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APPLYING LINEAR LEBESGUE DENSITY TO $\mathbb{R}^{n+1}$ WITH CONCENTRIC n-SPHERES

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Abstract: Density points in $\mathbb{R}^{n+1}$ are typically determined by the use of $n+1$-dimensional Lebesgue measure. We create an alternative by considering the linear density of complete and almost-complete n-spheres around a given point in $\mathbb{R}^{n+1}$. We explore several properties of our density operators and the topologies that can be created through their use. We show that several of these topologies are normal.

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1. Complete n-Spheres

We begin by investigating the properties of Φ_C, the circle density operator. It allows us to determine the density of a point in $\mathbb{R}^{n+1}$ by considering only complete n-spheres centered at that point. (Here we do not use complete in he topological sense, but rather to distinguish from the incomplete n-spheres we will address later.) Let $A \subseteq \mathbb{R}^{n+1}$ and define $K(A) \subseteq \mathbb{R}$ by

$$r \in K(A) \Longleftrightarrow \big\{x \in \mathbb{R}^{n+1} \colon ||x|| = r\big\} \backslash A = \emptyset \tag{1}$$

Note that we do not require that A or $K(A)$ be measurable. We define the operator Φ_C to determine whether the origin is included in $\Phi_C(A)$. Each other point is handled identically by translation. Let λ_1 be linear inner Lebesgue measure. Then

$$\mathbf{0} \in \Phi_C(A) \quad \Longleftrightarrow \quad \lim_{\varepsilon \to 0^+} \frac{\lambda_1\big(K(A) \cap (0,\varepsilon)\big)}{\varepsilon} = 1 \tag{2}$$

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That is, the origin is a circle density point of A if 0 is a density point of $K(A)$ with respect to inner linear Lebesgue measure. Using Φ_C we can define our first topology, $\mathcal{T}_C$.

$$\mathcal{T}_C = \{A \subseteq \mathbb{R}^n \colon A \subseteq \Phi_C(A)\} \tag{3}$$

It is clear that $\emptyset$ and $\mathbb{R}^n$ are elements of $\mathcal{T}_C$. In addition we demonstrate that $\mathcal{T}_C$ exhibits closure under arbitrary union and finite intersection [1].

Theorem 1. *Let $\{A_\alpha \in \mathcal{T}_C \colon \alpha \in \Gamma\}$ be an arbitrary subset of $\mathcal{T}_C$. Then $\bigcup_{\alpha\in\Gamma} A_\alpha \in \mathcal{T}_C$.*

Proof. Suppose without loss of generality that $\mathbf{0} \in A_\alpha$ for some α. Then

$$\lim_{\varepsilon\to 0^+} \frac{\lambda_1\big(K(A_\alpha) \cap (0,\varepsilon)\big)}{\varepsilon} = 1 \tag{4}$$

And clearly $K(A_\alpha) \subseteq \bigcup_{\alpha\in\Gamma} K(A_\alpha)$ for each α. Therefore

$$\lim_{\varepsilon\to 0^+} \frac{\lambda_1\Big(\bigcup_{\alpha\in\Gamma} K(A_\alpha) \ \cap \ (0,\varepsilon)\Big)}{\varepsilon} = 1 \tag{5}$$

So $\mathbf{0} \in \Phi_C\Big(\bigcup_{\alpha\in\Gamma} A_\alpha\Big)$. Hence $\mathcal{T}_C$ is closed under arbitrary union. □

Theorem 2. *Let $\{A_\alpha \in \mathcal{T}_C \colon \alpha \in \Gamma\}$ be a finite subset of $\mathcal{T}_C$. Then $\bigcap_{\alpha\in\Gamma} A_\alpha \in \mathcal{T}_C$.*

Proof. It suffices to show $A_1 \cap A_2 \subseteq \Phi_C\big(A_1 \cap A_2\big)$. Suppose without loss of generality that $\mathbf{0} \in A_1, A_2$. Then there exist two sequences of positive reals ε_{1_k} and ε_{2_k} such that $\varepsilon < \varepsilon_{1_k}$ and $\varepsilon < \varepsilon_{2_k}$ imply respectively

$$\frac{\lambda_1\big(K(A_1)\cap(0,\varepsilon)\big)}{\varepsilon} > 1 - \frac{1}{k} \quad \text{and} \quad \frac{\lambda_1\big(K(A_2)\cap(0,\varepsilon)\big)}{\varepsilon} > 1 - \frac{1}{k} \tag{6}$$

So if $\varepsilon < \varepsilon_{1_k}$ and $\varepsilon < \varepsilon_{2_k}$ then

$$\frac{\lambda_1\big(K(A_1) \cap K(A_2) \ \cap \ (0,\varepsilon)\big)}{\varepsilon} > 1 - \frac{2}{k} \tag{7}$$

So $\mathbf{0} \in \Phi_C\big(A_1 \cap A_2\big)$. Hence $\mathcal{T}_C$ is closed under finite intersection. □

Furthermore, it can be shown that circle density is strictly more selective than ordinary density. If λ_n is inner Lebesgue measure in $\mathbb{R}^n$, then the corresponding Lebesgue density operator is defined as

$$\mathbf{0} \in \Phi_n(A) \Longleftrightarrow \lim_{\varepsilon \to 0^+} \frac{\lambda_n\big(A \cap (-\varepsilon, \varepsilon)^n\big)}{(2\varepsilon)^n} = 1 \tag{8}$$

Theorem 3. *Let $A \subseteq \mathbb{R}^{n+1}$. If $A \subset \Phi_C(A)$ then $A \subset \Phi_{n+1}(A)$.*

Proof. Suppose $A \in \mathcal{T}_C$. Then there exists a sequence of positive reals ε_k such that if $\varepsilon < \varepsilon_k$ then

$$\frac{\lambda_1\big(K(A) \cap (0, \varepsilon)\big)}{\varepsilon} > 1 - \frac{1}{k} \tag{9}$$

At worst each point in $\mathbb{R} \backslash K(A)$ corresponds to a complete n-sphere in $\mathbb{R}^{n+1}$. So if $\varepsilon < \varepsilon_k$ then

$$\mathbf{0} \in \Phi_C(A) \Longrightarrow \frac{\lambda_n\big(A \cap (-\varepsilon, \varepsilon)^n\big)}{(2\varepsilon)^n} > 1 - \frac{S_n}{2^n} \frac{1}{k} \tag{10}$$

Where S_n be the hyper surface area of a unit n-sphere. It's clear that a no n-sphere can intersect a concentric hypercube of side length 2ε at a hyper surface area greater than $S_n \varepsilon^n$. This sequence converges as k becomes large so any circle density point is also an ordinary density point. □

The converse of Theorem 3 is not true, which we can see trivially by considering an example such as $A = \mathbb{R}^n \times \big(\mathbb{R} \backslash \mathbb{Q}\big)$, which demonstrates full Lebesgue density and null concentric circle density everywhere.

2. Incomplete n-Spheres

In (1) we used $K(A)$ to determine density points with respect to complete n-spheres. We now define two similar families as follows to determine density points with respect to nearly complete circles.

$$\begin{aligned} r \in K_0(A) \quad &\Longleftrightarrow \quad \big\{x \in \mathbb{R}^{n+1} \colon ||x|| = r\big\} \backslash A \text{ is null} \\ r \in K_1(A) \quad &\Longleftrightarrow \quad \big\{x \in \mathbb{R}^{n+1} \colon ||x|| = r\big\} \backslash A \text{ is of first category} \end{aligned} \tag{11}$$

Of course null refers to n dimensional Lebesgue measure and first category refers to a countable union of nowhere dense sets in $\mathbb{R}^n$. The comparison be made easily enough by stereographic projection. We construct $\mathcal{T}_0$ and $\mathcal{T}_1$ as in (3) but using the operators Φ_0 and Φ_1 respectively. For both topologies it is clear that Theorems 1 and 2 hold. The family of null sets is closed under finite union and arbitrary intersection, as is the family of first category sets. Hence their complements must be closed under arbitrary union and finite intersection.

For our topologies we can establish the following heirarchy, where $\mathcal{T}$ is the natural topology.

$$\mathcal{T} \subset \mathcal{T}_C \begin{array}{l} \subset \mathcal{T}_0 \\ \subset \mathcal{T}_1 \end{array} \tag{12}$$

The inclusions are obvious, and we can show that each inclusion is strict. We also see that $\mathcal{T}_0$ and $\mathcal{T}_1$ cannot be compared; this is a consequence of the existence of both null sets of second category and sets of first category which have full measure. We can also show that $\mathcal{T}_C \not\subset \mathcal{T}$.

Theorem 4. *There exists a plane set in $\mathcal{T}_C$ that is not an element of $\mathcal{T}$.*

Proof. Let A be the union of a sequence of open annuli centered at the origin such that $K(A)$ has full density at the origin. Then $A \cup \{\mathbf{0}\}$ an element of $\mathcal{T}_C$ but not of $\mathcal{T}$. □

3. Separation Axioms

It is easy to see that each topology defined here is Hausdorff. We show that under the Martin Axiom both $\mathcal{T}_0$ and $\mathcal{T}_1$ are also T_4, and therefore normal. We find that $\mathcal{T}_C$ is not normal. Further separation properties remain open.

Theorem 5. *If the union of fewer than continuum Lebesgue null sets (respectively, first category sets) is null (first category) then $\mathcal{T}_0$ ($\mathcal{T}_1$) is normal.*

Proof. Let $A_0, B_0 \subseteq \mathbb{R}^n$ be disjoint sets closed in $\mathcal{T}_0$. Then we must construct $A, B \in \mathcal{T}_0$ such that $A_0 \subseteq A$, $B_0 \subseteq B$, and $A \cap B = \emptyset$. We then proceed with a technique devised by Sierpinski [2].

Since n is finite, there are exactly $\mathfrak{c}$ n-spheres in $\mathbb{R}^{n+1}$. We well order them as $\{S_\alpha \colon \alpha < \omega_\mathfrak{c}\}$. Let $S'_0 = S_0$. Then for $\alpha < \omega_\mathfrak{c}$ let $S'_\alpha = S_\alpha \setminus \bigcup_{\beta<\alpha} S_\beta$. The intersection of two distinct n-spheres is null using n-dimensional Lebesgue measure and $\alpha < \omega_\mathfrak{c}$ so S'_α is an n-sphere missing only a null set. Then for $x \in \mathbb{R}^{n+1}$ and define

$$E_x = \bigcup S'_\alpha \colon S_\alpha \textit{ is centered at } x \tag{13}$$

Note that because the sets S'_α are pairwise disjoint, the sets E_x must be as well. We continue inductively by creating

$$\begin{aligned} A_{k+1} &= \bigcup_{x \in A_k} E_x \\ B_{k+1} &= \bigcup_{x \in B_k} E_x \end{aligned} \tag{14}$$

So $A_k \subseteq \Phi_0(A_{k+1})$ and $B_k \subseteq \Phi_0(B_{k+1})$ by construction. Finally, let

$$\begin{aligned} A &= \bigcup_{n=0}^{\infty} A_k \backslash B_0 \\ B &= \bigcup_{n=0}^{\infty} B_k \backslash A_0 \end{aligned} \tag{15}$$

Clearly $A_0 \subseteq A$, $B_0 \subseteq B$ and $A \cap B = \emptyset$. We must finally show that A and B are open in $\mathcal{T}_0$.

Consider some point $x \in A$. Then there exists k such that $x \in A_k$. So A_{k+1} includes E_x, meaning that all but a null set of points from each n-sphere centered at x are elements of A_{k+1}. Furthermore, $\mathbb{R}^{n+1}\backslash B_0$, of which x is an element, is open in $\mathcal{T}_0$. Hence the n-spheres centered at x in which B_0 has positive measure must be of null density at x. Hence $A \in \mathcal{T}_0$.

By symmetry, the same argument can be made to demonstrate that $B \in \mathcal{T}_0$. Hence $\mathcal{T}_0$ is T_4 and thus normal.

A nearly identical proof demonstrates that $\mathcal{T}_1$ is T_4 if we instead take the union of fewer than continuum sets of first category to be of first category. Whether this proof can be completed in ZFC remains an interesting question. □

Theorem 6. *$\mathcal{T}_C$ is not normal.*

Proof. Let X be the x axis in $\mathbb{R}^{n+1}$ and let Q and Q' be disjoint subsets of X which are dense on the line. We also require that Q and Q' be countable; as a result they must be closed in $\mathcal{T}_C$.

From Theorem 3 we see that for $A \subseteq \mathbb{R}^{n+1}$, $\Phi_C(A) \cap X \subseteq \Phi_1(A \cap X)$ where Φ_1 is the linear density operator on the x axis. So it suffices to show that Q and Q' cannot be separated in the ordinary density topology on the line, a proof which can be found in [3]. □

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