International Electronic Journal of Pure and Applied Mathematics
Volume 3 No. 4 2011, 279-287

EXACT SOLITON SOLUTIONS OF THE KP-MKP EQUATION VIA THE GENERALIZED TANH-FUNCTION AND THE (G'/G)-EXPANSION METHODS

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Abstract: The generalized tanh-function method is one of most direct and effective algebraic method for obtaining exact solutions of nonlinear partial differential equations. In this paper, the generalized tanh-function and the (G'/G)-expansion methods are used to construct exact solutions of the Kadomtsev-Petviashvili-modified Kadomtsev-Petviashvili (KP-mKP) equation.

AMS Subject Classification: 47F05, 35B05, 35Dxx
Key Words: generalized tanh-function method, Riccati equation, Kadomtsev-Petviashvili (KP)-modified KP equation, (G'/G)-expansion method

1. Introduction

It is well known that the nonlinear partial differential equations (NPDEs) are widely used to describe complex phenomena in various fields of sciences, such as physics, biology, chemistry, etc. Therefore, seeking exact solutions of NPDEs are very important and significant in the nonlinear sciences. In the past decades, a great effort has been made for this task and many powerful methods have been presented, such as homogeneous balance method [1-3], modified simplest equation method [4], tanh method [5-7], Jacobian elliptic function expansion method [8], first integral method [9-11], (G'/G)-expansion method [12] and so on.

Received: March 7, 2011

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International Electronic Journal of Pure and Applied Mathematics – IEJPAM, Volume 3, No. 4 (2011)

Very recently, Wang et al. [12] introduced a new method called the (G'/G)-expansion method to look for travelling wave solutions of nonlinear evolution equations. The (G'/G)-expansion method is based on the assumptions that the travelling wave solutions can be expressed by a polynomial in (G'/G), and that $G = G(\xi)$ satisfies a second order linear ordinary differential equation (ODE). The degree of the polynomial can be determined by considering the homogeneous balance between the highest order derivative and nonlinear terms appearing in the given nonlinear evolution equations. The coefficients of the polynomial can be obtained by solving a set of algebraic equations resulted from the process of using the method. By using the (G'/G)-expansion method, Wang et al. successfully obtained more travelling wave solutions of four nonlinear evolution equations.

Lately, work has been done on the extensions of the (G'/G)-expansion method. For example, in [13], the method was improved to deal with the mKdV equation with variable coefficients. In [14], the method was improved to find more types of non-travelling wave and coefficient function solutions.

The aim of this paper is to find exact soliton solutions of the Kadomtsev-Petviashvili (KP)-modified KP equation [15] by using the generalized tanh-function and the (G'/G)-expansion methods.

2. Methodology

Let us consider a PDE for $u(x, y, t)$ in the form

$$F(u, u_x, u_y, u_t, u_{xx}, u_{xt}, u_{xy}, ...) = 0, \tag{1}$$

where F is a polynomial in its arguments. By the transformation $u(x, y, t) = u(\xi),\ \xi = x + \alpha y + \beta t$, where α and β are non-zero arbitrary constants, equation (1) can be reduced into an ODE of the form

$$H(u, u', u'', ...) = 0 \tag{2}$$

where primes denote derivatives with respect to ξ.

2.1. The Generalized Tanh-Function Method

Then, the solution of (1) we are looking for is expressed in the form of a finite series of tanh functions

$$u(x, t) = u(\xi) = \sum_{i=0}^{M} a_i F^i, \tag{3}$$

where $F^i = \tanh^i(\xi)$, M is a positive integer that can be determined by balancing the highest order derivative with the highest nonlinear terms in equation.

$\alpha, \beta, a_0, ..., a_M$ are parameters to be determined. The crucial step of the method is to take full advantage of a Riccati equation that the tanh function satisfies and use its solutions F.

The required Riccati equation is written as

$$F' = k + F^2. \tag{4}$$

where $F' = \frac{dF}{d\xi}$ and k is a constant. The Riccati equation has the general solutions

(a) If $k < 0$

$$F = -\sqrt{-k}\ \tanh(\sqrt{-k}\xi), \tag{5}$$
$$F = -\sqrt{-k}\ \coth(\sqrt{-k}\xi).$$

(b) If $k = 0$

$$F = -\frac{1}{\xi}. \tag{6}$$

(c) If $k > 0$

$$F = \sqrt{k}\ \tan(\sqrt{k}\xi), \tag{7}$$
$$F = -\sqrt{k}\ \cot(\sqrt{k}\xi).$$

Substituting equation (3) into (2) by using (4) yields a set of algebraic equations for F^i, and all coefficients of F^i have to vanish. After this separated algebraic equations, we can find coefficients $\alpha, \beta, a_0, ..., a_M$.

2.2. The (G'/G)-Expansion Method

We initially predict the structure of the solution $u = u(\xi)$ to equation (2) in the finite series form

$$u = \sum_{i=0}^{M} a_i (\frac{G'}{G})^i, \quad G'' + \lambda G' + \mu G = 0 \tag{8}$$

where $G = G(\xi)$ and primes denote derivatives with respect to ξ; $a_0, ..., a_M, \lambda$, and μ are constants to be specified later. The positive integer M can be determined by the homogeneous balance method. Substituting (8) into equation (2) yields a system of nonlinear algebraic equations for $a_0, ..., a_M,\ \lambda,\ \mu,\ \alpha$ and β. Finally, substitution of the system's solutions into (8) gives traveling wave solutions to equation (1).

Remark 1. The second order LODE (8) has the following solutions:
When $\lambda^2 - 4\mu > 0$,

$$\frac{G'}{G} = -\frac{\lambda}{2} + \frac{\sqrt{\lambda^2 - 4\mu}}{2}(\frac{C_1 sinh(\frac{\sqrt{\lambda^2-4\mu}\xi}{2}) + C_2 cosh(\frac{\sqrt{\lambda^2-4\mu}\xi}{2})}{C_1 cosh(\frac{\sqrt{\lambda^2-4\mu}\xi}{2}) + C_2 sinh(\frac{\sqrt{\lambda^2-4\mu}\xi}{2})}).$$

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When $\lambda^2 - 4\mu < 0$,

$$\frac{G'}{G} = -\frac{\lambda}{2} + \frac{\sqrt{4\mu-\lambda^2}}{2}\left(\frac{-C_1 sin(\frac{\sqrt{4\mu-\lambda^2}\xi}{2}) + C_2 cos(\frac{\sqrt{4\mu-\lambda^2}\xi}{2})}{C_1 cos(\frac{\sqrt{4\mu-\lambda^2}\xi}{2}) + C_2 sin(\frac{\sqrt{4\mu-\lambda^2}\xi}{2})}\right). \tag{9}$$

When $\lambda^2 - 4\mu = 0$,

$$\frac{G'}{G} = -\frac{\lambda}{2} + \frac{C_2}{C_1 + C_2\xi}.$$

Where C_1 and C_2 are arbitrary constants.

3. KP-mKP equation

Let us consider the KP-mKP equation [15]

$$u_{xt} + [(a+bu)uu_x]_x + cu_{xxxx} + d^2u_{yy} = 0, \tag{10}$$

where a, b, c and d are real constants.

Taking travelling wave transformation

$$u(x,y,t) = u(\xi), \quad \xi = x + \alpha y + \beta t, \tag{11}$$

substituting equation (11) into equation (10), we obtain

$$(\beta + d^2\alpha^2)u'' + cu'''' + a(u')^2 + auu'' + 2bu(u')^2 + bu^2u'' = 0. \tag{12}$$

When balancing u'''' with u^2u'' then gives $M = 1$.

3.1. Using Generalized Tanh-Function Method

Therefore, we may choose

$$u = a_0 + a_1F. \tag{13}$$

Substituting (13) into (12) using the auxiliary equation (4) yields a set of algebraic equations for a_0, a_1, α, β. We can arrive at the following system:

$$24ca_1 + 4ba_1^3 = 0,$$

$$6ba_0a_1^2 + 3aa_1^2 = 0,$$

$$40cka_1 + 2ba_0^2a_1 + 6bka_1^3 + 2aa_0a_1 + 2(\beta + d^2\alpha^2)a_1 = 0,$$

$$8bka_0a_1^2 + 4aka_1^2 = 0,$$

$$16ck^2a_1 + 2(\beta + d^2\alpha^2)ka_1 + 2bk^2a_1^3 + 2bka_0^2a_1 + 2aka_0a_1 = 0,$$

$$2bk^2a_0a_1^2 + ak^2a_1^2 = 0. \tag{14}$$

Then, with the aid of Maple, we can find the solutions of the above system

$$a_0 = -\frac{a}{2b}, \quad a_1 = \pm\sqrt{-\frac{6c}{b}}, \quad \beta = \frac{a^2}{4b} - d^2\alpha^2 - 2kc, \tag{15}$$

where k and α are arbitrary constants.

Using (5)-(7), (13) and (15), we can obtain the following multiple soliton-like, triangular periodic and rational solutions of the KP-mKP equation (10).

Case 1. Choosing $k = -1,\ a_0 = -\frac{a}{2b},\ a_1 = \sqrt{-\frac{6c}{b}},\ \beta = \frac{a^2}{4b} - d^2\alpha^2 + 2c$, we can obtain the new multiple soliton-like solutions for equation (10) that

$$u_1(x,y,t) = -\frac{a}{2b} - \sqrt{-\frac{6c}{b}}\tanh[x + \alpha y + (\frac{a^2}{4b} - d^2\alpha^2 + 2c)t]. \tag{16}$$

$$u_2(x,y,t) = -\frac{a}{2b} - \sqrt{-\frac{6c}{b}}\coth[x + \alpha y + (\frac{a^2}{4b} - d^2\alpha^2 + 2c)t]. \tag{17}$$

Case 2. In this case, we choose $k = -1,\ a_0 = -\frac{a}{2b},\ a_1 = -\sqrt{-\frac{6c}{b}},\ \beta = \frac{a^2}{4b} - d^2\alpha^2 + 2c$, and the soliton-like solutions solutions for equation (10) will be

$$u_3(x,y,t) = -\frac{a}{2b} + \sqrt{-\frac{6c}{b}}\tanh[x + \alpha y + (\frac{a^2}{4b} - d^2\alpha^2 + 2c)t]. \tag{18}$$

$$u_4(x,y,t) = -\frac{a}{2b} + \sqrt{-\frac{6c}{b}}\coth[x + \alpha y + (\frac{a^2}{4b} - d^2\alpha^2 + 2c)t]. \tag{19}$$

Case 3. Again, when we choose $k = 1,\ a_0 = -\frac{a}{2b},\ a_1 = \sqrt{-\frac{6c}{b}},\ \beta = \frac{a^2}{4b} - d^2\alpha^2 - 2c$, therefore, we obtain the triangular periodic solutions for equation (10) that

$$u_5(x,y,t) = -\frac{a}{2b} + \sqrt{-\frac{6c}{b}}\tan[x + \alpha y + (\frac{a^2}{4b} - d^2\alpha^2 - 2c)t]. \tag{20}$$

$$u_6(x,y,t) = -\frac{a}{2b} - \sqrt{-\frac{6c}{b}}\cot[x + \alpha y + (\frac{a^2}{4b} - d^2\alpha^2 - 2c)t]. \tag{21}$$

Case 4. When we choose $k = 1$, then, the coefficients of (15) might be choose as $k = 1,\ a_0 = -\frac{a}{2b},\ a_1 = -\sqrt{-\frac{6c}{b}},\ \beta = \frac{a^2}{4b} - d^2\alpha^2 - 2c$, According to this ansatz, a pair of triangular periodic solutions are found

$$u_7(x,y,t) = -\frac{a}{2b} - \sqrt{-\frac{6c}{b}}\tan[x + \alpha y + (\frac{a^2}{4b} - d^2\alpha^2 - 2c)t]. \tag{22}$$

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$$u_8(x,y,t) = -\frac{a}{2b} + \sqrt{-\frac{6c}{b}}\cot[x + \alpha y + (\frac{a^2}{4b} - d^2\alpha^2 - 2c)t]. \qquad (23)$$

Case 5. When $k = 0$, we know that $a_0 = -\frac{a}{2b}$, $a_1 = \pm\sqrt{-\frac{6c}{b}}$, $\beta = \frac{a^2}{4b} - d^2\alpha^2$, from which we obtain the new rational solutions

$$u_9(x,y,t) = -\frac{a}{2b} - \sqrt{-\frac{6c}{b}}\frac{1}{x + \alpha y + (\frac{a^2}{4b} - d^2\alpha^2)t}. \qquad (24)$$

$$u_{10}(x,y,t) = -\frac{a}{2b} + \sqrt{-\frac{6c}{b}}\frac{1}{x + \alpha y + (\frac{a^2}{4b} - d^2\alpha^2)t}. \qquad (25)$$

4. Using the (G'/G)-Expansion Method

Now, we assume that the solution of equation (12) can be expressed as the ansatz (8). By the homogeneous balance principle, we determine that $M = 1$. Hence, we look for solutions to equation (12) in the form

$$u(\xi) = a_0 + a_1(\frac{G'}{G}), \quad G'' + \lambda G' + \mu G = 0. \qquad (26)$$

Substituting equation (26) into equation (12), the left-hand side of equation (12) is converted into a polynomial of $(\frac{G'}{G})^j$ $(j = 0,1,2,3,4,5)$, then setting each coefficient to zero, we get a set of under-determined algebraic equations for $a_i (i = 0,1), \alpha, \beta, \lambda$ and μ:

$$c\mu\lambda^3 a_1 + (\beta + d^2\alpha^2)\lambda\mu a_1 + b\lambda\mu a_0^2 a_1 + 8c\lambda\mu^2 a_1 + a\lambda\mu a_0 a_1 + a\mu^2 a_1^2 + 2b\mu^2 a_0 a_1^2 = 0,$$

$$16c\mu^2 a_1 + 2b\mu^2 a_1^3 + 2(\beta + d^2\alpha^2)\mu a_1 + 3a\lambda\mu a_1^2 + 6b\lambda\mu a_0 a_1^2 + 2a\mu a_0 a_1 + 22c\lambda^2\mu a_1 + 2b\mu a_0^2 a_1 + (\beta + d^2\alpha^2)\lambda^2 a_1 + a\lambda^2 a_0 a_1 + c\lambda^4 a_1 + b\lambda^2 a_0^2 a_1 = 0,$$

$$60c\lambda\mu a_1 + 4a\mu a_1^2 + 8b\mu a_0 a_1^2 + 5b\lambda\mu a_1^3 + 2a a_1^2\lambda^2 + 15c\lambda^3 a_1 + 4b\lambda^2 a_0 a_1^2 + 3(\beta + d^2\alpha^2)\lambda a_1 + 3b\lambda a_0^2 a_1 + 3a\lambda a_0 a_1 = 0.$$

$$6b\mu a_1^3 + 40c\mu a_1 + 2a a_0 a_1 + 5a\lambda a_1^2 + 10b\lambda a_0 a_1^2 + 3b\lambda^2 a_1^3 + 2(\beta + d^2\alpha^2)a_1 + 2b a_0^2 a_1 + 50c\lambda^2 a_1 = 0,$$

$$3a a_1^2 + 60c\lambda a_1 + 6b a_0 a_1^2 + 7b\lambda a_1^3 = 0,$$

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$$4ba_1^3 + 24ca_1 = 0. \tag{27}$$

Solving these under-determined algebraic equations, we get the following result:

$$a_0 = -\frac{a}{2b} \mp \frac{\lambda}{2}\sqrt{-\frac{6c}{b}}, \quad a_1 = \mp\sqrt{-\frac{6c}{b}}, \quad \beta = \frac{a^2}{4b} - d^2\alpha^2 + \frac{c}{2}(\lambda^2 - 4\mu), \tag{28}$$

where α, λ and μ are arbitrary constants.

Substituting equation (28) into equation (26), we have

$$u_I(\xi) = -\frac{a}{2b} \mp \frac{\lambda}{2}\sqrt{-\frac{6c}{b}} \mp \sqrt{-\frac{6c}{b}}(\frac{G'}{G}), \tag{29}$$

Substituting the general solutions of LODE (8) into equation (29); we have three types of travelling wave solutions of the KP-mKP equation as follows:

When $\lambda^2 - 4\mu > 0$,

$$u_{I_{1,2}}(\xi) = -\frac{a}{2b} \mp \sqrt{-\frac{6c}{b}} \times \frac{\sqrt{\lambda^2 - 4\mu}}{2}(\frac{C_1 sinh(\frac{\sqrt{\lambda^2-4\mu}\xi}{2}) + C_2 cosh(\frac{\sqrt{\lambda^2-4\mu}\xi}{2})}{C_1 cosh(\frac{\sqrt{\lambda^2-4\mu}\xi}{2}) + C_2 sinh(\frac{\sqrt{\lambda^2-4\mu}\xi}{2})}),$$

where $\xi = x + \alpha y + (\frac{a^2}{4b} - d^2\alpha^2 + \frac{c}{2}(\lambda^2 - 4\mu))t$.

When $\lambda^2 - 4\mu < 0$,

$$u_{I_{3,4}}(\xi) = -\frac{a}{2b} \mp \sqrt{-\frac{6c}{b}} \times \frac{\sqrt{4\mu - \lambda^2}}{2}(\frac{-C_1 sin(\frac{\sqrt{4\mu-\lambda^2}\xi}{2}) + C_2 cos(\frac{\sqrt{4\mu-\lambda^2}\xi}{2})}{C_1 cos(\frac{\sqrt{4\mu-\lambda^2}\xi}{2}) + C_2 sin(\frac{\sqrt{4\mu-\lambda^2}\xi}{2})}),$$

where $\xi = x + \alpha y + (\frac{a^2}{4b} - d^2\alpha^2 + \frac{c}{2}(\lambda^2 - 4\mu))t$.

When $\lambda^2 - 4\mu = 0$,

$$u_{I_{5,6}}(x,y,t) = -\frac{a}{2b} \mp \sqrt{-\frac{6c}{b}}(\frac{C_2}{C_1 + C_2(x + \alpha y + (\frac{a^2}{4b} - d^2\alpha^2)t)}).$$

In particular, if $C_1 \neq 0$, $C_2 = 0$, then $u_{I_{1,2}}$, $u_{I_{3,4}}$ becomes

$$u_{I_{1,2}}(x,y,t) = -\frac{a}{2b} \mp \sqrt{-\frac{6c}{b}} \times \frac{\sqrt{\lambda^2 - 4\mu}}{2} \tanh[\frac{\sqrt{\lambda^2 - 4\mu}}{2}(x + \alpha y + (\frac{a^2}{4b} - d^2\alpha^2 + \frac{c}{2}(\lambda^2 - 4\mu))t)].$$

$$u_{I_{3,4}}(x,y,t) = -\frac{a}{2b} \pm \sqrt{-\frac{6c}{b}} \times \frac{\sqrt{4\mu - \lambda^2}}{2} \tan[\frac{\sqrt{4\mu - \lambda^2}}{2}(x + \alpha y + (\frac{a^2}{4b} - d^2\alpha^2 + \frac{c}{2}(\lambda^2 - 4\mu))t)].$$

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