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MULTIPLE SOLUTIONS ON A BALL FOR A GENERALIZED LANE–EMDEN EQUATION

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Abstract: I examine the nonlinear ordinary differential equation corresponding to radial solutions of the Dirichlet problem $\nabla \cdot (|x|^{\alpha} |\nabla u|^{p-2} \nabla u) + |x|^{\beta} |u|^{q-1} u = 0$, $u|_{\partial B^n} = 0$ for $q > 1$, $1 < p \leq 2, \alpha, \beta \geq 0$, where $B^n = \{x \in R^n : |x| \leq 1\}$.

I establish the existence and non-uniqueness of infinitely many radially symmetric solutions, controlled by a Sobolev critical exponent, to the generalized form of the Lane-Emden equation (GLE). I prove the non-existence of solutions for $q + 1 \geq p^*$, $p^* = \frac{p(n+\beta)}{n+\alpha-p}$. I also prove that the class of all nontrivial solutions are bounded below for $p < q + 1 < p^*$, $1 < p \leq 2, q > 1, \alpha, \beta \geq 0$ by a constant dependent on p, n, α, β.

Key Words: p-laplacian, shooting method, critical Sobolev exponents, weighted Sobolev spaces

1. Introduction

In this paper, I examine radial solutions to the nonlinear ordinary differential equation

$$r^{1-n}(r^{n+\alpha-1} |u_r|^{p-2} u_r)_r + r^{\beta} |u|^{q+1} u = 0, \tag{1.1}$$

with boundary conditions

$$u(1) = u_r(0) = 0, \tag{1.2}$$

for $1 < p \leq 2 < n$ and $p < q + 1 < p^* = \frac{p(n+\beta)}{\alpha+n-p}$, corresponding to the Dirichlet

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problem

$$\nabla \cdot (|x|^{\alpha} |\nabla u|^{p-2} \nabla u) + |x|^{\beta} |u|^{q-1} u = 0, \tag{1.3}$$

$$u|_{\partial B^n} = 0,$$

on the unit ball $B^n = \{x \in R^n : |x| \leq 1\}$.

I use a shooting argument to prove the existence and nonuniqueness of radial solutions to (1.3) in $W^{1,p}(B^n, |x|^{\alpha}) \cap L^{q+1}(B^n, |x|^{\beta})$, where for $q+1 < p^*$, $p^* = \frac{p(n+\beta)}{n+\alpha-p}$, the following inequality $\|u\|_{L^{q+1}(B^n,|x|^{\beta})} \leq C \|\nabla u\|_{L^p(B^n,|x|^{\alpha})}$ holds and the embedding $W^{1,p}(B^n, |x|^{\alpha}) \hookrightarrow L^{q+1}(B^n, |x|^{\beta})$ is compact, [1]. This was established for the case p=2, $\alpha, \beta = 0$ in [2], [3] and the case $p \neq 2, \alpha, \beta = 0$ in [4]. I prove the nonexistence of radial and general solutions on a star shaped domain using Pohozaev's identity for $q + 1 \geq p^*$. I also prove that the class of all nontrivial solutions are bounded below for $p < q+1 < p^* = \frac{p(n+\beta)}{n+\alpha-p}$, $1 < p \leq 2, q > 1, \alpha, \beta \geq 0$ by a constant dependent on p, n, α, β.

Equation (1.3) serves as a generalization to the Lane-Emden equation which has attracted great interest in the literature and has undergone extensive research due to its frequent use in mathematics and astrophysics. The Lane-Emden equation was first introduced by Homer Lane in his attempt to compute both the temperature and the mass density in portions of the sun, [5]. In spite of the fact that the results he obtained were incorrect near the surface of the sun, the values for both quantities were reasonable at the interior, [6]. As a result the Lane-Emden equation is still in use today to compute the structure of the interior of polytropic stars.

Chandrasekhar subsequently introduced the Lane–Emden equation to astrophysics, for star equilibrium problems with $n = 3$, in 1937, [7]. Later, mathematicians including Nirenberg, Ni and Serrin, [8], [9], studied detailed properties of the Lane–Emden equation in $\mathbb{R}^n$, for general n. The Lane-Emden Dirichlet problem on a unit ball with the condition $u = 0$ on $\partial \mathbb{B}^n$ for $u > 0$, has also been studied extensively by Serrin, [10]. He was able to prove that the Sobolev critical exponent $\frac{n+2}{n-2}$ "sets up a dividing number" for the existence and nonexistence of positive solutions. He showed that only for $q < \frac{n+2}{n-2}$ there exist radial positive solutions, while for $q > \frac{n+2}{n-2}$, the Lane-Emden equation has neither radial nor non radial solutions on a ball of radius $r > 0$, [10]. Extensive research on the Lane-Emden equation followed after that by a number of researchers, [11], [12], [13], [14]. In 1973, Henon proposed a more general form of the Lane-Emden equation, involving the weight function $|x|^l$, to study rotating stellar systems, [13], referred to as the Emden-Fowler equation

$$\Delta u + |x|^l u^q = 0, \quad \text{in} \quad \Omega, \tag{1.4}$$

$$u > 0 \text{ in } \Omega \text{ and } u = 0 \text{ on } \partial\Omega,$$

where Ω is a bounded smooth domain in $\mathbb{R}^n$. It was shown that the Sobolev exponent in this case was changed to $q = \frac{n+2+2l}{n-2}$ and classical radial solutions on a ball exist for $q < \frac{n+2+2l}{n-2}$, [13]. Several other generalizations of the Lane-Emden equation followed to cover more applications arising in geometry and physics, astrophysics and engineering, [14], [15], [16], [17], [18], [19].

The generalized Lane-Emden equation appears as an application in fluid mechanics as a physical phenomena related to equilibrium of anisotropic media which possible are perfect insulators. For instance, if $\check{T}$ is the sheer stress and $\nabla_p u$ is the velocity gradient then these quantities obey a relation of the form $\check{T}(x) = a(x)\nabla_p u(x)$ where $\nabla_p u = |\nabla u|^{p-2}\nabla u$ is the p-laplacian operator, $p > 1$ is an arbitrary number. The case $p = 2$ (respectively $p < 2$, $p > 2$) corresponds to a Newtonian (respectively pseudo plastic, dilatant) fluid. The resulting equations of motion then involve the nonlinear, inhomogeneous expression $\nabla \cdot (a\nabla_p u)$, which reduces to $a\nabla \cdot (\nabla_p u)$ for a being a constant, [18], [20], [21].

In this paper I generalize the model proposed by Henon in (1.4) by considering the p-laplacian operator on the function u and introducing the two weight functions $|x|^\alpha$ and $|x|^\beta$ where $\alpha, \beta \geq 0$. Analyzing the generalized form of the Lane-Emden equation and its solutions develops a better understanding of how results involving non-weighted Sobolev spaces can be extended to the weighted context and for the application to new classes of physical models.

I start by providing terminology for the weighted Sobolev spaces involved followed by preliminary results in Section 2. In Section 3, I prove the non-existence of solutions, radial and general, for $q + 1 \geq p^*$ using Pohozaev's identity on a star shaped domain. A Proof of the existence and non-uniqueness of an infinite number of solutions for the GLE equation using a shooting argument is provided in Section 4. In Section 5, I prove that the class of all nontrivial solutions to the GLE, radial and general, are bounded below uniformly by a constant dependent on the parameters p, n, α and β.

2. Terminology and Preliminary Results

I define p^*, the critical exponent for the Lane-Emden equation ($p^* = \frac{2n}{n-2}$), [10], where $p^* = \frac{p(n+\beta)}{\alpha+n-p}$ for the generalized Lane-Emden equation.

Theorem 2.1. *Weighted Sobolev Embedding Theorem.*
Let $\alpha, \beta \geq 0$, $1 < p \leq 2$ and $p < q + 1 \leq p^*, p^* = \frac{p(n+\beta)}{\alpha+n-p}$. Then

$$\|u\|_{L^{q+1}(B^n,|x|^\beta)} \leq C \|\nabla u\|_{L^p(B^n,|x|^\alpha)}$$

and the embedding $W^{1,p}(B^n, |x|^\alpha) \hookrightarrow L^{q+1}(B^n, |x|^\beta)$ is continuous. If the upper bound for $q + 1$ is strict then the embedding is compact.

This theorem is a consequence of theorem 2.3 of [1].

Next, I define the weighted $q+1$ norm to be

$$\|u\|_{L^{q+1}(B^n,|x|^\beta)} = (\int_{B^n} |x|^\beta |u(x)|^{q+1} dx)^{\frac{1}{q+1}}$$

for $1 \leq q+1 < \infty$ and the weighted p norm as

$$\|\nabla u\|_{L^p(B^n,|x|^\alpha)} = (\int_{B^n} |x|^\alpha |\nabla u(x)|^p dx)^{\frac{1}{p}}$$

for $1 \leq p < \infty$.

Finally, I define the following function spaces.

Definition 2.1

1. $W^{1,p}(B^n, |x|^\alpha) \equiv \Big\{ u \in W^{1,p}_{loc}(B^n) : \|\nabla u\|_{L^p(B^n,|x|^\alpha)} = \int_{B^n} |x|^\alpha (|u|^p + |\nabla u|^p) dx)^{\frac{1}{p}} < \infty \Big\}.$

2. $L^{q+1}(B^n, |x|^\beta) \equiv \Big\{ u \in W^{q+1}_{loc}(B^n) : \|u\|_{L^{q+1}(B^n,|x|^\beta)} = \int_{B^n} |x|^\beta |u|^{q+1} dx)^{\frac{1}{q+1}} < \infty \Big\}.$

The generalized Lane-Emden equation (1.3) may be considered as the Euler-Lagrange equation for the functional

$$F(u) = \int_\Omega (\frac{1}{p} |x|^\alpha |\nabla u|^p - \frac{1}{q+1} |x|^\beta |u|^{q+1}) dx.. \tag{2.1}$$

Upon taking the first variation of the functional $F(u)$ in (2.1) we have

$$\delta F(u) = \delta \int_\Omega (\frac{1}{p} |x|^\alpha |\nabla u|^p - \frac{1}{q+1} |x|^\beta |u|^{q+1}) dx. \tag{2.2}$$

where $\delta F(u) = \frac{d}{d\epsilon} F(u + \epsilon w)|_{\epsilon=0}, \forall w \in C_0^\infty(\Omega)$ and $\epsilon w = \delta u$. Since $\delta |\nabla u|^p = p |\nabla u|^{p-2} \nabla u \cdot \nabla \delta u$ and $\delta |u|^{q+1} = (q+1) |u|^{q-1} u \delta u$, we have

$$\delta F(u) = \int_\Omega (|x|^\alpha |\nabla u|^{p-2} \nabla u \cdot \nabla \delta u - |x|^\beta |u|^{q-1} u \delta u) dx, \tag{2.3}$$

then

$$\delta F(u) = - \int_\Omega [\nabla \cdot (|x|^\alpha |\nabla u|^{p-2} \nabla u) + |x|^\beta |u|^{q-1} u] \delta u dx. \tag{2.4}$$

I consider radial solutions regular at the origin satisfying

$$< F'(u), u >= 0, \tag{2.5}$$

International Electronic Journal of Pure and Applied Mathematics – IEJPAM, Volume 3, No. 3 (2011)

that is

$$\int_{\Omega} u(\nabla \cdot (|x|^{\alpha} |\nabla u|^{p-2} \nabla u) + |x|^{\beta} |u|^{q-1} u)dx = 0. \tag{2.6}$$

This simplifies in to

$$\int_{\Omega} (-\nabla u \cdot (|x|^{\alpha} |\nabla u|^{p-2} \nabla u) + |x|^{\beta} |u|^{q+1})dx = 0, \tag{2.7}$$

hence

$$\int_{B^n} |x|^{\alpha} |\nabla u|^{p} \, dx = \int_{B^n} |x|^{\beta} |u|^{q+1} \, dx. \tag{2.8}$$

In this paper I examine solutions for which the terms in this relation are finite.

Next, the following analysis leads to introducing two lemmas. Consider expanding the radial form of the generalized Lane-Emden equation (1.1), we then have

$$(p-1)r^{\alpha} |u_r|^{p-2} u_{rr} + (\alpha + n - 1)r^{\alpha-1} |u_r|^{p-2} u_r + r^{\beta} |u|^{q-1} u = 0. \tag{2.9}$$

Multiplying (2.9) by u_r gives

$$(p-1)r^{\alpha} |u_r|^{p-2} u_{rr}u_r + (\alpha + n - 1)r^{\alpha-1} |u_r|^{p} + r^{\beta} |u|^{q-1} uu_r = 0. \tag{2.10}$$

Since

$$(p-1)r^{\alpha} |u_r|^{p-2} u_r u_{rr} = \frac{d}{dr}[\frac{p-1}{p} r^{\alpha} |u_r|^{p}] - \alpha(\frac{p-1}{p}) r^{\alpha-1} |u_r|^{p}, \tag{2.11}$$

and

$$r^{\beta} |u|^{q-1} uu_r = \frac{d}{dr}[\frac{1}{q+1} r^{\beta} |u|^{q+1}] - \frac{\beta}{q+1} r^{\beta-1} |u|^{q+1}, \tag{2.12}$$

equation (2.10) becomes

$$\frac{d}{dr}[\frac{p-1}{p} r^{\alpha} |u_r|^{p} + \frac{1}{q+1} r^{\beta} |u|^{q+1}] = -(\frac{\alpha}{p} + n - 1) r^{\alpha-1} |u_r|^{p} + \frac{\beta}{q+1} r^{\beta-1} |u|^{q+1}. \tag{2.13}$$

Integrating both sides of equation (2.13) with respect to r from 0 to r we obtain

$$\int_0^r \frac{d}{ds}[\frac{p-1}{p} s^{\alpha} |u_s|^{p} + \frac{1}{q+1} s^{\beta} |u|^{q+1}]ds = \int_0^r [-(\frac{\alpha}{p} + n - 1) s^{\alpha-1} |u_s|^{p} + \frac{\beta}{q+1} s^{\beta-1} |u|^{q+1}]ds \tag{2.14}$$

and letting $E(r) = \frac{p-1}{p} r^{\alpha} |u_r|^{p} + \frac{1}{q+1} r^{\beta} |u|^{q+1}$, we then have

$$E(r) = \int_0^r [-(\frac{\alpha}{p} + n - 1) s^{\alpha-1} |u_s|^{p} + \frac{\beta}{q+1} s^{\beta-1} |u|^{q+1}]ds + E(0). \tag{2.15}$$

I examine bounded solutions such that $u(0)$ and $u_r(0)$ are both finite, with $\alpha, \beta \geq 0$, $1 < p \leq 2, q > 1$. In particular, let $u(0) = \sigma, |\sigma| < \infty, u_r(0) = 0, u(1) = 0$, $u_r(1) = -\delta, -\delta \in (-\infty, \infty)$, consequently $E(0) = 0$ for $\alpha, \beta > 0$. For $\alpha, \beta = 0$, $E(r) = \frac{p-1}{p} |u_r|^p + \frac{1}{q+1} |u|^{q+1}$ and $E(0) = \frac{|\sigma|^{q+1}}{q+1}$. For reference on the case $\alpha, \beta = 0$ see [4].

Therefore

$$E(r,\sigma) = \int_0^r [-(\frac{\alpha}{p} + n - 1)s^{\alpha-1} |u_s|^p + \frac{\beta}{q+1} s^{\beta-1} |u|^{q+1}]ds. \quad (2.16)$$

Consider (2.13) again, integrating both sides from ϵ to 1 with respect to r we have

$$\frac{p-1}{p} r^{\alpha} |u_r|^p |_{\epsilon}^1 + \frac{1}{q+1} r^{\beta} |u|^{q+1} |_{\epsilon}^1 = \int_{\epsilon}^1 [-(\frac{\alpha}{p} + n - 1) r^{\alpha-1} |u_r|^p + \frac{\beta}{q+1} r^{\beta-1} |u|^{q+1}]dr. \quad (2.17)$$

Letting $\epsilon \to 0$ in (2.17), for $1 < p \leq 2, q > 1, u(0) = |\sigma| < \infty, u_r(0) = 0, u(1) = 0$, $u_r(1) = -\delta, -\delta \in (-\infty, \infty)$, the integrand $\int_0^1 -(\frac{\alpha}{p} + n - 1) r^{\alpha-1} |u_r|^p \, dr$ is finite for $\alpha \geq 0$ and the integrand $\int_0^1 \frac{\beta}{q+1} r^{\beta-1} |u|^{q+1} \, dr$ is finite for $\beta \geq 0$.

Let $E_1(r,\sigma) = r^{-\beta} E(r,\sigma) = \frac{p-1}{p} r^{\alpha-\beta} |u_r|^p + \frac{1}{q+1} |u|^{q+1}$, now I prove that $E_1(r,\sigma)$ is a monotone decreasing function for a fixed value of σ.

Lemma 2.1. *$\frac{d}{dr} E_1(r,\sigma) \leq 0$ for $\alpha, \beta \geq 0$, $1 < p \leq 2$, $q > 1$, $u(0) = |\sigma| < \infty$, $u_r(0) = 0$, $r \geq 0$.*

Proof. Consider multiplying (2.9) by $r^{-\beta} u_r$, we then have

$$(p-1) r^{\alpha-\beta} |u_r|^{p-2} u_r u_{rr} + (\alpha + n - 1) r^{\alpha-\beta-1} |u_r|^p + |u|^{q-1} u u_r = 0. \quad (2.18)$$

After simplification and collecting like terms equation (2.18) gives

$$\frac{d}{dr} [\frac{p-1}{p} r^{\alpha-\beta} |u_r|^p + \frac{1}{q+1} |u|^{q+1}] = -(n + \beta - 1 + \frac{\alpha - \beta}{p}) r^{\alpha-\beta-1} |u_r|^p. \quad (2.19)$$

Observe that $n - 1 + \frac{\alpha}{p} + \frac{(p-1)\beta}{p} > 0$ for $\alpha, \beta \geq 0$ and $1 < p \leq 2$, hence

$$\frac{dE_1}{dr} \leq 0 \quad \text{for} \quad r \geq 0. \quad (2.20)$$

Integrating (2.20) from 0 to r gives

$$E_1(r) \leq E_1(0) \quad \text{for} \quad r \geq 0. \quad (2.21)$$

Observe that the function, $E_1(r)$, is bounded provided that $\alpha, \beta \geq 0$, $1 < p \leq 2$, $q > 1$, $u_r(0) = 0$ and $u(0) = |\sigma|$ are finite, which in return implies that $u(r)$ and $u_r(r)$ are bounded. We therefore have

$$E_1(r) \leq \frac{1}{q+1} |\sigma|^{q+1} . \tag{2.22}$$

Integrating (2.20) with respect to r from $r = 0$ to $r = 1$ gives the following estimate for $u_r(1) = -\delta$

$$E_1(1) \leq E_1(0), \tag{2.23}$$

hence

$$|\delta|^p \leq \frac{p}{(p-1)(q+1)} |\sigma|^{q+1} . \tag{2.24}$$

Lemma 2.2. *Let $u(r)$ satisfy (1.1), $u(0) = \sigma$, $u_r(0) = 0, \alpha, \beta \geq 0$, $1 < p \leq 2$, $p^* = \frac{p(n+\beta)}{n+\alpha-p}$, then*

$$\frac{1}{n+\beta} r^n E(r,\sigma) = \lambda \int_0^r s^{n+\beta-1} |u|^{q+1} \, ds - \frac{1}{p^*} r^{n+\alpha-1} |u_r|^{p-2} u_r u, \tag{2.25}$$

for $r \in [0,1]$ and $\lambda = \frac{1}{q+1} - \frac{1}{p^*}$.

Proof. Consider

$$(r^n E)_r = r^{n-1}(rE_r + nE) \tag{2.26}$$

where

$$E_r = -(\frac{\alpha}{p} + n - 1) r^{\alpha-1} |u_r|^p + \frac{\beta}{q+1} r^{\beta-1} |u|^{q+1} \tag{2.27}$$

and

$$rE_r = -(\frac{\alpha}{p} + n - 1) r^{\alpha} |u_r|^p + \frac{\beta}{q+1} r^{\beta} |u|^{q+1} . \tag{2.28}$$

Hence

$$rE_r + nE = [-(\frac{\alpha}{p} + n - 1) + \frac{n(p-1)}{p}] r^{\alpha} |u_r|^p + \frac{n+\beta}{q+1} r^{\beta} |u|^{q+1} , \tag{2.29}$$

which simplifies in to

$$rE_r + nE = -(\frac{n+\alpha-p}{p}) r^{\alpha} |u_r|^p + \frac{n+\beta}{q+1} r^{\beta} |u|^{q+1} . \tag{2.30}$$

Then

$$(r^n E)_r = -(\frac{n+\alpha-p}{p}) r^{\alpha+n-1} |u_r|^p + \frac{n+\beta}{q+1} r^{\beta+n-1} |u|^{q+1} . \tag{2.31}$$

Now consider equation (2.9), upon multiplying by r^{n-1} we have

$$(p-1)r^{n+\alpha-1}\left|u_r\right|^{p-2}u_{rr}+(\alpha+n-1)r^{\alpha+n-2}\left|u_r\right|^{p-2}u_r=-r^{\beta+n-1}\left|u\right|^{q-1}u. \quad (2.32)$$

note also that

$$(r^{n+\alpha-1}\left|u_r\right|^{p-2}u_r)_r=(p-1)r^{n+\alpha-1}\left|u_r\right|^{p-2}u_{rr}+(n+\alpha-1)r^{n+\alpha-2}\left|u_r\right|^{p-2}u_r, \quad (2.33)$$

hence equation (2.32) can be written as

$$(r^{n+\alpha-1}\left|u_r\right|^{p-2}u_r)_r=-r^{\beta+n-1}\left|u\right|^{q-1}u. \quad (2.34)$$

And so

$$(r^{n+\alpha-1}\left|u_r\right|^{p-2}u_r)_ru=-r^{\beta+n-1}\left|u\right|^{q+1}. \quad (2.35)$$

Next

$$(r^{n-1+\alpha}\left|u_r\right|^{p-2}u_ru)_r=(r^{\alpha+n-1}\left|u_r\right|^{p-2}u_r)_ru+(r^{\alpha+n-1}\left|u_r\right|^{p-2}u_r)u_r, \quad (2.36)$$

and so

$$(r^{n-1+\alpha}\left|u_r\right|^{p-2}u_ru)_r=(r^{n-1+\alpha}\left|u_r\right|^{p-2}u_r)_ru+r^{\alpha+n-1}\left|u_r\right|^{p}. \quad (2.37)$$

Equations (2.35) and (2.37) together give

$$(r^{n-1+\alpha}\left|u_r\right|^{p-2}u_ru)_r=-r^{\beta+n-1}\left|u\right|^{q+1}+r^{n+\alpha-1}\left|u_r\right|^{p}. \quad (2.38)$$

Now

$$\begin{aligned}(\tfrac{1}{n+\beta}r^nE+\tfrac{1}{p^*}r^{n+\alpha-1}\left|u_r\right|^{p-2}u_ru)_r \\ &= \frac{1}{n+\beta}(r^nE)_r+\frac{1}{p^*}(r^{n-1+\alpha}\left|u_r\right|^{p-2}u_ru)_r \quad (2.39)\\ = \tfrac{1}{n+\beta}[-\tfrac{n+\alpha-p}{p}r^{n+\alpha-1}\left|u_r\right|^{p}+\tfrac{n+\beta}{q+1}r^{n+\beta-1}\left|u\right|^{q+1}] \\ &+\frac{1}{p^*}[r^{n+\alpha-1}\left|u_r\right|^{p}-r^{n+\beta-1}\left|u\right|^{q+1}]. \quad (2.40)\end{aligned}$$

Collecting like terms we have

$$(\tfrac{1}{n+\beta}r^nE+\tfrac{1}{p^*}r^{n+\alpha-1}\left|u_r\right|^{p-2}u_ru)_r= \\ [\frac{1}{p^*}-\frac{n+\alpha-p}{p(n+\beta)}]r^{n+\alpha-1}\left|u_r\right|^{p}+[\frac{1}{q+1}-\frac{1}{p^*}]r^{n+\beta-1}\left|u\right|^{q+1} \quad (2.41)$$

since $p^*=\frac{p(n+\beta)}{\alpha+n-p}$, then $[\frac{1}{p^*}-\frac{\alpha+n-p}{p(n+\beta)}]=0$. Hence we have

$$(\frac{1}{n+\beta}r^nE+\frac{1}{p^*}r^{n-1+\alpha}\left|u_r\right|^{p-2}u_ru)_r=[\frac{1}{q+1}-\frac{1}{p^*}]r^{n+\beta-1}\left|u\right|^{q+1}. \quad (2.42)$$

Integrating both sides of equation (2.42) from 0 to r with respect to r gives

$$\frac{1}{n+\beta}r^nE+\frac{1}{p^*}r^{n+\alpha-1}\left|u_r\right|^{p-2}u_ru=\int_0^r(\frac{1}{q+1}-\frac{1}{p^*})s^{n+\beta-1}\left|u(s)\right|^{q+1}ds. \quad (2.43)$$

Rearranging equation (2.43) gives the desired result.

3. Non-Existence Results

Theorem 3.1. *For $\alpha, \beta \geq 0, 1 < p \leq 2, q > 1, u(0) = |\sigma| < \infty, u_r(0) = 0$, $q+1 \geq p^*, p^* = \frac{p(n+\beta)}{\alpha+n-p}$, no radial solutions exist for the generalized Lane-Emden equation (1.3).*

Proof. Let $r = r_0 > 0$ denote the value of r at which $u(r)$ has its first zero such that $u(r_0) = 0$. Substituting this value in (2.25) gives the equation

$$\frac{1}{n+\beta} r_0^n E(r_0) = (\frac{1}{q+1} - \frac{1}{p^*}) \int_0^{r_0} r^{n+\beta-1} |u(r)|^{q+1} dr. \tag{3.1}$$

If $q+1 = p^*$, $p^* = \frac{p(n+\beta)}{n+\alpha-p}$, then $E(r_0) = \frac{p-1}{p} r_0^\alpha |u_r(r_0)|^p + \frac{1}{q+1} r_0^\beta |u(r_0)|^{q+1} = 0$ and consequently $u_r(r_0) = 0$. Since $r_0 > 0$, then $u(r_0) = 0$ on $(r_0 - \epsilon, r_0 + \epsilon)$ for some $\epsilon > 0$, which violates the assumption on r_0 being the first zero of $u(r)$. If $q+1 > p^*$, then $E(r_0) < 0$ for nontrivial solutions, which is a contradiction since $E(r)$ is positive for all $r > 0$ and $1 < p \leq 2$, $q > 1, \alpha, \beta \geq 0$. Therefore either $u(r) = 0$, which implies that all possible solutions are trivial solutions, or no radial solutions exist for $q+1 \geq p^*$. This leaves us with the case $q+1 < p^*$ to examine existence of solutions.

I also establish the nonexistence of more general solutions for $q+1 \geq p^*$ by applying Pohozaev's identity to the GLE equation (1.3).

Corollary 3.1. *For $\alpha, \beta \geq 0, 1 < p \leq 2, q > 1, q+1 \geq p^*$, $p^* = \frac{p(n+\beta)}{\alpha+n-p}$, there exist no solutions to the generalized Lane-Emden equation over any domain which is smooth and star-shaped with respect to the origin.*

Proof. Multiplying the generalized Lane-Emden equation (1.3) by $x \cdot \nabla u$ on both sides of the equation and integrating by parts gives (see appendix)

$$\frac{p-1}{p} \int_{\partial B^n} |x|^\alpha |\nabla u|^p x \cdot n dS + \frac{\alpha+n-p}{p} \int_{B^n} |x|^\alpha |\nabla u|^p dx = \int_{B^n} \frac{n+\beta}{q+1} |x|^\beta |u|^{q+1} dx. \tag{3.2}$$

Combining the two identities (2.8) and (3.2), we obtain

$$\frac{p-1}{p} \int_{\partial B^n} |x|^\alpha |\nabla u|^p x \cdot \nu dS + \frac{n+\alpha-p}{p} \int_{B^n} |x|^\beta |u|^{q+1} dx = \int_{B^n} \frac{n+\beta}{q+1} |x|^\beta |u|^{q+1} dx. \tag{3.3}$$

International Electronic Journal of Pure and Applied Mathematics – IEJPAM, Volume 3, No. 3 (2011)

Hence we have

$$\frac{p-1}{p}\int_{\partial B^n}|x|^{\alpha}|\nabla u|^p x\cdot\nu dS=-[\frac{n+\alpha-p}{p}-\frac{n+\beta}{q+1}]\int_{B^n}|x|^{\beta}|u|^{q+1}dx. \quad (3.4)$$

It is clear that for $\frac{n+\beta}{q+1}=\frac{n+\alpha-p}{p}$, that is for $q+1=p^*=\frac{p(n+\beta)}{n+\alpha-p}$, $q>1$, $1<p\le 2$, $n>2$, $\alpha,\beta\ge 0$, only trivial solutions exist for (1.3). For $(\frac{n+\alpha-p}{p}-\frac{n+\beta}{q+1})>0$, $q>1$, $1<p\le 2$, $\alpha,\beta\ge 0$, we have a contradiction since the right hand side of the equation is positive and so is the left hand side for the absolute values. This implies that for $\frac{n+\alpha-p}{p}>\frac{n+\beta}{q+1}$, that is for $q+1>\frac{p(n+\beta)}{n+\alpha-p}=p^*$, provided that $p<n+\alpha$, no nontrivial solutions exist.

4. Existence and Nonuniqueness Result

Proving the existence of infinitely many radially symmetric solutions to the boundary value problem

$$(p-1)r^{\alpha}|u_r|^{p-2}u_{rr}+(\alpha+n-1)r^{\alpha-1}|u_r|^{p-2}u_r+|r|^{\beta}|u|^{q-1}u=0 \quad (4.1)$$

$$u(1)=u_r(0)=0 \quad (4.2)$$

is done using the shooting argument which relates (4.1), (4.2) to the initial value problem (4.1) with initial conditions (4.3),

$$u(0)=\sigma, u_r(0)=0 \quad (4.3)$$

The value of σ is chosen in a manner that the solution to (4.1), (4.3) also satisfies $u(1)=0$. This solution exists for $1<p\le 2$, $\alpha,\beta\ge 0$, $p<q+1<p^*$ and is unique for $r\in[0,1]$, as will be shown next. Using the shooting argument I will be able to show that there exists a sequence $\{\sigma_n\}$, $n=1,2,3,...$ of values of σ, each of which gives rise to a corresponding solution of (4.1), (4.2). Thus the σ_n parametrize an infinite sequence of solutions, $\{u_n(r)\}$, n = 1,2,3, ... to the boundary value problem (4.1), (4.2), with $\sigma_n=u_n(0)\to\infty$ as $n\to\infty$. I start by proving the existence of a unique radial solution to the corresponding initial value problem (4.1), (4.3). This is done by defining an integral operator and proving the existence of a unique fixed point of this operator.

Definition 4.1. Let $\Phi_p(x)=|x|^{p-2}x$, for $x\in\Re, p>1$ and denote its inverse by $\Phi_p'(x)$, where $\frac{1}{p}+\frac{1}{p'}=1$.

Consider multiplying (1.1) by r^{n-1} and integrating with respect to r from 0 to r we then have,

$$|u_r|^{p-2}u_r=-\frac{1}{r^{n+\alpha-1}}\int_0^r s^{n+\beta-1}|u(s)|^{q-1}uds. \quad (4.4)$$

Note that $\Phi_p(u_r) = |u_r|^{p-2}\, u_r$ and $\Phi_{q+1}(u) = |u|^{q-1}\, u$, hence applying $\Phi_{p'}$ to both sides of equation (4.4) gives

$$u_r(r) = \Phi_{p'}(-\frac{1}{r^{n+\alpha-1}} \int_0^r s^{n+\beta-1}\Phi_{q+1}(u(s))ds). \tag{4.5}$$

Since

$$\Phi_{p'}(-\frac{1}{r^{n+\alpha-1}}) = \left|-\frac{1}{r^{n+\alpha-1}}\right|^{p'-2} (-\frac{1}{r^{n+\alpha-1}}) = -(\frac{1}{r^{\frac{n+\alpha-1}{p-1}}}), \tag{4.6}$$

then (4.5) gives

$$u_r(r) = -(\frac{1}{r^{\frac{n+\alpha-1}{p-1}}})\Phi_{p'}(\int_0^r s^{n+\beta-1}\Phi_{q+1}(u(s))ds). \tag{4.7}$$

Integrating (4.7) on the interval $(0, r)$ leads to defining the mapping T given by

$$Tu(r) = \sigma - \int_0^r (\frac{1}{t^{\frac{n+\alpha-1}{p-1}}})\Phi_{p'}(\int_0^r s^{n+\beta-1}\Phi_{q+1}(u(s))ds)dt. \tag{4.8}$$

Next, I prove the existence of a fixed point of $T(u(r))$ of (4.8), for the initial value problem (4.1), (4.3), and show that this fixed point is continuously dependent on the initial data uniformly on $[0, 1]$.

Let $R > 0$, σ be a real number and $B^{\epsilon}_R(\sigma) = \{u : u \in C[0, \epsilon], \|u - \sigma\|_\infty \le R\}$ where $\epsilon < \sqrt[p+\beta-\alpha]{\frac{(n+\beta)(p+\beta-\alpha)^{p-1}}{R^{q+1-p}}}\ \sqrt[p+\beta-\alpha]{\frac{1}{q^{p-1}}}$, such that for $\beta-\alpha+1 > 0$, $u_r(0) = 0$. It is clear that $B^{\epsilon}_R(\sigma)$ is a closed subset of $C[0, \epsilon]$. I show that T leaves $B^{\epsilon}_R(\sigma)$ invariant and is a contraction on $B^{\epsilon}_R(\sigma)$ with respect to the sup-norm.

Suppose that $u \in B^{\epsilon}_R(\sigma)$ then by (4.8)

$$|Tu(r) - \sigma| \le \left|\int_0^r \frac{1}{t^{\frac{n+\alpha-1}{p-1}}}\Phi_{p'}(\int_0^t s^{n+\beta-1}\Phi_{q+1}u(s)ds)dt\right| \tag{4.9}$$

$$\le \frac{1}{(n+\beta)^{\frac{1}{p-1}}}(\frac{p-1}{p+\beta-\alpha})r^{\frac{\beta+p-\alpha}{p-1}}(\|u\|_\infty)^{\frac{q}{p-1}} \tag{4.10}$$

$$\le \frac{1}{(n+\beta)^{\frac{1}{p-1}}}(\frac{p-1}{p+\beta-\alpha})\epsilon^{\frac{\beta+p-\alpha}{p-1}}R^{\frac{q}{p-1}} \le R. \tag{4.11}$$

This inequality holds if and only if $\epsilon \le \sqrt[p+\beta-\alpha]{\frac{(n+\beta)(p+\beta-\alpha)^{p-1}}{R^{q+1-p}}}\ \sqrt[p+\beta-\alpha]{\frac{1}{(p-1)^{p-1}}}$ which follows since $\sqrt[p+\beta-\alpha]{\frac{1}{(p-1)^{p-1}}} > \sqrt[p+\beta-\alpha]{\frac{1}{q^{p-1}}}$ and $\epsilon < \sqrt[p+\beta-\alpha]{\frac{(n+\beta)(p+\beta-\alpha)^{p-1}}{R^{q+1-p}}}\ \sqrt[p+\beta-\alpha]{\frac{1}{q^{p-1}}}$ therefore T leaves $B^{\epsilon}_R(\sigma)$ invariant.

Let $u, v \in B^{\epsilon}_R(\sigma)$, then

$$Tu(r)-Tv(r)=\int_0^r \frac{1}{t^{\frac{n+\alpha-1}{p-1}}}[\Phi_{p'}(\int_0^t s^{n+\beta-1}\Phi_{q+1}(u(s))ds)-$$
$$\Phi_{p'}(\int_0^t s^{n+\beta-1}\Phi_{q+1}(v(s))ds)]dt, \tag{4.12}$$

and so

$$|Tu(r)-Tv(r)| \leq \left|\int_0^r \frac{1}{t^{\frac{n+\alpha-1}{p-1}}}(\chi[u](t)-\chi[v](t))dt\right| \tag{4.13}$$

where $\chi[u](t)=\Phi_{p'}(\int_0^t s^{n+\beta-1}\Phi_{q+1}(u(s))ds)$ and $\chi[v](t)$ similarly.

Let $G(\lambda;t)=\chi[\lambda u+(1-\lambda)v](t)$ then $\chi[u](t)-\chi[v](t)=G(1;t)-G(0;t)$.

$$G(\lambda;t)=\Phi_{p'}(\int_0^t s^{n+\beta-1}\Phi_{q+1}(\lambda u(s)+(1-\lambda)v(s))ds). \tag{4.14}$$

Since $\frac{d}{dz}\Phi_{p'}(z)=(p'-1)\,|z|^{p'-2}$ and $p'=\frac{p}{p-1}$ then

$$G_\lambda(\lambda;t)=\frac{1}{p-1}\left|\int_0^t s^{n+\beta-1}\Phi_{q+1}(\lambda u(s)+(1-\lambda)v(s))ds\right|^{\frac{2-p}{p-1}}$$
$$\cdot\int_0^t s^{n+\beta-1}q\,|\lambda u(s)+(1-\lambda)v(s)|^{q-1}(u(s)-v(s))ds. \tag{4.15}$$

By the mean value theorem

$$\chi[u](t)-\chi[v](t)=G_\lambda(\lambda;t) \quad \textit{for some} \quad 0<\lambda<1. \tag{4.16}$$

Therefore by (4.13), (4.15), (4.16) we have

$$|Tu(r)-Tv(r)| \leq \frac{q}{p-1}\int_0^r \frac{1}{t^{\frac{n+\alpha-1}{p-1}}}[\left|\int_0^t s^{n+\beta-1}\Phi_{q+1}(\lambda u+(1-\lambda)v)ds\right|^{\frac{2-p}{p-1}}$$
$$\cdot\int_0^t s^{n+\beta-1}\,|\lambda u(s)+(1-\lambda)v(s)|^{q-1}\,|u(s)-v(s)|\,ds]dt. \tag{4.17}$$

Therefore

$$|Tu(r)-Tv(r)| \leq \frac{q}{p+\beta-\alpha}\frac{1}{(n+\beta)^{\frac{1}{p-1}}}R^{\frac{q+1-p}{p-1}}r^{\frac{p+\beta-\alpha}{p-1}}\,\|u-v\|_\infty \tag{4.18}$$

for $r\in[0,\epsilon]$. Hence,

$$\|Tu(r)-Tv(r)\|_\infty \leq C\,\|u-v\|_\infty \tag{4.19}$$

where

$$C=\frac{q}{p+\beta-\alpha}\frac{1}{(n+\beta)^{\frac{1}{p-1}}}R^{\frac{q+1-p}{p-1}}\epsilon^{\frac{p+\beta-\alpha}{p-1}}<1,$$

since $\epsilon < \sqrt[p+\beta-\alpha]{\frac{(n+\beta)(p+\beta-\alpha)^{p-1}}{R^{q+1-p}}}\ \sqrt[p+\beta-\alpha]{\frac{1}{q^{p-1}}}$.

So T is a contraction on $B_R^\epsilon(\sigma)$ and has a unique fixed point in $B_R^\epsilon(\sigma)$. It can be verified that a fixed point of (4.8) is a solution of (4.1), (4.2). Using the monotonic property of the energy inequality, $E_{1r}(r,\alpha) \le 0$, in lemma 2.1, observe that this solution is uniformly bounded on $[0,r]$ for any r and therefore can be extended to $[0,1]$.

Suppose that $u(r)$, $v(r)$ satisfy (1.1), then

$$u(r) - v(r) = u(0) - v(0) + \int_0^r \frac{1}{t^{\frac{n+\alpha-1}{p-1}}} [\Phi_{p'}(\int_0^t s^{n+\beta-1}\Phi_{q+1}(u(s))ds) - \Phi_{p'}(\int_0^t s^{n+\beta-1}\Phi_{q+1}v(s)ds)]dt. \quad (4.20)$$

And so

$$|u(r) - v(r)| \le |u(0) - v(0)| + \int_0^r \frac{1}{t^{\frac{n+\alpha-1}{p-1}}} |\chi[u](t) - \chi[v](t)|\, dt. \quad (4.21)$$

Hence

$$|u(r) - v(r)| \le |u(0) - v(0)| + \frac{q}{p-1}\int_0^r \frac{1}{t^{\frac{n+\alpha-1}{p-1}}}[\left|\int_0^t s^{n+\beta-1}\Phi_{q+1}(\lambda u(s) + (1-\lambda)v(s))ds\right|^{\frac{2-p}{p-1}} \times \int_0^t s^{n+\beta-1} |\lambda u(s) + (1-\lambda)v(s)|^{q-1} |u(s) - v(s)|\, ds]dt. \quad (4.22)$$

Using Lemma 2.1, we may bound $u(s)$ and $v(s)$ uniformly on $[0,1]$ by a constant C and so by (4.22) we have

$$|u(r) - v(r)| \le |u(0) - v(0)| + C\int_0^r \frac{1}{t^{\frac{n+\alpha-1}{p-1}}}[\left|\int_0^t s^{n+\beta-1}ds\right|^{\frac{2-p}{p-1}} \cdot \int_0^t s^{n+\beta-1} |u(s) - v(s)|\, ds]dt \quad (4.23)$$

$$\le |u(0) - v(0)| + C\int_0^r t^{-n-\frac{(\alpha-1)-\beta(p-2)}{p-1}}[\int_0^t s^{n+\beta-1} |u(s) - v(s)|\, ds]dt \quad (4.24)$$

$$\le |u(0) - v(0)| + C\int_0^r t^{\frac{1+\beta-\alpha}{p-1}} \int_0^t |u(s) - v(s)|\, dsdt \quad (4.25)$$

$$\le |u(0) - v(0)| + Cr^{\frac{p+\beta-\alpha}{p-1}} \int_0^r |u(t) - v(t)|\, dt \quad (4.26)$$

where C is a generic constant. By Gronwall's inequality and (4.26) we have

$$|u(r) - v(r)| \le C\,|u(0) - v(0)| \quad (4.27)$$

for all $r \in [0,1]$ and the proof is complete.

Next I introduce a quantity measure which I refer to as

$$\chi(r,\sigma) = (r^{2\alpha} |u_r|^{2(p-1)} + r^{2\beta} u^2)^{1/2} \tag{4.28}$$

I will use the property that

$$\chi(r,\sigma) \to \infty, \quad \text{as} \quad |\sigma| \to \infty, \tag{4.29}$$

uniformly for $r \in [0,1]$. It is clear from the definitions of (4.28) and the Energy function $E(r,\sigma)$ that $\chi(r,\sigma) \to \infty$ if and only if $E(r,\sigma) \to \infty$, hence one may establish (4.29) by showing

$$E(r,\sigma) \to \infty, \quad \text{as} \quad |\sigma| \to \infty, \tag{4.30}$$

uniformly in $r \geq 0$.

Suppose that $0 \leq \rho \leq 1$ and $r = r_\rho(\sigma)$ are such that, for a finite σ, then

$$u(r_\rho(\sigma)) = \rho\sigma, |u(r)| \geq |\sigma|\, \rho, \forall 0 \leq r, \leq r_\rho(\sigma) \tag{4.31}$$

$$u(r_0(\sigma)) = 0, \quad r_1(\sigma) = 0. \tag{4.32}$$

As a result, I introduce the following estimate for $r_\rho(\sigma)$.

Lemma 4.1. *Let $r_\rho(\sigma)$ be as defined above. Then for some positive constant $C = C(p,n,\alpha,\beta)$, $r_\rho(\sigma)$ satisfies*

$$|\sigma|^{-\frac{1}{\theta}} (1-\rho)^{\frac{p-1}{\beta+p-\alpha}} \leq C r_\rho(\sigma) \leq |\sigma|^{-\frac{1}{\theta}} \rho^{-\frac{q}{p+\beta-\alpha}} (1-\rho)^{\frac{p-1}{\beta+p-\alpha}} \tag{4.33}$$

where $\theta = \frac{\beta+p-\alpha}{q+1-p}$ and $C = (\frac{p-1}{p+\beta-\alpha})^{\frac{p-1}{\beta+p-\alpha}} (\frac{1}{n+\beta})^{\frac{1}{\beta+p-\alpha}}$.

Proof. Integrating with respect to r both sides of equation (4.7) from 0 to $r_\rho(\sigma)$ we have

$$\int_0^{r_\rho(\sigma)} u_r(r)dr = -\int_0^{r_\rho(\sigma)} \frac{1}{r^{\frac{n+\alpha-1}{p-1}}} \Phi_{p'}(\int_0^r s^{n+\beta-1} \Phi_{q+1}(u(s))ds)dr \tag{4.34}$$

where

$$\int_0^{r_\rho(\sigma)} u_r(r)dr = u(r_\rho(\sigma)) - u(0) = \rho\sigma - \sigma = -\sigma(1-\rho). \tag{4.35}$$

Hence we have

$$\sigma(1-\rho) = \int_0^{r_\rho(\sigma)} \frac{1}{r^{\frac{n+\alpha-1}{p-1}}} \Phi_{p'}(\int_0^r s^{n+\beta-1} \Phi_{q+1}(u(s))ds)dr. \tag{4.36}$$

Since $|u(r)| \geq |\sigma|\,\rho$ for all $0 \leq r \leq r_\rho(\sigma)$, then $|u(0)| \geq |u(r)| \geq |u(r_\rho(\sigma))|$ and therefore $|\sigma|\,\rho \leq |u(r)| \leq |\sigma|$ for $r \in [0, r_\rho(\sigma)]$. Hence we have

$$\int_0^{r_\rho(\sigma)} \frac{1}{r^{\frac{n+\alpha-1}{p-1}}} \Phi_{p'}\Big(\int_0^r s^{n+\beta-1}(|\sigma|\,\rho)^q ds\Big)dr \leq |\sigma|\,(1-\rho)$$
$$\leq \int_0^{r_\rho(\sigma)} \frac{1}{r^{\frac{n+\alpha-1}{p-1}}} \Phi_{p'}\Big(\int_0^r s^{n+\beta-1}(|\sigma|^q)ds\Big)dr, \quad (4.37)$$

where $\Phi_{q+1}(|\sigma|\,\rho) = |\sigma\rho|^{q-1}\,|\sigma|\,\rho = (|\sigma|\,\rho)^q$, similarly $\Phi_{q+1}(|\sigma|) = |\sigma|^q$.

Evaluating the inside integrals of equation (4.37) gives

$$\int_0^{r_\rho(\sigma)} \frac{1}{r^{\frac{n+\alpha-1}{p-1}}} \Phi_{p'}\Big[\frac{(|\sigma|\,\rho)^q}{n+\beta} s^{n+\beta}|_0^r\Big]dr \leq |\sigma|\,(1-\rho)$$
$$\leq \int_0^{r_\rho(\sigma)} \frac{1}{r^{\frac{n+\alpha-1}{p-1}}} \Phi_{p'}\Big[\frac{|\sigma|^q}{n+\beta} s^{n+\beta}|_0^r\Big]dr, \quad (4.38)$$

which for $n+\beta > 0$ gives

$$\int_0^{r_\rho(\sigma)} \frac{1}{r^{\frac{n+\alpha-1}{p-1}}} \Phi_{p'}\Big[\frac{(|\sigma|\,\rho)^q}{n+\beta} r^{n+\beta}\Big]dr \leq |\sigma|\,(1-\rho)$$
$$\leq \int_0^{r_\rho(\sigma)} \frac{1}{r^{\frac{n+\alpha-1}{p-1}}} \Phi_{p'}\Big[\frac{|\sigma|^q}{n+\beta} r^{n+\beta}\Big]dr. \quad (4.39)$$

Equation (4.39) in turn simplifies to

$$\int_0^{r_\rho(\sigma)} \frac{1}{r^{\frac{n+\alpha-1}{p-1}}} \Big[\frac{(|\sigma|\,\rho)^q}{n+\beta} r^{n+\beta}\Big]^{\frac{1}{p-1}} dr \leq |\sigma|\,(1-\rho)$$
$$\leq \int_0^{r_\rho(\sigma)} \frac{1}{r^{\frac{n+\alpha-1}{p-1}}} \Big[\frac{|\sigma|^q}{n+\beta} r^{n+\beta}\Big]^{\frac{1}{p-1}} dr \quad (4.40)$$

where

$$\Phi_{p'}\Big[\frac{(|\sigma|\,\rho)^q}{n+\beta} r^{n+\beta}\Big] = \Big|\frac{(|\sigma|\,\rho)^q}{n+\beta} r^{n+\beta}\Big]\Big|^{\frac{p}{p-1}-2}\Big(\frac{(|\sigma|\,\rho)^q}{n+\beta} r^{n+\beta}\Big) = \Big[\frac{(|\sigma|\,\rho)^q}{n+\beta} r^{n+\beta}\Big]^{\frac{1}{p-1}} \quad (4.41)$$

and

$$\Phi_{p'}\Big[\frac{|\sigma|^q}{n+\beta} r^{n+\beta}\Big] = \Big[\frac{|\sigma|^q}{n+\beta} r^{n+\beta}\Big]^{\frac{1}{p-1}}. \quad (4.42)$$

Hence we have

$$\Big(\frac{(|\sigma|\,\rho)^q}{n+\beta}\Big)^{\frac{1}{p-1}} \int_0^{r_\rho(\sigma)} r^{\frac{(\beta-\alpha+1)}{(p-1)}} dr \leq |\sigma|\,(1-\rho) \leq \Big(\frac{|\sigma|^q}{n+\beta}\Big)^{\frac{1}{p-1}} \int_0^{r_\rho(\sigma)} r^{\frac{\beta-\alpha+1}{p-1}} dr. \quad (4.43)$$

For $p > \alpha - \beta, p > 1$, we have

$$(\frac{p-1}{p+\beta-\alpha})(\frac{(|\sigma|\,\rho)^q}{n+\beta})^{\frac{1}{p-1}}(r_\rho(\sigma))^{\frac{(p+\beta-\alpha)}{p-1}} \leq |\sigma|\,(1-\rho)$$
$$\leq (\frac{p-1}{p+\beta-\alpha})(\frac{|\sigma|^q}{n+\beta})^{\frac{1}{p-1}}(r_\rho(\sigma))^{\frac{(p+\beta-\alpha)}{p-1}}, \quad (4.44)$$

hence

$$|\sigma|^{-\frac{1}{\theta}}(1-\rho)^{\frac{(p-1)}{(\beta+p-\alpha)}} \leq Cr_\rho(\sigma) \leq |\sigma|^{-\frac{1}{\theta}}\rho^{-\frac{q}{(\beta+p-\alpha)}}(1-\rho)^{\frac{(p-1)}{(\beta+p-\alpha)}} \quad (4.45)$$

where $C = (\frac{1}{n+\beta})^{\frac{1}{p+\beta-\alpha}}(\frac{p-1}{p+\beta-\alpha})^{\frac{p-1}{p+\beta-\alpha}}$ and $\theta = \frac{p+\beta-\alpha}{q+1-p}$.

It is evident now that as $\sigma \to \infty$, $|\sigma|^{-1/\theta} \to 0$ for $\theta > 0$ or $|\sigma|^{-1/\theta} \to \infty$ for $\theta < 0$. Hence $r_\rho(\sigma) \to \infty$ or 0 as $|\sigma| \to \infty$, for $\rho \in [0,1]$, depending on the sign of θ. Next, I prove that radial bounded solutions approach the origin with zero slope for $\beta - \alpha + 1 > 0$, which in turn implies that $\theta = \frac{p+\beta-\alpha}{q+1-p}$ is positive for $1 < p \leq 2$, $q > 1$, $\alpha, \beta \geq 0$.

Theorem 4.1. *For $\beta - \alpha + 1 > 0, 1 < p \leq 2, q > 1$, radial solutions of (1.1) satisfying the boundary condition in (1.2) approach the origin with zero slope such that $u_r(0) = 0$.*

Proof. For simplicity use the convention $|a|^{\gamma-1}a = a^\gamma$ in (1.1), we then have

$$(r^{n+\alpha-1}u_r^{p-1})_r = -r^{n+\beta-1}u^q. \quad (4.46)$$

Integrating both sides of equation (4.46) from ϵ to r with respect to r gives

$$r^{n+\alpha-1}u_r^{p-1}(r) - \epsilon^{n+\alpha-1}u_r^{p-1}(\epsilon) = -\int_\epsilon^r t^{n+\beta-1}u^q(t)dt. \quad (4.47)$$

I am interested in bounded radial solutions in $C^1[0,1]$, in particular, bounded at the origin with $u_r(0) = 0$. Letting $lim_{\epsilon\to 0}\epsilon^{n+\alpha-1}u_r^{p-1}(\epsilon) = 0$ as $\epsilon \to 0$, therefore

$$r^{n+\alpha-1}u_r^{p-1}(r) = -\int_0^r t^{n+\beta-1}u^q(t)dt. \quad (4.48)$$

Adding $\int_0^r t^{n+\beta-1}u^q(0)dt$ and applying the absolute value to both sides of equation (4.48) gives

$$\left|r^{n+\alpha-1}u_r^{p-1}(r) + \int_0^r t^{n+\beta-1}u^q(0)dt\right| \leq \left|\int_0^r t^{n+\beta-1}(u^q(t) - u^q(0))dt\right|, \quad (4.49)$$

which for $n + \beta > 0$ and near zero gives

$$\left|r^{n+\alpha-1}u_r^{p-1}(r) + \frac{\sigma^q}{n+\beta}r^{n+\beta}\right| \leq Cr^{n+\beta}. \quad (4.50)$$

Therefore

$$\left|u_r^{p-1}(r) + \frac{\sigma^q}{n+\beta} r^{\beta-\alpha+1}\right| \leq C r^{\beta-\alpha+1}. \tag{4.51}$$

The condition $u_r(0) = 0$ requires $u_r^{p-1}(r)$ to decay at least as fast as $\frac{\sigma^q}{n+\beta} r^{\beta-\alpha+1}$ and $\beta - \alpha + 1 > 0$, hence

$$|u_r(r)| \leq O(r^{\frac{\beta-\alpha+1}{p-1}}). \tag{4.52}$$

Using Lemmas 2.2 and 4.1, I now prove (4.30).

Corollary 4.1. *Let $u(r)$ satisfy (1.1), (4.2), (4.3) and $\alpha, \beta \geq 0, 1 < p \leq 2$, $q+1 \neq p, \theta = \frac{p+\beta-\alpha}{q+1-p} < 0$, then*

$$E(r,\sigma) = \tfrac{p-1}{p} r^\alpha |u_r|^p + \tfrac{1}{q+1} r^\beta |u|^{q+1} \to \infty \text{ as } |\sigma| \to \infty$$

uniformly for $r \in [0,1]$.

Proof. By (2.25), for all $r \in [0,1]$

$$(n+\beta)\lambda \int_0^r s^{\beta+n-1} |u|^{q+1} \, ds = \frac{n+\beta}{p^*} r^{n+\alpha-1} |u_r|^{p-2} u_r u + r^n E(r,\sigma). \tag{4.53}$$

Therefore for $r \in [0,1]$ and $\alpha \geq 0$ then $\left|r^{n+\alpha-1}\right| \leq |r^\alpha|$, we have

$$(n+\beta)\lambda \int_0^r s^{\beta+n-1} |u|^{q+1} \, ds \leq \frac{n+\beta}{p^*} r^\alpha |u_r|^{p-1} |u| + E(r,\sigma). \tag{4.54}$$

Using the arithmetic-geometric inequality $a^c b^d \leq ca + db$, $c + d = 1$, $a, b \geq 0$, taking $a = |u_r|^p$, $b = |u|^p$, $c = 1 - \frac{1}{p}$, and $d = \frac{1}{p}$ we have,

$$(n+\beta)\lambda \int_0^r s^{\beta+n-1} |u|^{q+1} \, ds \leq \frac{n+\beta}{p^*} r^\alpha [\frac{p-1}{p} |u_r|^p + \frac{1}{p} |u|^p] + E(r,\sigma). \tag{4.55}$$

Multiplying by $\frac{1}{n+\beta}$ and using the arithmetic-geometric inequality again taking $a = |u|^{q+1}$, $c = \frac{p}{q+1}$,$b = 1$, $d = 1 - \frac{p}{q+1}$, for $\alpha \geq \beta$, and $r \in [0,1]$, we have

$$\begin{aligned} &\lambda \int_0^r s^{\beta+n-1} |u|^{q+1} \, ds \\ &\quad \leq \frac{1}{p^*} [\frac{p-1}{p} r^\alpha |u_r|^p + \frac{1}{q+1} r^\beta |u|^{q+1} + \frac{1}{p} r^\beta (1 - \frac{p}{q+1})] + \frac{1}{n+\beta} E(r,\sigma). \end{aligned} \tag{4.56}$$

Equation (4.56) implies that for any $r \in [0,1]$ and arbitrary $r_\rho(\sigma) \leq r$

$$\lambda \int_0^r s^{\beta+n-1} |u|^{q+1} \, ds \leq [\frac{n+\alpha}{(n+\beta)p}] E(r,\sigma) + C' \tag{4.57}$$

with $C' = \frac{1}{p^*}[\frac{1}{p} - \frac{1}{q+1}]$.

By (4.31), then for all $r \geq r_\rho(\sigma)$, and $n+\beta > 0$ we have

$$\frac{\lambda}{n+\beta}(r_\rho(\sigma))^{n+\beta}(|\sigma|\,\rho)^{q+1} \leq [\frac{n+\alpha}{(n+\beta)p}]E(r,\sigma) + C'. \tag{4.58}$$

Using the inequality (4.33), for all $r \geq r_\rho(\sigma)$, we have

$$\frac{\lambda}{n+\beta}\,|\sigma|^{q+1}\,|\rho|^{q+1}\,(1-\rho)^{\frac{(\beta+n)(p-1)}{\beta+p-\alpha}}(|\sigma|^{-\frac{1}{\theta}})^{\beta+n} \leq C([\frac{n+\alpha}{(n+\beta)p}]E(r,\sigma) + C'). \tag{4.59}$$

It is clear that for $\theta < 0, q+1 > 0, \beta+n > 0$, the energy function of solutions $E(r,\sigma) \to \infty$ as $|\sigma| \to \infty$ for any $r_\rho(\sigma) \leq r$ since $\rho \in [0,1]$.

The fixed point $u(r,\sigma)$ of T in (4.8) is a continuous function of the initial data σ and r since the conditions $1 < p \leq 2 < n, \beta \geq 0$ and $p < q+1 < p^*$ guarantee that the functional $\Phi_{p'}(\int_0^r s^{n+\beta-1}\Phi_{q+1}(u)ds)$ is locally lipschitz. This and equation (4.7) imply that $u_r(r,\sigma)$ is also a continuous function of σ and r.

Let $\chi = (r^{2\alpha}\,|u_r|^{2(p-1)} + r^{2\beta}u^2)^{1/2}$. Define $\Theta(r,\sigma)$ such that

$$r^\beta u(r,\sigma) = \chi cos\Theta(r,\sigma) \tag{4.60}$$

and

$$-r^\alpha\,|u_r(r,\sigma)|^{p-2}\,u_r(r,\sigma) = \chi sin\Theta(r,\sigma). \tag{4.61}$$

Lemma 4.2. $\Theta(r,\sigma)$ *is a uniformly continuous function of* σ *for* $0 \leq r \leq 1$.

Proof. A natural consequence of the uniform continuity of $u(r,\sigma)$ and $u_r(r,\sigma)$ on $r \in [0,1]$.

Lemma 4.3. $\Theta(r,\sigma) \geq 0$ *for all* $r \in [0,1]$ *and* $\sigma > 0$.

Proof. Suppose that there exists an $r \in (0,1]$ such that $\Theta(r,\sigma) < 0$ (note that $\Theta(0,\sigma) = \frac{\pi}{2}$). Since $u(r,\sigma)$ is continuous in r and $u(0,\sigma) = \sigma > 0$ then by (4.4) there exists $\epsilon > 0$ such that $\Theta(r,\sigma) > 0$ for $0 < r \leq \epsilon$. Then for a fixed $\sigma > 0$, by the continuity of $\Theta(r,\sigma)$ and the Mean Value Theorem, there exists $r^* > \epsilon > 0$ satisfying $\Theta(r^*,\sigma) = 0$ and thus some $\delta > 0$ such that

1. in $(r^* - \delta, r^*)$, $u(r,\sigma) > 0$ and $u_r(r,\sigma) < 0$.

2. in $(r^*, r^* + \delta)$, $u(r,\sigma) > 0$ and $u_r(r,\sigma) > 0$.

3. $u(r^*,\sigma) > 0$, $u_r(r^*,\sigma) = 0$.

If $u(r) > 0$ for all $r \in (0, r^*)$, then let $\hat{r} = 0$. If $u(r,\sigma) \leq 0$ for some $r \in (0, r^*)$, then (1) implies that u has a local maximum at some $s \in (0, r^*)$, where $u_r(s,\sigma) = 0$, and we let $\hat{r}$ be the largest of possible values of s. Therefore there exists an $\hat{r}$ in $(0, r^*)$ such that $u_r(\hat{r},\sigma) = 0$ and $u(r,\sigma) > 0$ in $[\hat{r}, r^*]$.

Using (4.4) we have

$$\Phi_p(u_r(r^*,\sigma)) = -\frac{1}{r^{*(n+\alpha-1)}}\int_0^{r^*} s^{n+\beta-1}\Phi_{q+1}(u(s))ds$$
$$= -\frac{1}{r^{*(n+\alpha-1)}}\int_0^{\hat{r}} s^{n+\beta-1}\Phi_{q+1}(u)ds - \frac{1}{r^{*(n+\alpha-1)}}\int_{\hat{r}}^{r*} s^{n+\beta-1}\Phi_{q+1}(u)ds. \quad (4.62)$$

And so

$$\Phi_p(u_r(r^*,\sigma)) = \frac{\hat{r}^{n+\alpha-1}}{r^{*(n+\alpha-1)}}\Phi_p(u_r(\hat{r},\sigma)) - \frac{1}{r^{*(n+\alpha-1)}}\int_{\hat{r}}^{r*} s^{n+\beta-1}\Phi_{q+1}(u)ds. \quad (4.63)$$

Since both $u_r(\hat{r},\sigma)$ and $u_r(r^*,\sigma) = 0$ we have,

$$0 = -\frac{1}{r^{*(n+\alpha-1)}}\int_{\hat{r}}^{r^*} s^{n+\beta-1}\Phi_{q+1}(u)ds, \quad (4.64)$$

which implies $u(r) = 0$. Since $u(r) > 0$ by assumption on $[\hat{r}, r^*]$, this leads to $\Theta(r,\sigma) \geq 0$ by contradiction.

By differentiating $\Theta(r,\sigma)$ with respect to r and using (1.1) we obtain:

$$\Theta_r(r,\sigma) = \frac{(n+\beta-1)r^{\alpha+\beta-1}|u_r|^{p-2}uu_r + r^{2\beta}|u|^{q+1} + r^{\alpha+\beta}|u_r|^p}{\chi^2}. \quad (4.65)$$

Theorem 4.2. $\Theta(1,\sigma) \to \infty$ *as* $\sigma \to \infty$.

Proof. Write $[s_0, 1]$ as $U_{k=1}^m[r_k, r_{k+1}]$, where m is a positive integer, $0 < s_0 < 1$, $r_k < r_{k+1}$, $s_0 = r_1, r_{m+1} = 1$ and the set $\{r_k\}_{k=1}^m$ contains all the zeros of u in $[s_0, 1]$. Since as a function of r, Θ is continuous in $[r_k, r_{k+1}]$ and differentiable in (r_k, r_{k+1}), we have,

$$\Theta(1,\sigma) - \Theta(s_0,\sigma) = \sum_{k=1}^m(\Theta(r_{k+1},\sigma) - \Theta(r_k,\sigma)) \quad (4.66)$$

$$= \lim_{\delta\to 0}\sum_{k=1}^m(\Theta(r_{k+1}-\delta,\sigma) - \Theta(r_k+\delta,\sigma)) \quad (4.67)$$

$$= \lim_{\delta\to 0}\sum_{k=1}^m\int_{r_k+\delta}^{r_{k+1}-\delta}\frac{(n+\beta-1)r^{\alpha+\beta-1}|u_r|^{p-2}uu_r + r^{2\beta}|u|^{q+1} + r^{\alpha+\beta}|u_r|^p}{\chi^2}dr \quad (4.68)$$

$$= \int_{s_0}^{1} \frac{(n+\beta-1)r^{\alpha+\beta-1}|u_r|^{p-2}uu_r + r^{2\beta}|u|^{q+1} + r^{\alpha+\beta}|u_r|^p}{\chi^2}dr \tag{4.69}$$

$$\geq \int_{s_0}^{1} \frac{r^{\alpha+\beta}|u_r|^p + r^{2\beta}|u|^{q+1} - [\frac{n+\beta-1}{2s_0}][r^{2\alpha}|u_r|^{2(p-1)} + r^{2\beta}u^2]}{\chi^2}dr \tag{4.70}$$

$$= \int_{s_0}^{1} \frac{r^{\alpha+\beta}|u_r|^p + r^{2\beta}|u|^{q+1}}{\chi^2}dr - [\frac{n+\beta-1}{2}](\frac{1}{s_0}-1) \tag{4.71}$$

which can be rewritten as

$$\Theta(1,\sigma) + \frac{n+\beta-1}{2}(\frac{1}{s_0}-1) \geq \int_{s_0}^{1} \frac{r^{\alpha+\beta}|u_r|^p + r^{2\beta}|u|^{q+1}}{\chi^2}dr. \tag{4.72}$$

The integrand in (4.73) equals

$$I = \frac{r^{\alpha+\beta}|u_r|^p + r^{2\beta}|u|^{q+1}}{r^{2\alpha}|u_r|^{2(p-1)} + r^{2\beta}u^2} \tag{4.73}$$

by definition of χ^2. Since $p > 2(p-1)$ and $|u_r(r)|, |u(r)| \to \infty$ if and only if $E(r,\sigma) \to \infty$ then, from corollary 4.1 and lemma 4.2, the integrand above approaches infinity uniformly on $[s_0, 1]$ as $\sigma \to \infty$ for $q > 1$ and $\alpha, \beta \geq 0$. Therefore

$$\Theta(1,\sigma) \to \infty \quad as \quad \sigma \to \infty. \tag{4.74}$$

By the continuity of $\Theta(1,\sigma)$ in σ and by (4.75) there exists a sequence of pairs $\{\sigma_k, k\}$ such that $\Theta(1,\sigma_k) = \frac{(2k-1)\pi}{2}$ as a result we have this theorem.

Remark. $\Theta(1,\sigma) \to \infty$ as $\sigma \to -\infty$ can be shown in the same manner.

Theorem 4.3. *Let $\alpha, \beta \geq 0, 1 < p \leq 2, p < q+1 < p^* = \frac{p(n+\beta)}{\alpha+n-p}$. Then (1.1),(1.2) has infinitely many radially symmetric solutions lying in*

$$W^{1,p}(B^n, |x|^\alpha) \cap L^{q+1}(B^n, |x|^\beta).$$

5. Bounded Solutions

In this section I prove that the class of all nontrivial solutions of (1.3), general and radial, are bounded below for $p < q+1 < p^* = \frac{p(n+\beta)}{n+\alpha-p}$, $1 < p \leq 2, q > 1, \alpha, \beta \geq 0$ by a constant dependent on p, n, α, β.

Theorem 5.1. *General solutions for the generalized Lane-Emden equation (1.3) are bounded below by a constant $C = C(p, n, \alpha, \beta)$ for $\alpha, \beta \geq 0, 1 < p \leq 2$, $p < q+1 < p^*$.*

Proof. Consider (2.8)

$$\textstyle\int_{B^n}(|x|^\alpha |\nabla u|^p)dx = \int_{B^n}(|x|^\beta |u|^{q+1})dx. \qquad (2.8)$$

When both sides of equation (2.8) are raised to the power $\frac{1}{q+1}$ we have

$$(\int_{B^n}(|x|^\alpha |\nabla u|^p)dx)^{\frac{1}{q+1}} = \|u\|_{L^{q+1}(B^n,|x|^\beta)} \,. \tag{5.1}$$

The weighted Sobolev inequality $\|u\|_{L^{q+1}(B^n,|x|^\beta)} \leq C\,\|\nabla u\|_{L^p(B^n,|x|^\alpha)}$ for $p < q+1 < p^*$, $p^* = \frac{p(n+\beta)}{n+\alpha-p}, \alpha, \beta \geq 0, 1 < p \leq 2 < n$, where C is a constant dependent on p, n, α and β, is in fact the inequality

$$(\int_{B^n} |x|^\beta |u|^{q+1}\, dx) \leq C(\int_{B^n} |x|^\alpha |\nabla u|^p\, dx)^{\frac{q+1}{p}}. \tag{5.2}$$

Putting the equations (2.8) and (5.2) together we have,

$$(\int_{B^n} |x|^\alpha |\nabla u|^p\, dx) \leq C(\int_{B^n} |x|^\alpha |\nabla u|^p\, dx)^{\frac{q+1}{p}} \tag{5.3}$$

Therefore, for u not equal to zero

$$(\int_{B^n} |x|^\alpha |\nabla u|^p)dx)^{1-(\frac{q+1}{p})} \leq C \tag{5.4}$$

and

$$(\int_{B^n} |x|^\beta |u|^{q+1}\, dx)^{1-(\frac{q+1}{p})} \leq C. \tag{5.5}$$

The exponent simplifies to $-(\frac{q+1-p}{p})$. Observe that for $q+1 > p$ the exponent is negative and hence the class of all non trivial solutions are bounded below by a constant.

I also prove in a similar manner the same property for radial solutions of (1.3) for $\alpha, \beta \geq 0, 1 < p \leq 2, p < q+1 < p^*$.

Corollary 5.1. *Radial solutions for the generalized Lane-Emden equation (1.3) are bounded below for* $\alpha, \beta \geq 0, 1 < p \leq 2, p < q+1 < p^*$ *by a constant* $C(n, p, \alpha, \beta)$.

Proof. The weighted radial norm is defined as

$$\|u\|_{L^{q+1}(B^n,|x|^\beta)} = (\int_0^1 r^{\beta+n-1}\, |u|^{q+1}\, dr)^{\frac{1}{q+1}} \tag{5.6}$$

where $u = u(r)$, and

$$\|\nabla u\|_{L^p(B^n,|x|^\alpha)} = (\int_0^1 r^{\alpha+n-1}\, |u_r|^p\, dr)^{\frac{1}{p}}. \tag{5.7}$$

The weighted Sobolev inequality (5.2) in radial form for $p < q+1 < p^*$ and C, a constant dependent on p, n, α and β, is in fact the inequality

$$(\int_0^1 r^{\beta+n-1}\,|u|^{q+1}\,dr)^{\frac{1}{q+1}} \leq C(\int_0^1 r^{\alpha+n-1}\,|u_r|^p\,dr)^{\frac{1}{p}}. \tag{5.8}$$

Applying the same analysis used for the proof of theorem 5.1, Observe that the class of all nontrivial radial solutions in $L^{q+1}(B^n, |x|^\beta)$ is bounded below by a constant dependent on p, n, α and β.

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Appendix

Consider the GLE equation, [22]

$$\nabla \cdot (|x|^{\alpha} |\nabla u|^{p-2} \nabla u) = -|x|^{\beta} |u|^{q-1} u \tag{A-1}$$

on the domain $\Omega_\delta = \Omega \backslash \{x \in \Re^n : |x| \leq \delta\}$. Multiplying by $x \cdot \nabla u$ and integrating both sides of the equation we have

$$\int_{\Omega_\delta} \nabla \cdot (|x|^{\alpha} |\nabla u|^{p-2} \nabla u)(x \cdot \nabla u) dx = -\int_{\Omega_\delta} |x|^{\beta} |u|^{q-1} u(x \cdot \nabla u) dx. \tag{A-2}$$

Integrating by parts the left hand side of (A-2) gives the result of the LHS

$$= \int_{\partial\Omega_\delta} |x|^{\alpha} |\nabla u|^{p-2} (\nabla u \cdot \nu)(x \cdot \nabla u) dS - \int_{\Omega_\delta} |x|^{\alpha} |\nabla u|^{p-2} \nabla u \cdot \nabla(x \cdot \nabla u) dx \tag{A-3}$$

where ν is a unit outer normal vector. Consider part I of (A-3)

$$\int_{\partial\Omega_\delta} |x|^{\alpha} |\nabla u|^{p-2} (\nabla u \cdot \nu)(x \cdot \nabla u) dS$$
$$= \int_{\partial\Omega_\delta} |x|^{\alpha} |\nabla u|^{p} (x \cdot \nu) dS + \int_{|x|=\delta} \delta^{\alpha} |\nabla u|^{p-2} (\nabla u \cdot \nu)(x \cdot \nabla u) dS. \tag{A-4}$$

Part II of (A-3) is simplified as follows

$$\int_{\Omega_\delta} |x|^{\alpha} |\nabla u|^{p-2} \nabla u \cdot \nabla(x \cdot \nabla u) dx = \int_{\Omega_\delta} |x|^{\alpha} |\nabla u|^{p-2} (|\nabla u|^2 + (x \cdot \nabla) |\nabla u|^2) dx \tag{A-5}$$

$$= \int_{\Omega_\delta} |x|^{\alpha} |\nabla u|^{p} dx + \int_{\Omega_\delta} |x|^{\alpha} (x \cdot \nabla) \frac{1}{p} |\nabla u|^{p} dx \tag{A-6}$$

$$= \int_{\Omega_\delta} |x|^{\alpha} |\nabla u|^{p} dx + \int_{\partial\Omega_\delta} |x|^{\alpha} (x \cdot \nu) \frac{1}{p} |\nabla u|^{p} dS - (\alpha + n) \int_{\Omega_\delta} |x|^{\alpha} \frac{1}{p} |\nabla u|^{p} dx \tag{A-7}$$

$$= (1 - \frac{\alpha + n}{p}) \int_{\Omega_\delta} |x|^{\alpha} |\nabla u|^{p} dx + \frac{1}{p} \int_{\partial\Omega_\delta} |x|^{\alpha} (x \cdot \nu) |\nabla u|^{p} dS. \tag{A-8}$$

Then the LHS (I-II) gives

$$(1 - \frac{1}{p}) \int_{\partial\Omega_\delta} |x|^{\alpha} |\nabla u|^{p} (x \cdot \nu) dS + \int_{|x|=\delta} \delta^{\alpha} |\nabla u|^{p-2} (\nabla u \cdot \nu)(x \cdot \nabla u) dS$$
$$- \frac{1}{p} \int_{|x|=\delta} \delta^{\alpha} (x \cdot \nu) |\nabla u|^{p} dS - (1 - \frac{\alpha + n}{p}) \int_{\Omega_\delta} |x|^{\alpha} |\nabla u|^{p} dx. \tag{A-9}$$

Now consider the right hand side of (A-2)

$$R.H.S = -\int_{\Omega_\delta} |x|^\beta |u|^{q-1} u(x\cdot\nabla u)dx = -\int_{\Omega_\delta} |x|^\beta (x\cdot\nabla)\frac{|u|^{q+1}}{q+1}dx \quad \text{(A-10)}$$

$$= -\int_{\partial\Omega_\delta} |x|^\beta (x\cdot\nu)\frac{|u|^{q+1}}{q+1}dS + \int_{\Omega_\delta} \nabla\cdot(|x|^\beta x)\frac{|u|^{q+1}}{q+1}dx \quad \text{(A-11)}$$

$$= -\int_{\partial\Omega_\delta} |x|^\beta (x\cdot\nu)\tfrac{|u|^{q+1}}{q+1}dS + (n+\beta)\int_{\Omega_\delta} |x|^\beta \tfrac{|u|^{q+1}}{q+1}dx - \int_{|x|=\delta} \delta^\beta(x\cdot\nu)\frac{|u|^{q+1}}{q+1}dS. \quad \text{(A-12)}$$

On $|x| = \delta$, $x = -\delta\nu$, then $x\cdot\nu = -\delta$ and therefore $\delta^\alpha(x\cdot\nu) = -\delta^{\alpha+1}$ and $\delta^\beta(x\cdot\nu) = -\delta^{\beta+1}$, hence equations (A-2), (A-9) and (A-12) give

$$(1-\frac{1}{p})\int_{\partial\Omega_\delta} |x|^\alpha |\nabla u|^p (x\cdot\nu)dS + \int_{|x|=\delta} \delta^\alpha |\nabla u|^{p-2} (\nabla u\cdot\nu)(x\cdot\nabla u)dS + \frac{1}{p}\int_{|x|=\delta} \delta^{\alpha+1} |\nabla u|^p dS - (1-\frac{\alpha+n}{p})\int_{\Omega_\delta} |x|^\alpha |\nabla u|^p dx$$
$$= -\frac{1}{q+1}\int_{\partial\Omega_\delta} |x|^\beta (x\cdot\nu) |u|^{q+1} dS + \frac{n+\beta}{q+1}\int_{\Omega_\delta} |x|^\beta |u|^{q+1} dx + \frac{1}{q+1}\int_{|x|=\delta} \delta^{\beta+1} |u|^{q+1} dS. \quad \text{(A-13)}$$

Next we need to get rid of the boundary integrals along $|x| = \delta$ in (A-13). In fact let u be a solution of (1.3), from Caffarelli-Kohn-Nirenberg inequality

$$(\int_{\Re^n} |x|^{-bq} |u|^q dx)^{\frac{p}{q}} \le C_{a,b}\int_{\Re^n} |x|^{-ap} |Du|^p dx \quad \text{(A-14)}$$

and the compact embedding theorem, we know that $\int_\Omega |x|^\alpha |\nabla u|^p dx$ and

$$\int_\Omega |x|^\beta |u|^{q+1} dx$$

are finite. Therefore by the mean value theorem there exists a sequence $\{\delta_m\}$, $\delta_m \to 0$ such that integrals $\int_{|x|=\delta} |x|^\alpha |\nabla u|^p (x\cdot\nu)dS$ and $\int_{|x|=\delta} |x|^\beta |u|^{q+1} (x\cdot\nu)dS$ go to zero as $m\to\infty$. Letting $m\to\infty$ in (A-13) we obtain

$$\frac{p-1}{p}\int_{\partial\Omega} |x|^\alpha |\nabla u|^p (x\cdot v)dS + \frac{n+\alpha-p}{p}\int_\Omega |x|^\alpha |\nabla u|^p dx = \frac{n+\beta}{q+1}\int_\Omega |x|^\beta |u|^{q+1} dx. \quad \text{(A-15)}$$