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CYCLICITY OF THE MULTIPLICATION OPERATOR ON $H^p(\beta)$

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Abstract: In this paper, we give some sufficient conditions for cyclicity of the multiplication operator on $H^p(\beta)$.

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1. Introduction

Suppose that $1 < p < \infty$ and $\{\beta(n)\}$ denotes a sequence of nonzero complex numbers with $\beta(0) = 1$. For a sequence $f = \{\hat{f}(n)\}_{n=0}^{\infty}$, we define

$$||f|| = ||f||_p = (\sum_{n=0}^{\infty} |\hat{f}(n)|^p |\beta(n)|^p)^{\frac{1}{p}}.$$

Furthermore, we shall use the notation $f(z) = \sum_{n=0}^{\infty} \hat{f}(n) z^n$ regardless whether the series converges for any complex value of z. Throughout this article, by the space $H^p(\beta)$ we mean

$$H^p(\beta) = \{f : f(z) = \sum_{n=0}^{\infty} \hat{f}(n) z^n, ||f||_p < \infty\}$$

which is called a *weighted Hardy space.*

Let $\hat{f}_k(n) = \delta_{nk}$. So $f_k(z) = z^k$ and then $\{f_k\}_k$ is a basis such that $\|f_k\| = |\beta(k)|$. The operator M_z on $H^p(\beta)$ given by $(M_z f)(\zeta) = \zeta f(\zeta)$ is called the *unilateral*

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forward shift operator. Clearly M_z shifts the basis $\{f_k\}_k$. The operator M_z is bounded if and only if $\{\beta(k+1)/\beta(k)\}_k$ is bounded and in this case $\|M_z^n\| = \sup_k |\frac{\beta(n+k)}{\beta(k)}|$ for each positive integer n. The *unilateral backward shift operator* on $H^p(\beta)$ is defined by $Bf(z) = \sum_{k=0}^{\infty} \hat{f}(k+1)z^k$ where $f = \sum_{k=0}^{\infty} \hat{f}(k)f_k \in H^p(\beta)$. The operator B is bounded if and only if $\{\frac{\beta(k)}{\beta(k+1)}\}_k$ is bounded and in this case we have $\|B^n\| = \sup_k |\frac{\beta(k)}{\beta(k+n)}|$ for each positive integer N. Without loss of generality, by equivalence relations we can consider that $\{\beta(n)\} \subseteq \mathbb{R}^+$.

Let E be a Banach space. It is convenient and helpful to introduce the notation $< x, x^* >$ to stand for $x^*(x)$, for $x \in E$ and $x^* \in E$.

The space $H^p(\beta)$ is a reflexive Banach space and the dual of $H^p(\beta)$ is $H^q(\beta^{\frac{p}{q}})$ where $\frac{1}{p} + \frac{1}{q} = 1$ and $\beta^{\frac{p}{q}} = \{\beta(n)^{\frac{p}{q}}\}_n$ ([9]). If $f = \sum_{n=0}^{\infty} \hat{f}(n)f_n \in H^p(\beta)$ and $g = \sum_{n=0}^{\infty} \hat{g}(n)f_n \in (H^p(\beta))^*$, then

$$||g||_q^q = \sum_{n=0}^{\infty} |\hat{g}(n)|^q (\beta(n)^{\frac{p}{q}})^q = \sum_{n=0}^{\infty} |\hat{g}(n)|^q \beta(n)^p$$

and $g(f) =< f, g >= \sum_n \hat{f}(n)\overline{\hat{g}(n)}\beta(n)^p$. Here for simplicity we use $||g||_q$ instead of $||g||_{H^q(\beta^{\frac{p}{q}})}$. For some works on these spaces see [2–6].

We say that a vector x in a Banach space X is a *cyclic vector* of a bounded operator A on X if $X = \text{span}\{A^n x : n = 0, 1, 2, \ldots\}$. Here span$\{\cdot\}$ is the closed linear span of the set.

2. Main Results

In the main theorem of this paper we give some sufficient conditions for a vector f in $H^p(\beta)$ to be cyclic for the multiplication operator acting on $H^p(\beta)$.

From now on we suppose that $\{\beta(n)\} \subseteq \mathbb{R}^+$ and also we suppose that M_z is bounded on $H^p(\beta)$.

Theorem 2.1. *Let $\frac{1}{p} + \frac{1}{q} = 1$ and M_z be power bounded. Also, let $f \in H^p(\beta)$ and $f(0) \neq 0$. If $\{\frac{\beta(n+j_0)}{\beta(n)}\}_n \in \ell^q$ for some $j_0 \in \mathbb{N}$, then f is a cyclic vector of M_z on $H^p(\beta)$.*

Proof. Without loss of generality, assume that $f(0) = 1$. So there exists a formal power series $g(z) = \sum_{m=0}^{\infty} \hat{g}(m)z^m$ such that $g(z)f(z) = 1$ ([8]). Also let s_n be the

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nth partial sum of g, i.e., $s_n(z) = \sum_{m=0}^{n} \hat{g}(m)z^m$. Since

$$\{M_z^n(s_j(M_z)f) : n \geq 0\} \subseteq \{M_z^n f : n \geq 0\} \quad (j \in \mathbb{N}),$$

it is sufficient to show that for some $j \in \mathbb{N}$:

$$\text{span}\{M_z^n(s_j(M_z)f) : n \geq 0\} = H^p(\beta). \tag{$*$}$$

Note that the relation $f(z)g(z) = 1$ implies that $f(M_z)g(M_z) = I$ where I is the identity operator on $H^p(\beta)$. Recall that $f_i(z) = z^i$ for all nonnegative integer i. So we can see that

$$M_z^n(s_j(M_z) + \sum_{k \geq j+1} \hat{g}(k)M_z^k)f(M_z)f_0 = M_z^n f_0 = f_n.$$

Since $f(M_z)f_0 = f$, we get

$$M_z^n s_j(M_z)f - f_n = \sum_{k \geq j+1} \hat{g}(k)M_z^{k+n} f \tag{$**$}$$

and so $M_z^n s_j(M_z)f - f_n \in \mathcal{M}_{n+j+1}$ where for all nonnegative integer k, $\mathcal{M}_k$ is the closed linear span of the set $\{f_j : j \geq k\}$ in $H^p(\beta)$. Define the projection $P_i : H^p(\beta) \longrightarrow \mathcal{M}_i$ by

$$P_i(\sum_{m \geq 0} \hat{h}(m)z^m) = \sum_{m \geq i} \hat{h}(m)z^m.$$

Note that $P_i M_z^m = M_z^m P_{i-m}$. Thus by the relation $(**)$ we get

$$\begin{aligned}
\frac{1}{\beta(n)}\|M_z^n s_j(M_z)f - f_n\| &= \frac{1}{\beta(n)}\|P_{j+n+1}(M_z^n s_j(M_z)f - f_n)\| \\
&= \frac{1}{\beta(n)}\|P_{j+n+1}M_z^n s_j(M_z)f\| \\
&\leq \frac{1}{\beta(n)}\sum_{m=0}^{j} |\hat{g}(m)|\|M_z^{m+n}P_{j-m+1}f\| \\
&\leq \frac{\|f\|}{\beta(n)}\sum_{m=0}^{j} |\hat{g}(m)|\|M_z^{m+n}|_{\mathcal{M}_{j-m+1}}\| \\
&= \|f\|\sum_{m=0}^{j} |\hat{g}(m)| \sup_i \frac{\beta(j+n+i+1)}{\beta(n)\beta(j+i+1-m)}.
\end{aligned}$$

Note that $\|M_z^n\| = \sup_i \frac{\beta(n+i)}{\beta(i)}$ and M_z is power bounded, thus

$$\sup\{\frac{\beta(n+i)}{\beta(i)} : i \geq 1,\ n \geq 1\} < \infty.$$

Clearly, for all j we obtain that

$$\alpha_j = \sup\{\frac{\beta(n+i)\beta(j)}{\beta(n+j)\beta(i)} : n \geq 1,\ i \geq j+1\} < \infty.$$

Now for the positive integer j_0 in the condition of the theorem we have

$$\begin{aligned}
&\sup_i \frac{\beta(j_0+n+i+1)}{\beta(n)\beta(j_0+i+1-m)} \\
&= \sup_i \frac{\beta(j_0+n+i+1)\beta(j_0)}{\beta(j_0+n)\beta(j_0+i+1)} \frac{\beta(j_0+i+1)\beta(j_0+n)}{\beta(n)\beta(j_0)\beta(j_0+i+1-m)} \\
&\leq \alpha_{j_0} \frac{\beta(j_0+n)}{\beta(n)\beta(j_0)} \sup_i \frac{\beta(j_0+i+1)}{\beta(j_0+i+1-m)}.
\end{aligned}$$

Also, since

$$\|M_z^m|_{\mathcal{M}_{j_{0+1-m}}}\| = \sup_i \frac{\beta(j_0+i+1)}{\beta(j_0+i+1-m)} < \infty$$

for $m = 0, 1, \ldots, j_0$, there exists a positive number c such that

$$\sum_{m=0}^{j_0} |\hat{g}(m)| \sup_i \frac{\beta(j_0+i+1)}{\beta(j_0+i+1-m)} < c.$$

Thus we have

$$\frac{1}{\beta(n)} \|M_z^n s_{j_0}(M_z)f - f_n\| \leq c_n$$

where

$$c_n = \|f\| \alpha_{j_0} \frac{c}{\beta(j_0)} \frac{\beta(j_0+n)}{\beta(n)}.$$

But $\{\frac{\beta(n+j_0)}{\beta(n)}\}_n \in \ell^q$, hence $\{c_n\} \in \ell^q$ and so there exists a positive integer $n_0 > j_0$ such that $\alpha = (\sum_{n \geq n_0} c_n^q)^{\frac{1}{q}} < 1$. Now for all nonnegative integer m define

$$e_m = \frac{1}{\beta(n_0+m)} M_z^{n_0+m} s_{j_0}(M_z)f\ , \qquad b_m = \frac{1}{\beta(n_0+m)} f_{n_0+m}$$

and consider any finite linear combinations

$$\varphi = \sum_m d_m e_m\ , \qquad \psi = \sum_m d_m b_m.$$

By using the Hölder inequality we get

$$\|\varphi - \psi\| \leq \|\psi\| (\sum_{n \geq n_0} c_n^q)^{\frac{1}{q}} = \alpha \|\psi\|.$$

Define $Sb_m = e_m$, $(m \geq 0)$. Then $\|(S-I)\psi\| \leq \alpha\|\psi\|$. Since $0 \leq \alpha < 1$, the operator S extends to a bounded operator on $\mathcal{M}_{n_0}$ with $\|S - I\| < 1$. Thus S is invertible on $\mathcal{M}_{n_0}$, so the sequence $\{e_m\}$ inherits the basis property of $\{b_m\}$ ([1, page 100]). Consequently, since $\{f_{n_0+m}\}_{m\geq 0}$ is a basis for $\mathcal{M}_{n_0}$, it follows immediately that $\{M_z^n s_{j_0}(M_z)f : n \geq n_0\}$ is a complete set, i.e.,

$$\mathcal{M}_{n_0} = \text{span}\{M_z^n s_{j_0}(M_z)f : n \geq n_0\}.$$

Now clearly one can see that

$$\mathcal{M}_{n_0-1} = \text{span}\{M_z^n s_{j_0}(M_z)f : n \geq n_0 - 1\}.$$

By continuing this process, we conclude that $(*)$ holds and this completes the proof. □

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