

3. A Priori Error Estimates

From the construction of new elements, we know that $W_h \subset W = L^2(\Omega)$ but $\mathbf{V}_h \not\subseteq \mathbf{V} = H(\mathrm{div}\,, \Omega)$. Then we cannot guarantee that the bilinear forms $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$

make sense for functions of $\mathbf{V}_h$. Hence, we need to introduce the larger space $\mathbf{V}_h \cup H(\text{div}\,,\Omega) = \mathbf{X}_h$ on a triangulation $\mathcal{T}_h = \{K_i\}$ of Ω. Let

$$a_h(\mathbf{u},\mathbf{v}) = \sum_i \int_{K_i} \kappa^{-1}\mathbf{u}\mathbf{v}d\mathbf{x}, \tag{11}$$

$$b_h(p,\mathbf{v}) = -\sum_i \int_{K_i} p \text{ div}\,\mathbf{v}d\mathbf{x}. \tag{12}$$

Since dim $\mathbf{V}_h(K) < \infty$ and $\mathbf{V}_h(K) = \{\mathbf{v} \in (L^2(\Omega))^3 : \mathbf{v}|_{K_i} \in H(\text{div}\,,K_i),\ \forall K_i \in \mathcal{T}_h\}$, we know that (11) and (12) are valid definitions. Moreover the extended forms coincide with the original ones if their restrictions to $\mathbf{V} = H(\text{div}\,,\Omega)$ are considered:

$$\begin{aligned} a_h(\mathbf{u},\mathbf{v}) &= a(\mathbf{u},\mathbf{v}), \quad \forall \mathbf{u},\ \mathbf{v} \in \mathbf{V}, \\ b_h(p,\mathbf{v}) &= b(p,\mathbf{v}), \quad \forall p \in W,\ \mathbf{v} \in \mathbf{V}. \end{aligned} \tag{13}$$

Then $\mathbf{X}_h$ is equipped with a norm that is a reasonable extension of $\|\cdot\|_{H(\text{div}\,,\Omega)}$, i.e.

$$\| \mathbf{v} \|_{\mathbf{X}}^2 = \sum_i \| \mathbf{v} \|^2_{H(\text{div}\,,K_i)}\ .$$

The discrete variational problem is to find $(\mathbf{u}_h, p_h) \in \mathbf{X}_h \times W_h$ such that

$$\begin{aligned} a_h(\mathbf{u}_h,\mathbf{v}_h) + b_h(p_h,\mathbf{v}_h) &= 0, & \forall \mathbf{v}_h \in \mathbf{V}_h, \\ b_h(\mathbf{u}_h,q_h) &= -(f,q_h), & \forall q_h \in W_h. \end{aligned} \tag{14}$$

Where the bilinear forms $a_h(\cdot,\cdot)$ and $b_h(\cdot,\cdot)$ have the following continuous property:

$$a_h(\mathbf{u}_h,\mathbf{v}_h) \le A \| \mathbf{u}_h \|_X \| \mathbf{v}_h \|_X,$$
$$b_h(p_h,\mathbf{v}_h) \le B \| p_h \|_W \| \mathbf{v}_h \|_X,$$

with positive constants A and B. To confirm the existence of a unique discrete solution of (14), we need to check the following inf-sup condition(Babuška-Brezzi conditions) (see [6], [7]):

$$\sup_{\mathbf{v}_h \in\ N(B_h)} \frac{a_h(\mathbf{v}_h,\mathbf{w}_h)}{\| \mathbf{v}_h \|_{\mathbf{X}}} \ge \alpha \| \mathbf{w}_h \|_{\mathbf{X}},\ \forall \mathbf{w}_h \in N(B_h)$$
$$= \{\mathbf{v}_h \in \mathbf{X}_h \mid b_h(\mathbf{v}_h,q_h) = 0,\ \forall q_h \in W_h\},$$

$$\sup_{\mathbf{v}_h \in\ \mathbf{X}_h} \frac{b_h(q_h,\mathbf{v}_h)}{\| \mathbf{v}_h \|_{\mathbf{X}}} \ge \beta \inf_{r_h \in N(B_h^*)} \| r_h + q_h \|_W = \beta \| q_h \|_{W/N(B_h^*)},\ \forall q_h \in W_h,$$

with positive constants $\alpha, \beta > 0$ independent of the mesh size h.

First, we define an interpolation operator $\mathbf{\Pi}_h : \mathbf{H}^{k+1}(K) \to \mathbf{V}_h(K)$ satisfying

$$\int_{f_H} (\mathbf{u} - \mathbf{\Pi}_h \mathbf{u}) \cdot \mathbf{n}\, q\, dA = 0,\ \forall q \in Q_{k,k}(f_H)\backslash\{x^k y^k\}, \quad \text{for each horizontal faces } f_H,$$

$$\int_{f_V} (\mathbf{u} - \mathbf{\Pi}_h \mathbf{u}) \cdot \mathbf{n}\, q\, dA = 0,\ \forall q \in Q_{k,k}(f_V),\ \text{for each vertical faces } f_V,$$

$$\int_K (u_3 - \mathbf{\Pi}_h u_3)\phi_3\, d\mathbf{x} = 0, \quad \forall \phi_3 \in Q''_{k,k,k-1},$$

$$\int_K \{(u_1 - \mathbf{\Pi}_h u_1)\phi_1 + (u_2 - \mathbf{\Pi}_h u_2)\phi_2\}\ d\mathbf{x} = 0, \quad \forall \boldsymbol{\phi} = (\phi_1, \phi_2) \in \Psi_k(K).$$

Then this operator has the following property:

Lemma 4. *If $\mathbf{\Pi}_h \mathbf{u}$ is the interpolation of $\mathbf{u}$, then we have*

$$(\operatorname{div}(\mathbf{u} - \mathbf{\Pi}_h \mathbf{u}), q)_K = 0, \quad \forall\ \mathbf{u} \in \mathbf{V}_h(K), \quad \forall q \in W_h(K).$$

Proof. First, let $q \in W_h(K)$. Then we know that $q|_{f_H} \in Q_{k,k}(f_H)\backslash\{x^k y^k\}$ for each horizontal faces f_H and $q|_{f_V} \in Q_{k,k}(f_H)$ for each vertical faces f_V. Also,

$$\nabla q = \begin{pmatrix} \nabla q_1 \\ \nabla q_2 \\ \nabla q_3 \end{pmatrix} = \begin{pmatrix} Q'_{k-1,k,k}\backslash\{x^{k-1}y^k\} \\ Q'_{k,k-1,k}\backslash\{x^k y^{k-1}\} \\ Q'_{k,k-1,k-1}\backslash\{x^k y^k\} \end{pmatrix}.$$

Since $(\nabla q_2, \nabla q_3) \in \Psi_k(K)$ and $\nabla q_3 \in Q''_{k,k,k-1}(K)$, we have by the definition of $\mathbf{\Pi}_h$,

$$\begin{aligned}
(\operatorname{div}\mathbf{\Pi}_h \mathbf{u}, q)_K &= \int_{\partial K} \mathbf{\Pi}_h \mathbf{u} \cdot \mathbf{n}\ q\ dA - \int_K \mathbf{\Pi}_h \mathbf{u} \cdot \nabla q\ d\mathbf{x} \\
&= \left\{ \int_{\partial f_H} \mathbf{\Pi}_h \mathbf{u} \cdot \mathbf{n}\ q\ dA + \int_{\partial f_V} \mathbf{\Pi}_h \mathbf{u} \cdot \mathbf{n}\ q\ dA \right\} - \\
&\quad \left\{ \int_K \mathbf{\Pi}_h u_3 \nabla q_3\ d\mathbf{x} + \int_K (\mathbf{\Pi}_h u_1 \nabla q_1 + \mathbf{\Pi}_h u_2 \nabla q_2)\ d\mathbf{x} \right\} \\
&= \left\{ \int_{\partial f_H} \mathbf{u} \cdot \mathbf{n}\ q\ dA + \int_{\partial f_V} \mathbf{u} \cdot \mathbf{n}\ q\ dA \right\} - \\
&\quad \left\{ \int_K u_3 \nabla q_3\ d\mathbf{x} + \int_K (u_1 \nabla q_1 + u_2 \nabla q_2)\ d\mathbf{x} \right\} \\
&= \int_{\partial K} \mathbf{u} \cdot \mathbf{n}\ q\ dA - \int_K \mathbf{u} \cdot \nabla q\ d\mathbf{x} \\
&= (\operatorname{div}\mathbf{u}, q)_K.
\end{aligned}$$

□

From Lemma 2 and 4, we know that the inf-sup conditions are satisfied [8]. Now we show the analogy of Strang lemma.

Theorem 5. *Let* $(\mathbf{u}, p) \in \mathbf{V} \times W$ *be the solution of* (6) *and* $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times W_h$ *be the discrete solution of* (14). *Then we obtain the following error estimates:*

$$\begin{aligned} \| \mathbf{u} - \mathbf{u}_h \|_{\mathbf{X}} + \| p - p_h \|_W \quad \leq \quad & c \left(\inf_{\mathbf{v}_h \in \mathbf{V}_h} \| \mathbf{u} - \mathbf{v}_h \|_{\mathbf{V}} + \inf_{q_h \in W_h} \| p - q_h \|_W + \right. \\ & \left. \sup_{\mathbf{v}_h \in \mathbf{V}_h} \frac{| \; a_h(\mathbf{u}, \mathbf{v}_h) + b_h(p, \mathbf{v}_h) \; |}{\| \mathbf{v}_h \|_{\mathbf{X}}} \right), \end{aligned}$$

where c *is a constant independent of* h.

Proof. For an arbitrary $\mathbf{v}_h \in \mathbf{V}_h$, we let $\mathbf{w}_h \in \mathbf{V}_h$ be such that

$$b_h(q_h, \mathbf{w}_h) = b_h(q_h, \mathbf{u} - \mathbf{v}_h), \;\; \forall q_h \in W_h. \tag{15}$$

Let $\mathbf{x}_h = \mathbf{v}_h + \mathbf{w}_h$. Then we have

$$b_h(q_h, \mathbf{x}_h) = b_h(q_h, \mathbf{u}), \;\; \forall q_h \in W_h.$$

Using $\mathbf{u}_h - \mathbf{x}_h \in N(B_h)$, we obtain from the first inequality of inf-sup condition

$$\begin{aligned} & \alpha \| \mathbf{u}_h - \mathbf{x}_h \|_X \leq \sup_{\mathbf{v}_h \in N(B_h)} \frac{1}{\| \mathbf{v}_h \|_X} \mid a_h(\mathbf{u}_h - \mathbf{x}_h, \mathbf{v}_h) \mid \\ & \leq \sup_{\mathbf{v}_h \in N(B_h)} \frac{1}{\| \mathbf{v}_h \|_X} \mid a_h(\mathbf{u} - \mathbf{x}_h, \mathbf{v}_h) + a_h(\mathbf{u}_h - \mathbf{u}, \mathbf{v}_h) \mid \\ & \leq \sup_{\mathbf{v}_h \in N(B_h)} \frac{1}{\| \mathbf{v}_h \|_X} \mid a_h(\mathbf{u} - \mathbf{x}_h, \mathbf{v}_h) - b_h(q_h - p, \mathbf{v}_h) - \{a_h(\mathbf{u}, \mathbf{v}_h) + b_h(p, \mathbf{v}_h)\} \mid, \end{aligned}$$

for any $q_h \in W_h$. Since both bilinear forms are bounded, we get

$$\alpha \| \mathbf{u}_h - \mathbf{x}_h \|_X \leq A \| \mathbf{u} - \mathbf{x}_h \|_X + B \| p - q_h \|_W + \sup_{\mathbf{v}_h \in N(B_h)} \frac{| \, a_h(\mathbf{u}, \mathbf{v}_h) + b_h(p, \mathbf{v}_h) \, |}{\| \mathbf{v}_h \|_X}.$$

From the inverse triangle inequality, we have

$$\begin{aligned} \alpha(\| \mathbf{u} - \mathbf{u}_h \|_X - \| \mathbf{u} - \mathbf{x}_h \|_X) \leq A \| \mathbf{u} - \mathbf{x}_h \|_X + B \| p - q_h \|_W & \\ + \sup_{\mathbf{v}_h \in N(B_h)} \frac{| \, a_h(\mathbf{u}, \mathbf{v}_h) + b_h(p, \mathbf{v}_h) \, |}{\| \mathbf{v}_h \|_X}. & \end{aligned}$$

Therefore

$$\| \mathbf{u} - \mathbf{u}_h \|_X \quad \leq \quad \left(1 + \frac{A}{\alpha}\right) \| \mathbf{u} - \mathbf{x}_h \|_X + \frac{B}{\alpha} \| p - q_h \|_W +$$

$$\frac{1}{\alpha}\sup_{\mathbf{v}_h\in N(B_h)}\frac{\mid a_h(\mathbf{u},\mathbf{v}_h)+b_h(p,\mathbf{v}_h)\mid}{\parallel \mathbf{v}_h\parallel_X}. \tag{16}$$

From the second inequality of inf-sup condition and equation (15), we also have

$$\parallel \mathbf{u}-\mathbf{x}_h\parallel_X\leq\parallel \mathbf{u}-\mathbf{v}_h\parallel_X+\parallel \mathbf{w}_h\parallel_X\leq\left(1+\frac{B}{\beta}\right)\parallel \mathbf{u}-\mathbf{v}_h\parallel_X . \tag{17}$$

By (16) and (17), we have the following estimate:

$$\begin{aligned}\parallel \mathbf{u}-\mathbf{u}_h\parallel_X \quad\leq\quad & \left(1+\frac{A}{\alpha}\right)\left(1+\frac{B}{\beta}\right)\parallel \mathbf{u}-\mathbf{v}_h\parallel_X+\frac{B}{\alpha}\parallel p-q_h\parallel_W+\\ & \frac{1}{\alpha}\sup_{\mathbf{v}_h\in N(B_h)}\frac{\mid a_h(\mathbf{u},\mathbf{v}_h)+b_h(p,\mathbf{v}_h)\mid}{\parallel \mathbf{v}_h\parallel_X}.\end{aligned} \tag{18}$$

Using the first equation of (14), we know that

$$a_h(\mathbf{u}-\mathbf{u}_h,\mathbf{v}_h)+b_h(p-p_h,\mathbf{v}_h)=a_h(\mathbf{u},\mathbf{v}_h)+b_h(p,\mathbf{v}_h),\ \forall\mathbf{v}_h\in\mathbf{V}_h.$$

Then

$$b_h(p-p_h,\mathbf{v}_h)=-a_h(\mathbf{u}-\mathbf{u}_h,\mathbf{v}_h)+a_h(\mathbf{u},\mathbf{v}_h)+b_h(p,\mathbf{v}_h),\ \forall\mathbf{v}_h\in\mathbf{V}_h.$$

From the second inequality of inf-sup condition, we obtain

$$\begin{aligned}\parallel p-p_h\parallel_{W_h/N(B_h^*)}&\leq\frac{1}{\beta}\sup_{\mathbf{v}_h\in\mathbf{V}_h}\frac{\mid b_h(p-p_h,\mathbf{v}_h)\mid}{\parallel \mathbf{v}_h\parallel_X}\\ &\leq\frac{1}{\beta}\left(A\parallel \mathbf{u}-\mathbf{u}_h\parallel_X+\sup_{\mathbf{v}_h\in\mathbf{V}_h}\frac{\mid a_h(\mathbf{u},\mathbf{v}_h)+b_h(p,\mathbf{v}_h)\mid}{\parallel \mathbf{v}_h\parallel_X}\right).\end{aligned} \tag{19}$$

Joining the estimates (18) and (19), we obtain the desired result:

$$\begin{aligned}\parallel \mathbf{u}-\mathbf{u}_h\parallel_{\mathbf{X}}+\parallel p-p_h\parallel_W \quad\leq\quad & c\ \Bigg(\inf_{\mathbf{v}_h\in\mathbf{V}_h}\parallel \mathbf{u}-\mathbf{v}_h\parallel_{\mathbf{V}}+\inf_{q_h\in W_h}\parallel p-q_h\parallel_W+\\ & \sup_{\mathbf{v}_h\in\mathbf{V}_h}\frac{\mid\ a_h(\mathbf{u},\mathbf{v}_h)+b_h(p,\mathbf{v}_h)\ \mid}{\parallel \mathbf{v}_h\parallel_{\mathbf{X}}}\Bigg),\end{aligned}$$

where c is a constant depending only on A, B, α and β. □

From above theorem, we can show that the nonconforming approximation spaces $\mathbf{V}_h$ defined Definition 1 along with proper space W_h give a convergent nonconforming mixed discretization of (6), for $k\geq 1$(cf. [8], [9], [10]). In fact, as we see in the next section, our scheme exhibits optimal order of convergence for all variables.

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4. Numerical Results

In this section, we report some results of numerical experiment using our new element. We solve the problem (14) on the unit cube $\Omega = [0,1]^3$. When $\kappa = I$, the function

$$\begin{aligned} p(x,y,z) &= x(1-x)y(1-y)z(1-z), \\ p(x,y,z) &= sin(\pi x)y(1-y)z(1-z) \end{aligned}$$

are chosen as the exact solution of example 1 and 2. As example 3, we choose example 1 with the diffusion coefficient $\kappa = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0.2 \end{pmatrix}$. When $k = 1$, there are thirty five degrees of freedom for the velocity space, and six for the pressure space on each element. Then our new element has optimal second order accuracy for all variables, when we checked error with twenty seven Gaussian points.

	Pressure Error		Velocity Error	
Level	error	order	error	order
0	0.00395026	*	0.0184307	*
1	0.00142836	1.46759	0.00566302	1.70247
2	0.00040067	1.83387	0.00152621	1.89162
3	0.000102982	1.96002	0.000388931	1.97237
4	2.59223e-005	1.99013	9.77005e-005	1.99308
5	6.49164e-006	1.99754	2.44545e-005	1.99827

Table 1: L^2-error for pressure and velocity of example 1

	Pressure Error		Velocity Error	
Level	error	order	error	order
0	0.0159991	*	0.0742989	*
1	0.00561319	1.5111	0.0220259	1.75414
2	0.00153736	1.86837	0.00581344	1.92174
3	0.000392801	1.96858	0.00147434	1.97932
4	9.87306e-005	1.99223	0.000369932	1.99474
5	2.47158e-005	1.99806	9.25677e-005	1.99868

Table 2: L^2-error for pressure and velocity of example 2

	Pressure Error		Velocity Error	
Level	error	order	error	order
0	0.00429096	*	0.0182764	*
1	0.00142836	1.58694	0.00400256	2.19099
2	0.00040067	1.83387	0.00100935	1.9875
3	0.000102982	1.96002	0.000252824	1.99722
4	2.59223e-005	1.99013	6.32353e-005	1.99933
5	6.49164e-006	1.99754	1.58107e-005	1.99983

Table 3: L^2-error for pressure and velocity of example 3

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