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ALLEN SHIELD'S REFLEXIVITY QUESTION

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Abstract: This paper presents sufficient conditions to a reflexivity problem considered by A.L. Shields.

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1. Introduction

Let $\{\beta(n)\}_{n=-\infty}^{\infty}$ be a sequence of positive numbers satisfying $\beta(0) = 1$. If $1 < p < \infty$, the space $L^p(\beta)$ consists of all *formal Laurent series* $f(z) = \sum_{n=-\infty}^{\infty} \hat{f}(n)z^n$ such that the norm

$$\|f\|^p = \|f\|_\beta^p = \sum_{n=-\infty}^{\infty} |\hat{f}(n)|^p \beta(n)^p$$

is finite. These are reflexive Banach spaces with the norm $\|\cdot\|_\beta$. Let $\hat{f}_k(n) = \delta_k(n)$. So $f_k(z) = z^k$ and then $\{f_k\}_{k\in\mathbb{Z}}$ is a basis for $L^p(\beta)$ such that $\|f_k\| = \beta(k)$. We denote the set of multipliers

$$\{\varphi \in L^p(\beta) : \varphi L^p(\beta) \subseteq L^p(\beta)\}$$

by $L^p_\infty(\beta)$ and the linear operator of multiplication by φ on $L^p(\beta)$ by M_φ.

We say that a complex number λ is a *bounded point evaluation* on $L^p(\beta)$ if the functional $e(\lambda) : L^p(\beta) \longrightarrow \mathbb{C}$ defined by $e(\lambda)(f) = f(\lambda)$ is bounded.

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By the same method used in [12] we can see that $L^p(\beta)^* = L^q(\beta^{\frac{p}{q}})$, where $\frac{1}{p} + \frac{1}{q} = 1$. Also, if $f(z) = \sum_n \hat{f}(n) z^n \in L^p(\beta)$ and $g(z) = \sum_n \hat{g}(n) z^n \in L^q(\beta^{\frac{p}{q}})$, then clearly

$$< f, g >= \sum_n \hat{f}(n)\overline{\hat{g}(n)}\beta(n)^p$$

where the notation $< f, g >$ stands for $g(f)$. For a source in formal series, we refer the reader to [12 – 16].

Recall that if E is a separable Banach space and $A \in B(E)$, then $Lat(A)$ is by definition the *lattice of all invariant subspaces* of A, and $AlgLat(A)$ is the algebra of all operators B in $B(E)$ such that $Lat(A) \subset Lat(B)$. For the algebra $B(E)$, the *weak operator topology* is the one induced by the family of seminorms $p_{x^*,x}(A) = |< Ax, x^* >|$ where $x \in E$, $x^* \in E^*$ and $A \in B(E)$. Hence $A_\alpha \longrightarrow A$ in the weak operator topology if and only if $A_\alpha x \longrightarrow Ax$ weakly. Also similarly $A_\alpha \longrightarrow A$ in the *strong operator topology* if and only if $A_\alpha x \longrightarrow Ax$ in the norm topology. An operator A in $B(E)$ is said to be *reflexive* if $AlgLat(A) = W(A)$, where $W(A)$ is the smallest subalgebra of $B(E)$ that contains A and the identity I and is closed in the weak operator topology.

2. Main Results

The operator M_z has been the focus of attention for several decades and many of its properties have been studied (e.g. [2,10]). In [8] Sarason proved that normal operators are reflexive. It was shown by J. Deddens ([3]) that every isometry is reflexive. Also, R. Olin and J. Thomson ([6]) have shown that subnormal operators are reflexive. H. Bercovici, C. Foias, J. Langsam, and C. Pearcy ([1]) have shown that (BCP)-operators are reflexive. The reflexive operators on a finite dimensional space were characterized by J. Deddens and P. A. Fillmore ([4]). In [9,17,18,19] we investigated some sufficient conditions for the reflexivity of multiplication operators on some function spaces. Also, reflexivity of canonical models were studied in [5]. In this article we would like to give some sufficient conditions so that the operator M_z, acting on $L^p(\beta)$, becomes reflexive (for a good source of reflexivity see [7]). This paper is a continuation of our work in [9,18] and presents a sufficient condition to a problem that has been posed by A.L. Shields:

Question 2.1. Which shifts are reflexive?

Our result in this paper improves Theorem 2.4 in [18]. Throughout this paper we suppose that M_z is bounded on $L^p(\beta)$. We will use the following notations:

$$\begin{array}{lcl} r_{12} = r(M_z^{-1})^{-1} & , & \Omega_{12} = \{z \in \mathbb{C} : |z| > r_{12}\} \\ r_{22} = r(M_z) & , & \Omega_{22} = \{z \in \mathbb{C} : |z| < r_{22}\} \\ \Omega_2 = \Omega_{12} \cap \Omega_{22} & = & \{z \in \mathbb{C} : r_{12} < |z| < r_{22}\}. \end{array}$$

In the sequel, we need the following Lemma for the proof of our main theorem. By $H(\Omega)$ and $H^\infty(\Omega)$ we will mean respectively the set of analytic functions on a plane domain Ω and the set of bounded analytic functions on Ω.

Lemma 2.2. *Let M_z be invertible and $r_{12} < r_{22}$. Then $L^p_\infty(\beta) \subset H^\infty(\Omega_2)$.*

Proof. If $\varphi(z) = \sum_{k=-\infty}^{\infty} \hat{\varphi}(k) z^k \in L^p_\infty(\beta)$, then we have $< f_m\varphi, f_n >= \hat{\varphi}(n-m)\beta(n)^p$ where f_m and f_n are considered respectively as the elements of $L^p(\beta)$ and $L^q(\beta^{\frac{p}{q}})$. Thus for all n and m in $\mathbb{Z}$:

$$|\hat{\varphi}(n-m)| \leq \|M_\varphi\| \frac{\beta(m)}{\beta(n)}, \qquad (*)$$

which implies that

$$|\hat{\varphi}(k)| \leq \|M_\varphi\|. \inf_m \frac{\beta(m)}{\beta(m+k)} = \|M_\varphi\|.\|M_z^k\|^{-1}.$$

Thus, $|\varphi(z)| \leq \|M_\varphi\| \sum_{k=-\infty}^{\infty} \frac{|z|^k}{\|M_z^k\|}$. Let

$$\sum_{k=-\infty}^{\infty} \frac{z^k}{\|M_z^k\|} = \sum_{k=1}^{\infty} \frac{z^{-k}}{\|M_z^{-k}\|} + \sum_{k=0}^{\infty} \frac{z^k}{\|M_z\|^k} = g_1(z) + g_2(z).$$

Since $\lim_k \|M_z^{-k}\|^{\frac{1}{k}} = r(M_z^{-1})$, by the root test $g_1(\frac{1}{z})$ converges on $\{z : |z| < r(M_z^{-1})\}$ and so $g_1(z)$ converges on Ω_{12}. Also, since $\lim_k \|M_z^k\|^{\frac{1}{k}} = r(M_z) = r_{22}$, $g_2(z)$ converges on Ω_{22}. Thus indeed φ converges on $\Omega_{12} \cap \Omega_{22} = \Omega_2$ and so $L^p_\infty(\beta) \subset H(\Omega_2)$. But $L^p_\infty(\beta)$ is a commutative Banach algebra in the norm $\|\varphi\|_\infty = \|M_\varphi\|$ with identity and evaluation at λ is multiplicative, so the functional e_λ must have norm one and so $\frac{|\varphi(\lambda)|}{\|\varphi\|_\infty} = | < \frac{\varphi}{\|\varphi\|_\infty}, e_\lambda > | \leq \|e_\lambda\| = 1$. Therefore, $|\varphi(\lambda)| \leq \|\varphi\|_\infty = \|M_\varphi\|$ and so $\varphi \in H^\infty(\Omega_2)$. □

Note that the operator M_z is bounded on $L^p(\beta)$ if and only if $\beta(k+1)/\beta(k)$ is bounded and in this case $\|M_z^n\| = \sup_k[\beta(k+n)/\beta(k)]$, $n = 0, 1, 2, \ldots$. Also, M_z is invertible if and only if $\beta(k)/\beta(k+1)$ is bounded and in this case $\|M_z^{-n}\| = \sup_k[\beta(k)/\beta(k+1)]$, $n = 0, 1, 2, \ldots$.

Theorem 2.3. *Let M_z be invertible, $r_{12} < r_{22}$ and $L^p_\infty(\beta) = L^p(\beta)$. Then M_z is reflexive on $L^p(\beta)$.*

Proof. First, note that by Lemma 2.2, $L^p_\infty(\beta) \subset H^\infty(\Omega_2)$. Since $L^p_\infty(\beta) = L^p(\beta)$, thus $L^p(\beta) \subset H(\Omega_2)$ and so each point of Ω_2 is a bounded point evaluation on $L^p(\beta)$. Now, let $A \in AlgLat(M_z)$. Then $Lat(M_z) \subset Let(A)$. Since $M_z^* e(\lambda) = \lambda e(\lambda)$ for all λ in Ω_2, the one dimensional span of $e(\lambda)$ is invariant under M_z^*. Therefore it is invariant under A^* and we can write $A^* e(\lambda) = \varphi(\lambda) e(\lambda),\ \lambda \in \Omega_2$. So

$$< Af, e(\lambda) >=< f, A^* e(\lambda) >= \varphi(\lambda) f(\lambda)$$

for all $f \in L^p(\beta)$ and $\lambda \in \Omega_2$. This implies that $A = M_\varphi$ and $\varphi \in L^p_\infty(\beta)$. Choose $\varepsilon_0 > 0$ and $\varepsilon_0 > 0$ such that the circles

$$\Gamma_0 = \{z \in \mathbb{C} : |z| = r_{12} + \varepsilon_0\}$$

and

$$\Gamma_1 = \{z \in \mathbb{C} : |z| = r_{22} - \varepsilon_1\}$$

lying in Ω_2 and they do not meet each other. Set

$$L_1 = \{f \in L^p(\beta) : \int_{\Gamma_1} z^n f(z) dz = 0 \quad , n = 0, 1, 2, \cdots\}.$$

Note that L_1 is a subspace of $L^p(\beta)$. To see that L_1 is closed let $\{g_k\}$ be a sequence in L_1 such that $g_k \to g$ in $L^p(\beta)$. By the principle of uniform boundedness theorem, we can conclude that convergence in $L^p(\beta)$ implies uniform convergence on compact subsets of Ω_2. Since Γ_1 is a compact subset of Ω_2, it is now easy to see that $g \in L_1$ and so L_1 is closed in $L^p(\beta)$. Also, clearly L_1 is invariant under M_z and contains the constants. Since $L_1 \in Lat(M_z)$, we have $AL_1 \subset L_1$, so $A1 = \varphi \in L_1$. By applying the Cauchy integral formula we can write $\varphi = \varphi_0 + \varphi_1$ where $\varphi_0 \in H_0(\Omega_{12})$ and $\varphi_1 \in H(\Omega_{22})$ ($H_0(\Omega_{12})$ denotes the space of all functions in $H(\Omega_{12})$ that vanishes at ∞). Therefore φ_0 is analytic in $ext(\Gamma_2)$ the unbounded component of $\mathbf{C} \setminus \Gamma_2$ where the circle Γ_2 is chosen sufficiently close to Γ_1 with smaller radius so that Γ_1 lies in $ext(\Gamma_2)$. Now, we can write

$$\varphi_0(z) = \sum_{n=-\infty}^{-1} \hat{\varphi}_0(n) z^n \quad , \quad z \in ext(\Gamma_2).$$

Note that

$$\hat{\varphi}_0(n) = \frac{1}{2\pi i} \int_{\Gamma_1} \varphi_0(z) z^{-(n+1)} dz \ \ , \ n < 0.$$

Since $\varphi_1 \in H(\Omega_{22})$, we have

$$\int_{\Gamma_1} z^n \varphi_1(z) dz = 0 \quad , \ n = 0, 1, , 2, \cdots .$$

But $\varphi \in L_1$, thus we get

$$\int_{\Gamma_1} z^n \varphi_0(z)dz = 0 \quad , \quad n = 0, 1, 2, \cdots .$$

From this it follows that $\hat{\varphi}_0(n) = 0$ for all integers $n \leq -1$ and so $\varphi_0(z) = 0$, $z \in ext(\Gamma_2)$. Hence $\varphi_0 \equiv 0$ which implies that $\varphi = \varphi_1 \in H(\Omega_{22})$. Since Ω_{22} is a Caratheodory domain, we can represent φ by the formal power series $\sum_{k=0}^{\infty} \hat{\varphi}(k)z^k$. By the same method used in [11, page 90, proof of Theorem 12], if we define

$$P_n(z) = \sum_{k=0}^{n}(1 - \frac{k}{n+1})\hat{\varphi}(k)z^k \quad , \quad n \geq 0$$

for all $n \geq 0$, then $M_{p_n} \to M_\varphi$ in the weak operator topology. Since p_n is a polynomial and $M_{p_n} = p_n(M_z)$, we conclude that $A = M_\varphi \in W(M_z)$. Thus, $AlgLat(M_z) \subset W(M_z)$. But, clearly $W(M_z) \subset AlgLat(M_z)$, hence the proof is complete. □

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